Introduction to Nuclear Fusion as An Energy Source

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Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 7

2026 spring semester

Tuesday 9:00-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://reurl.cc/MMnkOL

2026/4/16 updated

Note

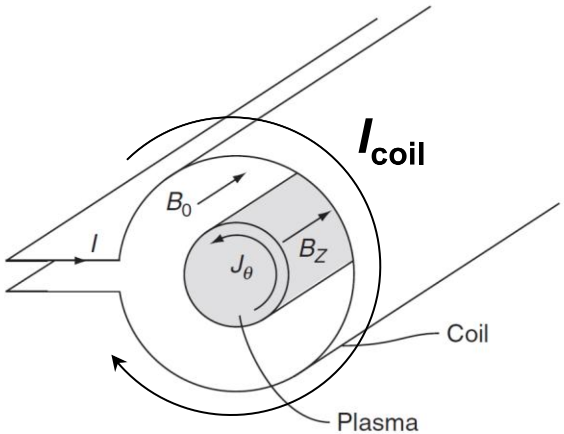
- 課程公告:Tokamak Physics
- 講者:陳騮 院士
- 課程日期:2023/4/23 至 6 月底
- 課程時間:每周二與周四,上午 10 點至 12 點
- 課程地點:科學四館 203 教室
- 報名方式:無須報名,於上課時間前往 203 教室即可
- 相關資訊:此次課程延續 2023 年 10 月至 12 月的內容,上課影片可以至下列
- Youtube 連結觀看 https://youtube.com/playlist?list=PLB7ffAhbDy-lcd4oWWcWsPpEmzGTfXmeZ&si=dgQVIX_GBxyYjTDM

- Fusion research lecture by DerPlasma (University of Stuttgart)
- https://youtube.com/playlist?list=PL9F2aQG5CnOdw9MXqS309tojBUvChQ9jm&si=Hyr9YttCucjhGuej

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z \hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}(\vec{B} \cdot \vec{\nabla})\vec{B} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

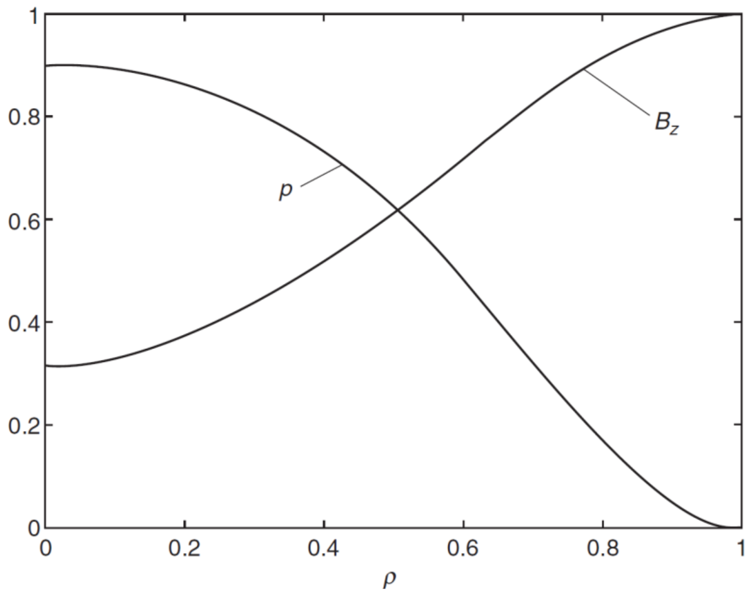
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma

- **Example:**

$$\frac{2\mu_o p(r)}{B_o^{\,2}} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$



$$j_\theta B_z = \frac{dp}{dr} \quad \Rightarrow \quad \frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{B_o^{\,2}} \qquad \rho = \frac{r}{a}$$

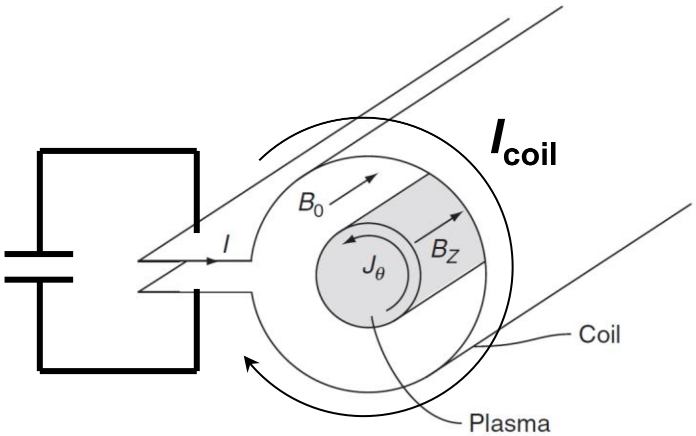
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{B_o^{\,2}} = \frac{4\mu_o}{a^2 B_o^{\,2}}\int_0^a p r dr = 2\int_0^1\left(1-\frac{B_z^{\,2}}{B_o^{\,2}}\right)\rho d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_o}{2}\ , \beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad \Rightarrow \quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$$\boxed{0 < \beta < 1}$$

Theta pinches provide good radial confinement but NOT axially

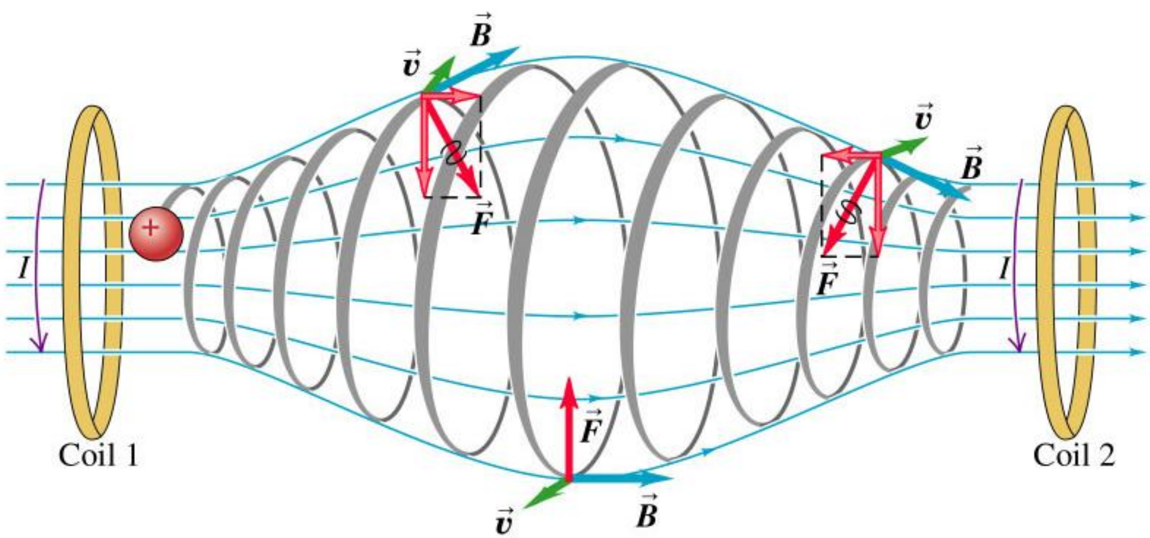


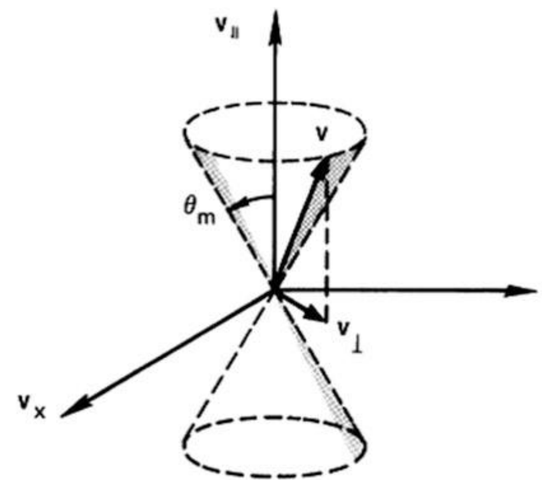
- The gas is initially preionized.
- The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.
- The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.
- This energy is ultimately converted to heat after repeated reflections off the converging piston.
- T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.
- The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{\mathrm{Ti}} \sim 10\mu s$ for L = 5 m.

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- **Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.**





$$v'_\perp{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_\perp{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv sin^2\theta$$

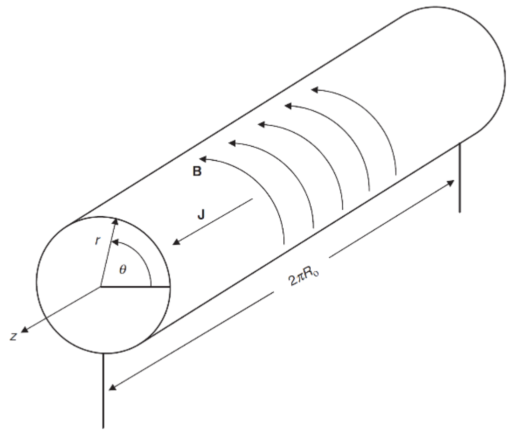
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_\perp{}^2 \qquad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_\perp{}^2}{B}$$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = sin^2\theta_m$$

- **Large $v_{||}$ may occur from collisions between particles.**
- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

- **Symmetry:** $\partial_\theta = \partial_z = 0$

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$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

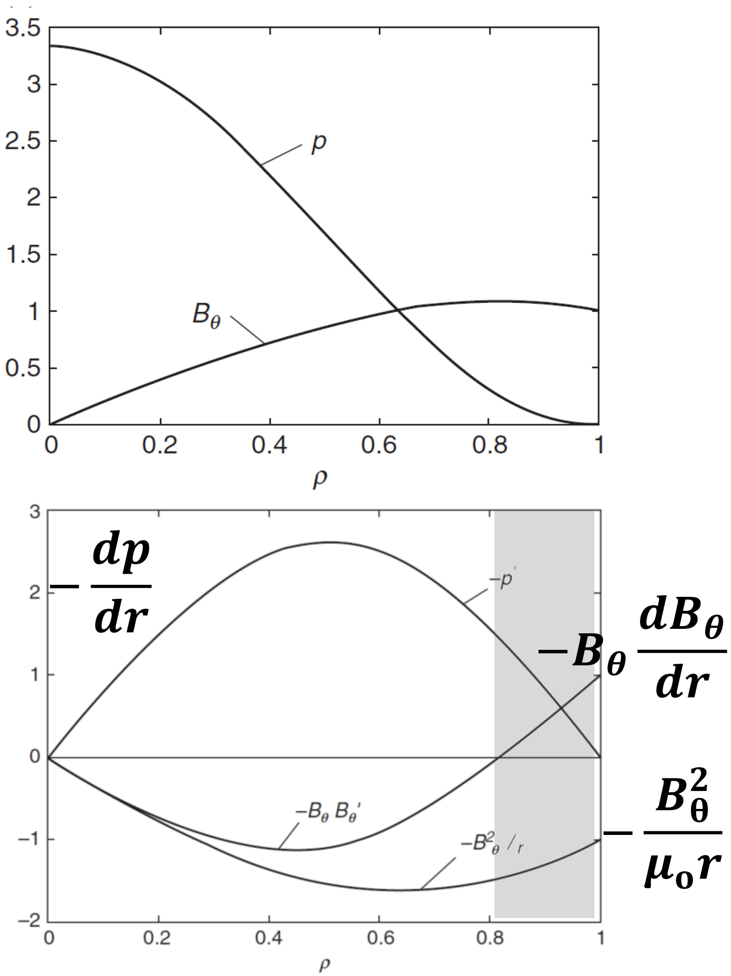
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β



$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$\frac{2\mu_o p(r)}{B_{\theta a}{}^2} = \frac{2}{3}(5 - 2\rho^2)(1 - \rho^2)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}} = 2\rho\left(1 - \frac{\rho^2}{2}\right)$$

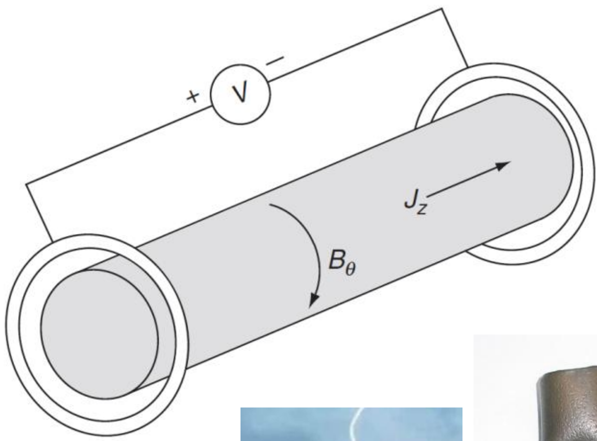
$$\frac{a\mu_o j_z(r)}{B_{\theta a}} = 4(1 - \rho^2)$$

$$B_{\theta a} \equiv B_\theta(a) = \frac{\mu_o I}{2\pi a}$$

$$\beta \equiv \beta_p = \frac{2\mu_o\langle p\rangle}{B_{\theta a}{}^2} = \frac{4\mu_o}{a^2 B_{\theta a}{}^2}\int_0^a prdr = 1$$

Bennett pinch relation: $\beta = 1$

Huge instabilities occur in a z pinch

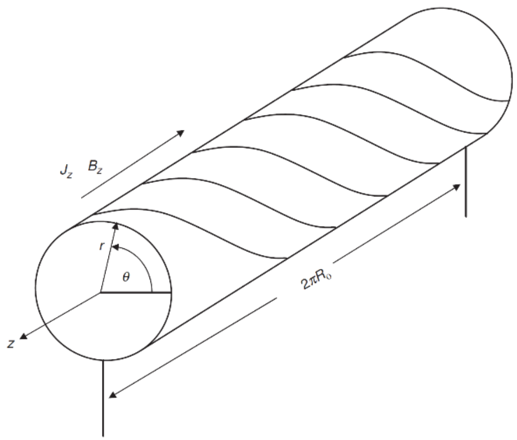


- **A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.**
- **The gas is ionized by the high voltage and produces a z current flowing along the plasma.**
- **Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.**

Main issue: unstable.

Facebook@鄭明典 (https://www.facebook.com/share/p/18N8x8kFmu/)
https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

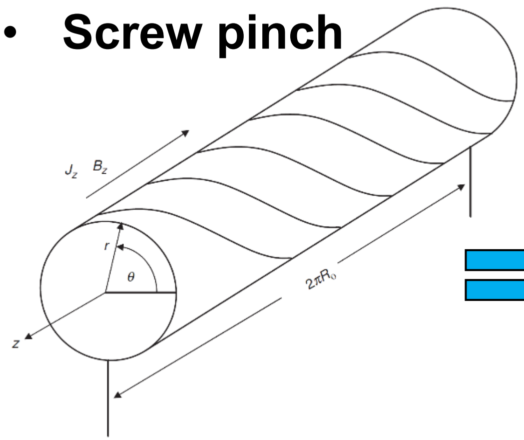
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch is flexible with varies range of $\boldsymbol{\beta}$

- **Screw pinch**



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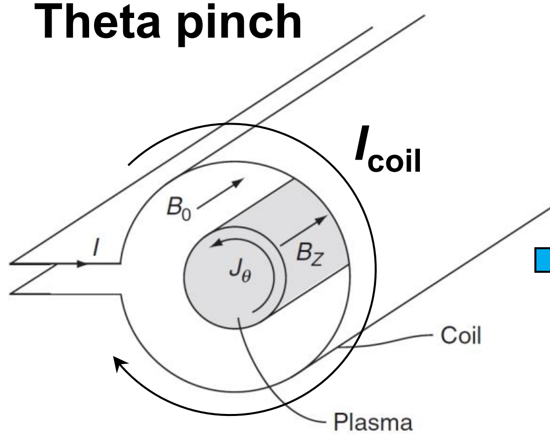
- **Theta pinch**



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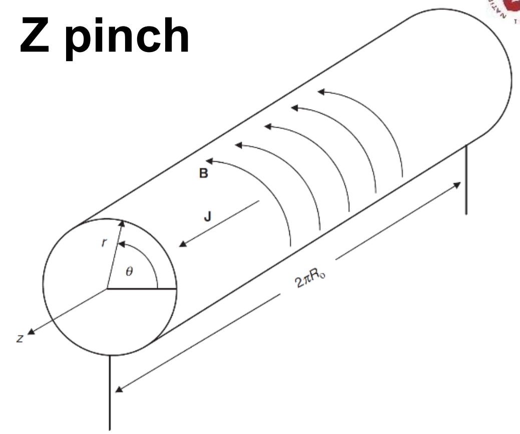
- **Z pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\vec{B} = B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta}$$

$$\vec{B} = B_\theta \hat{\theta}$$
$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

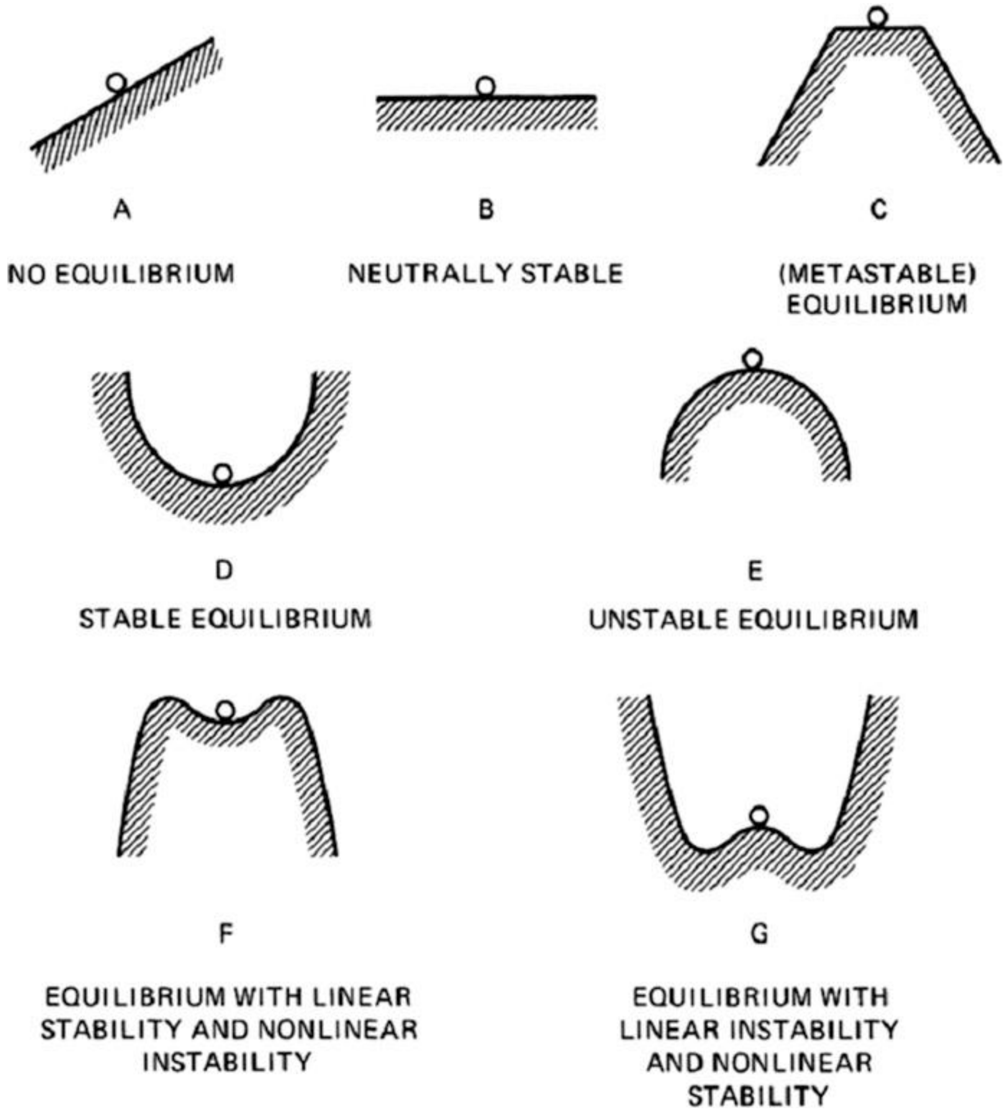
$$\int_0^a \pi r^2 dr \left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2 \rangle\right)$$

$$\beta_t = \frac{2\mu_o \langle p \rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o \langle p \rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_o \langle p \rangle}{B_o^2 + {B_{\theta a}}^2}$$

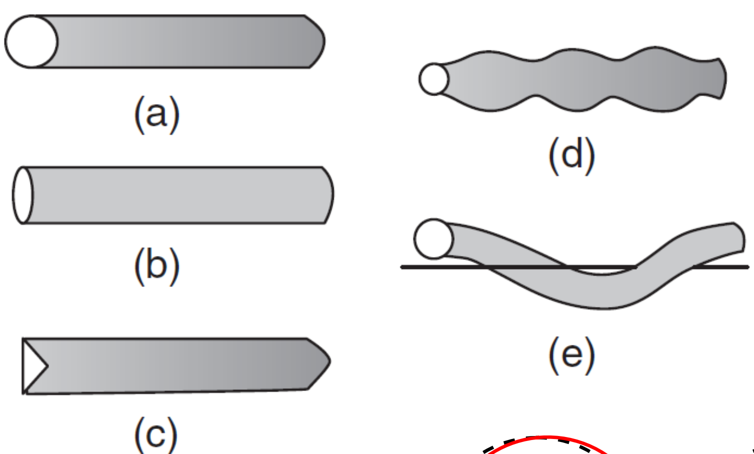
$$\boxed{0 \le \langle \beta \rangle \le 1}$$

An equilibrium state may not be stable

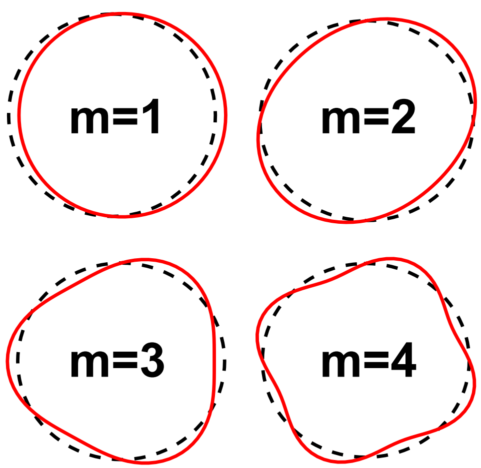


A cylindrical plasma column may not be stable

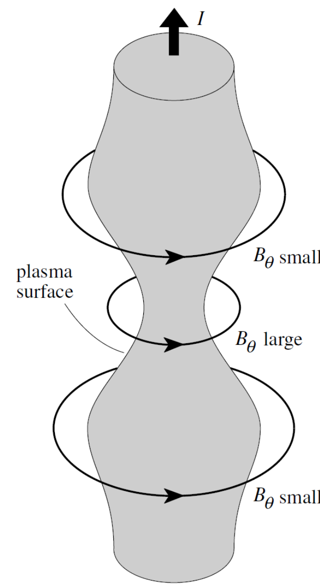
- **Instabilities of theta pinch**



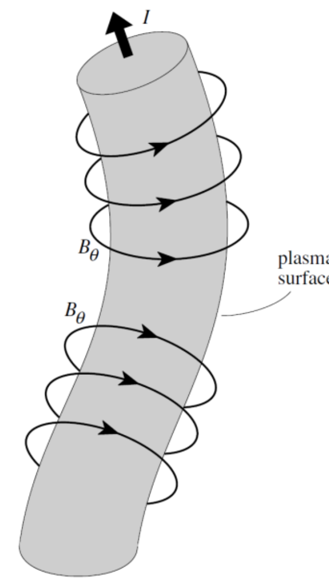
(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0



- **Instabilities of z pinch**



Sausage instability (m=0)

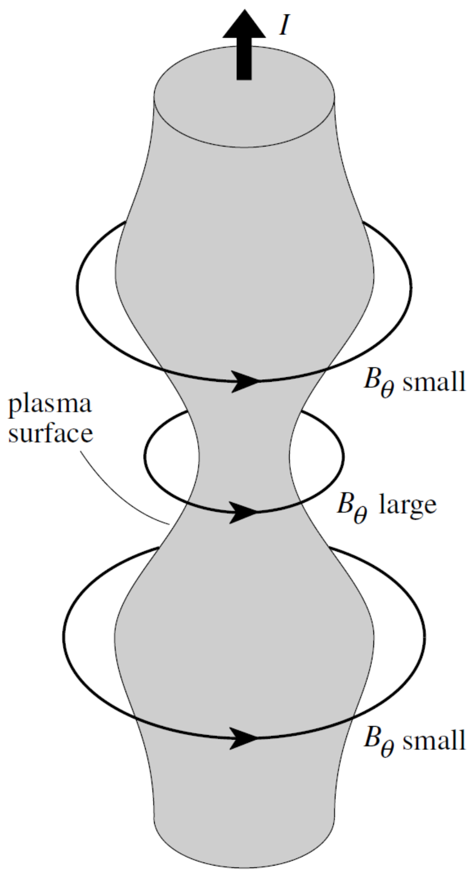


Kink instability (m=1)

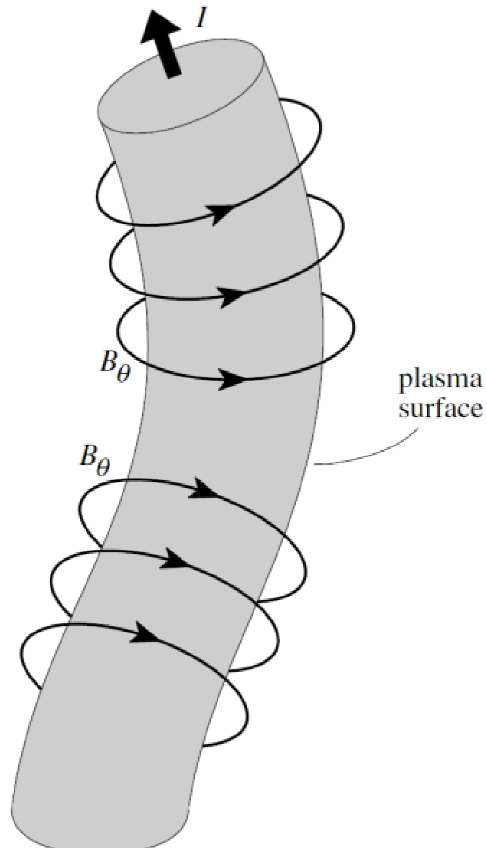
$$\zeta(\vec{r}) = \zeta(r)exp(im\theta + ikz)$$

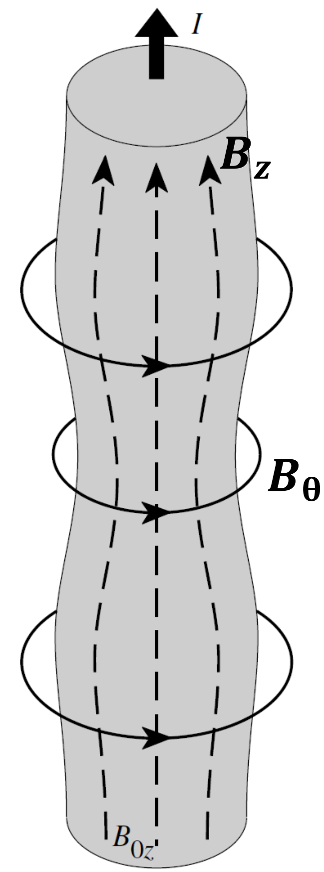
A cylindrical plasma column is stable when the safety factor is greater than unity (Kruskal–Shafranov limit)

- **Sausage instability (m=0)**



- **Kink instability**



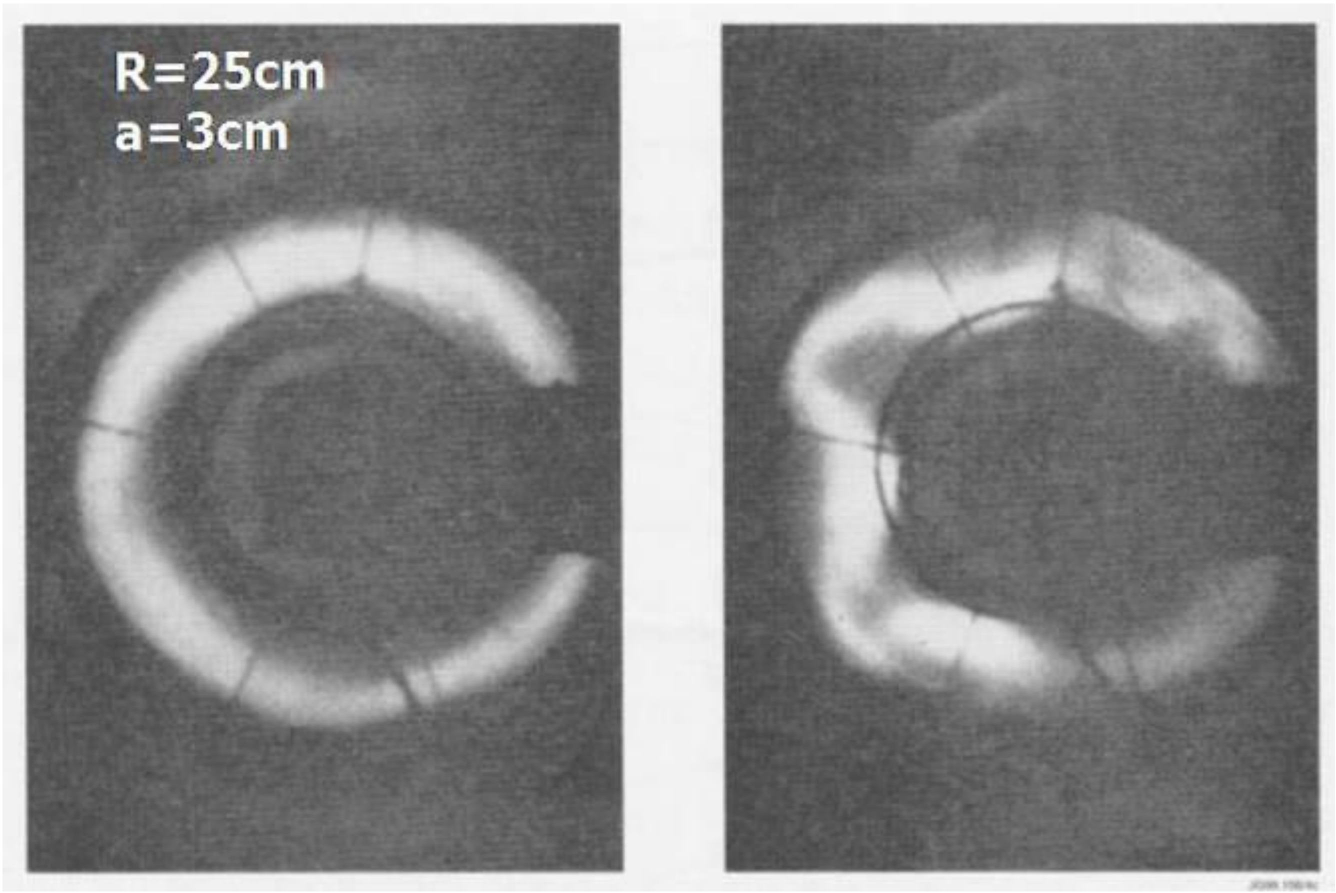


- **The tension of B_z provides the stabilizing force and suppresses the instabilities.**

- **MHD Safety factor:** $$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

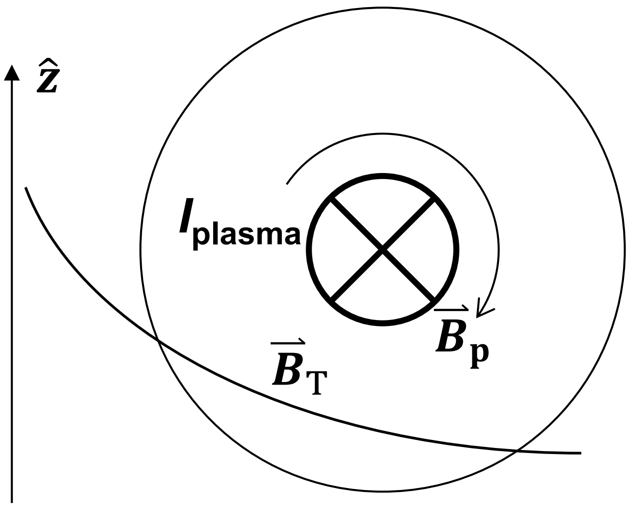
- **Kruskal–Shafranov limit:** $q(r) > 0$ **: stable**

Kink instability in action in a 3 by 25-cm pyrex tube at Aldermaston



https://en.wikipedia.org/wiki/Kink_instability
R A Bingham et al 2026 Plasma Phys. Control. Fusion 68 030201

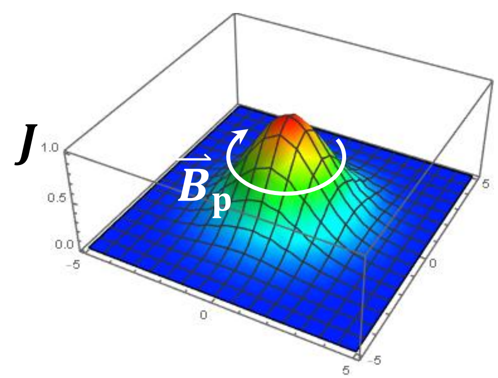
Sawtooth oscillation is initiated when the core is heated

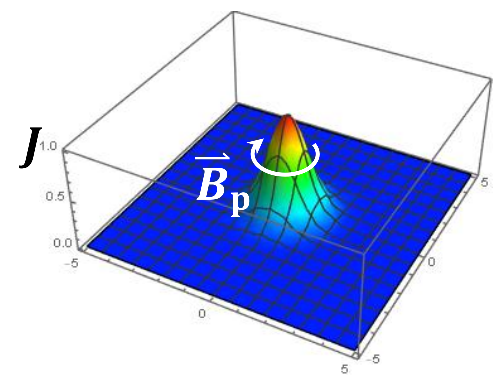


$$\eta \propto T_{\mathrm{e}}^{-3/2} \qquad E_{\parallel} = \eta J$$

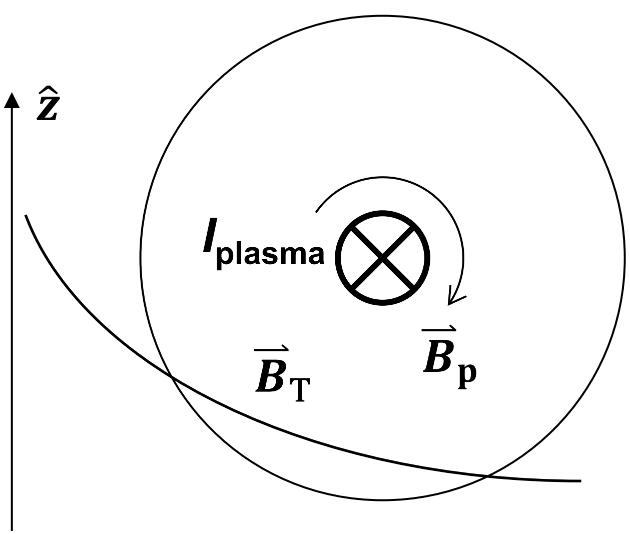
$$T_{\mathrm{e}} \uparrow \Rightarrow \eta \downarrow \Rightarrow J \uparrow$$

- Current is less diffused.
- Current density with a higher peak is formed.





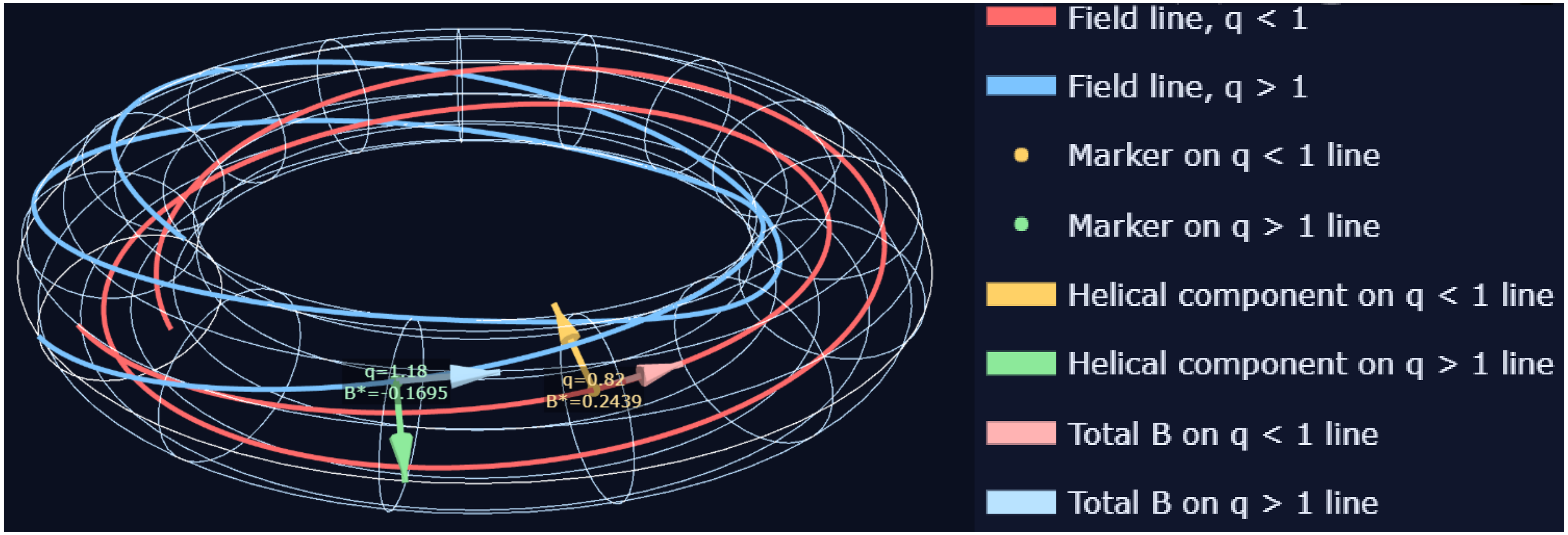
$$J \uparrow \Rightarrow \vec{B}_{\mathrm{p}} \uparrow \Rightarrow q(r) = \frac{r B_{\mathrm{T}}(r)}{R_{\mathrm{o}} B_{\mathrm{p}}(r)} \downarrow$$



- For a certain radius a,

$$q(a) = \frac{r B_{\mathrm{T}}(a)}{R_{\mathrm{o}} B_{\mathrm{p}}(a)} < 0 \ \Rightarrow \text{kink unstable}$$

The helical field component B* is opposite across the surface of q=1



$$B_* = \vec{B} \cdot \nabla(m\theta - n\phi)$$

$$B_* = \vec{B} \cdot \nabla(\theta - \phi) = \frac{B_\mathrm{p}}{r} - \frac{B_\mathrm{T}}{R} = \frac{B_\mathrm{T}}{R}\left(\frac{1}{q} - 1\right)$$

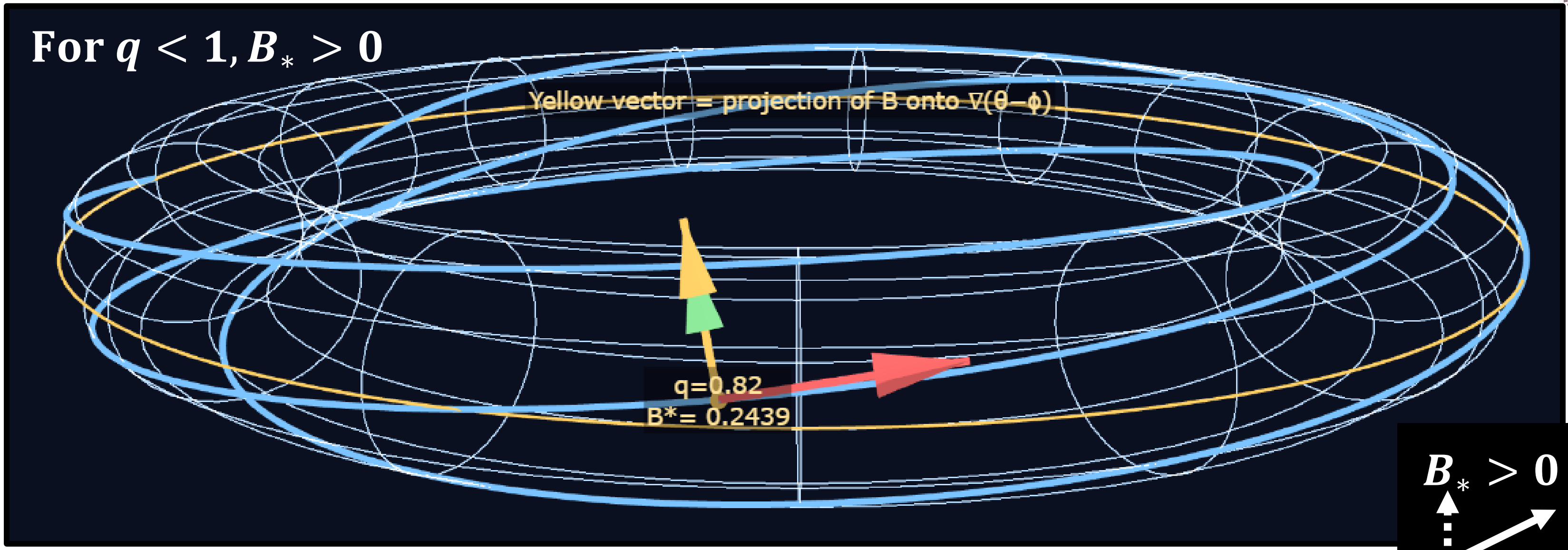
$$q(r) = \frac{rB_\mathrm{T}(a)}{R_\mathrm{o}B_\mathrm{p}(a)}$$

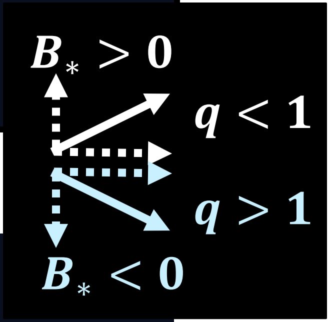
For $q < 1$, $\quad B_* > 0$

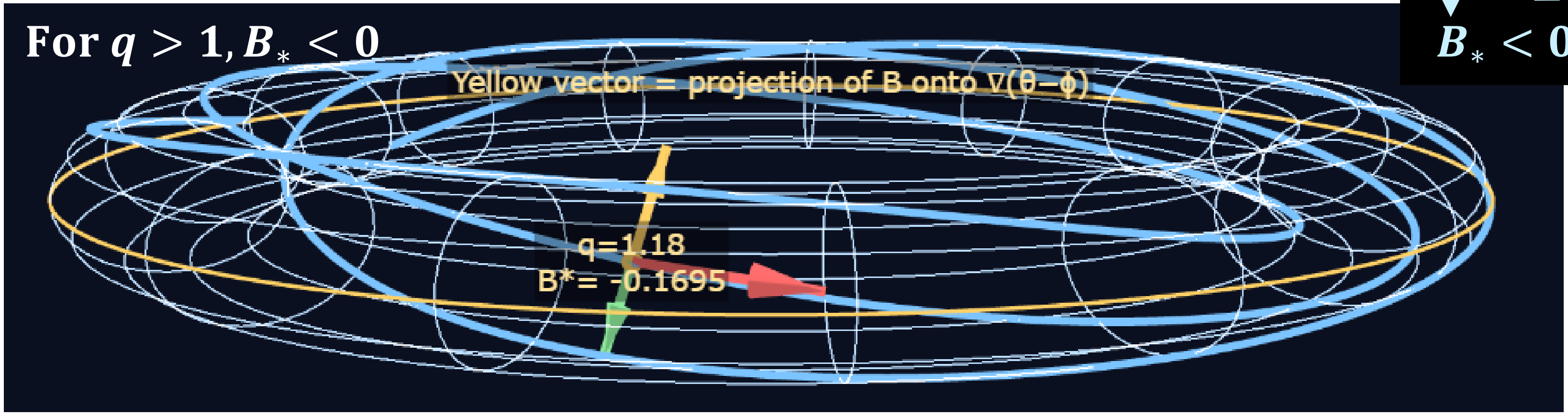
For $q = 1$, $\quad B_* = 0$

For $q > 1$, $\quad B_* < 0$

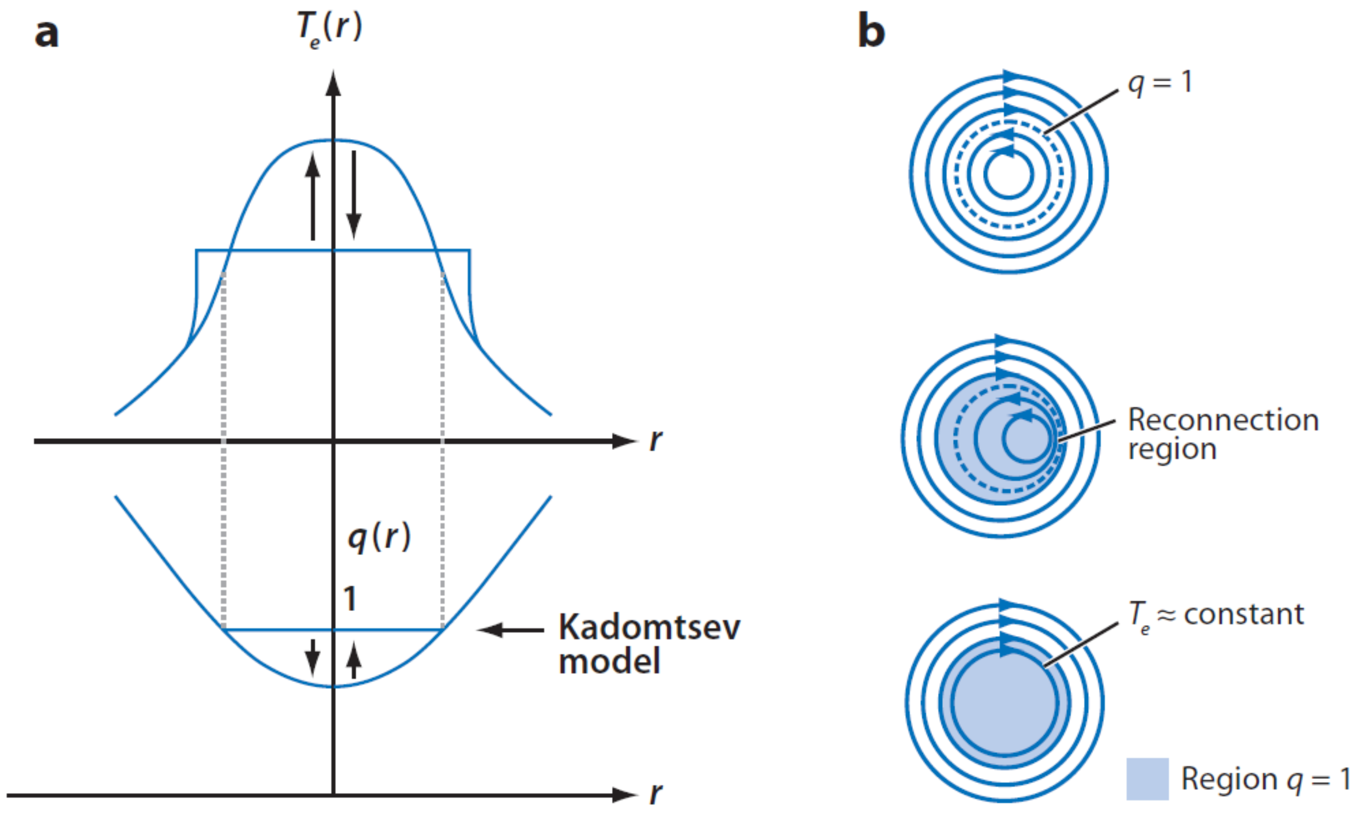
The helical field component B* is opposite across the surface of q=1





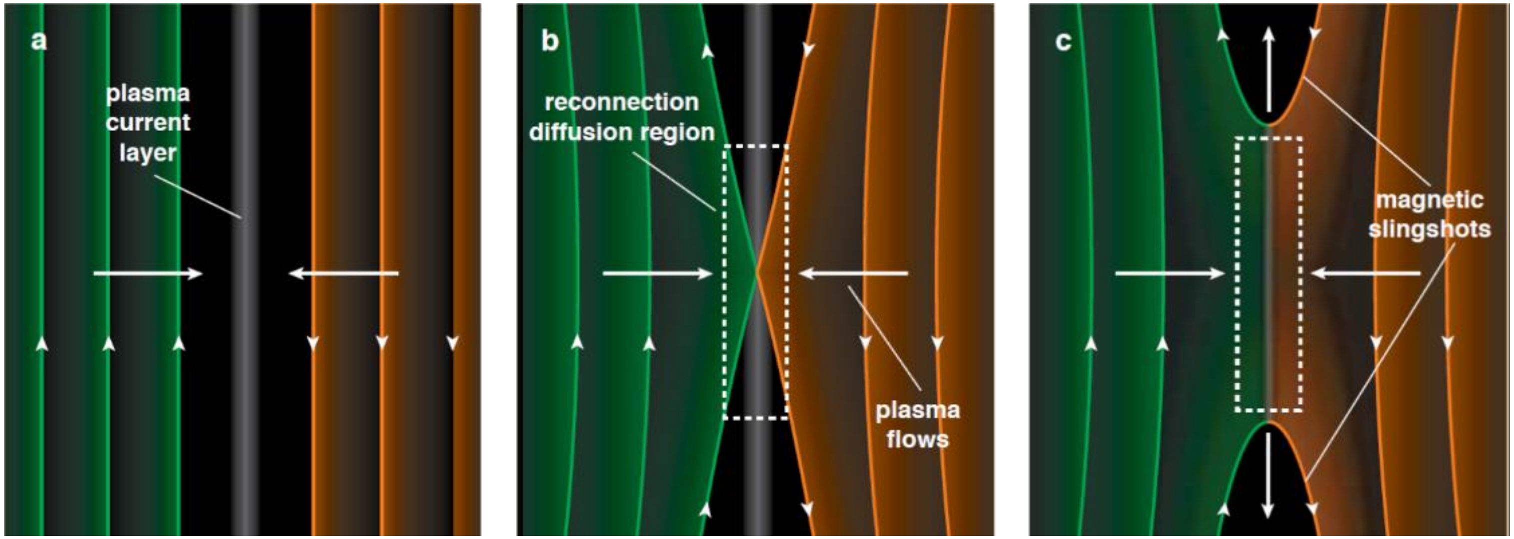


Kink instabilities occur at q <0 leading to reconnection events



E. G. Zweibel and M. Yamada, Annu. Rev. Astron. Astrophys., 47, 291 (2009)

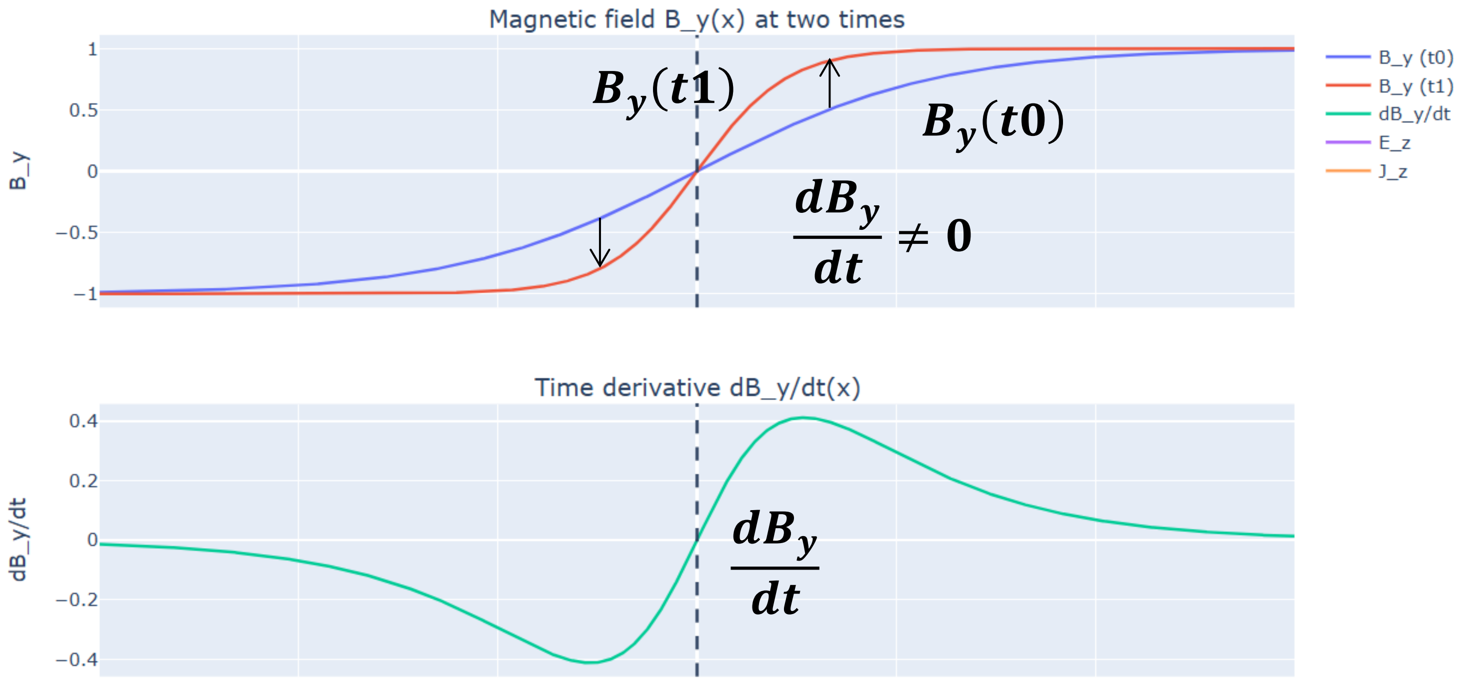
Reconnection converts the magnetic field energy to kinetic energy of particles



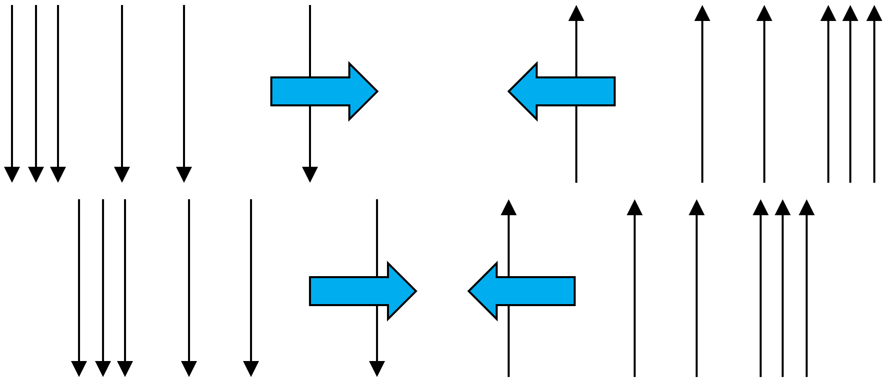
https://www.youtube.com/watch?v=7sS3Lpzh0Zw

- **Energetic particles are thermalized when they collide with the surrounding particles.**

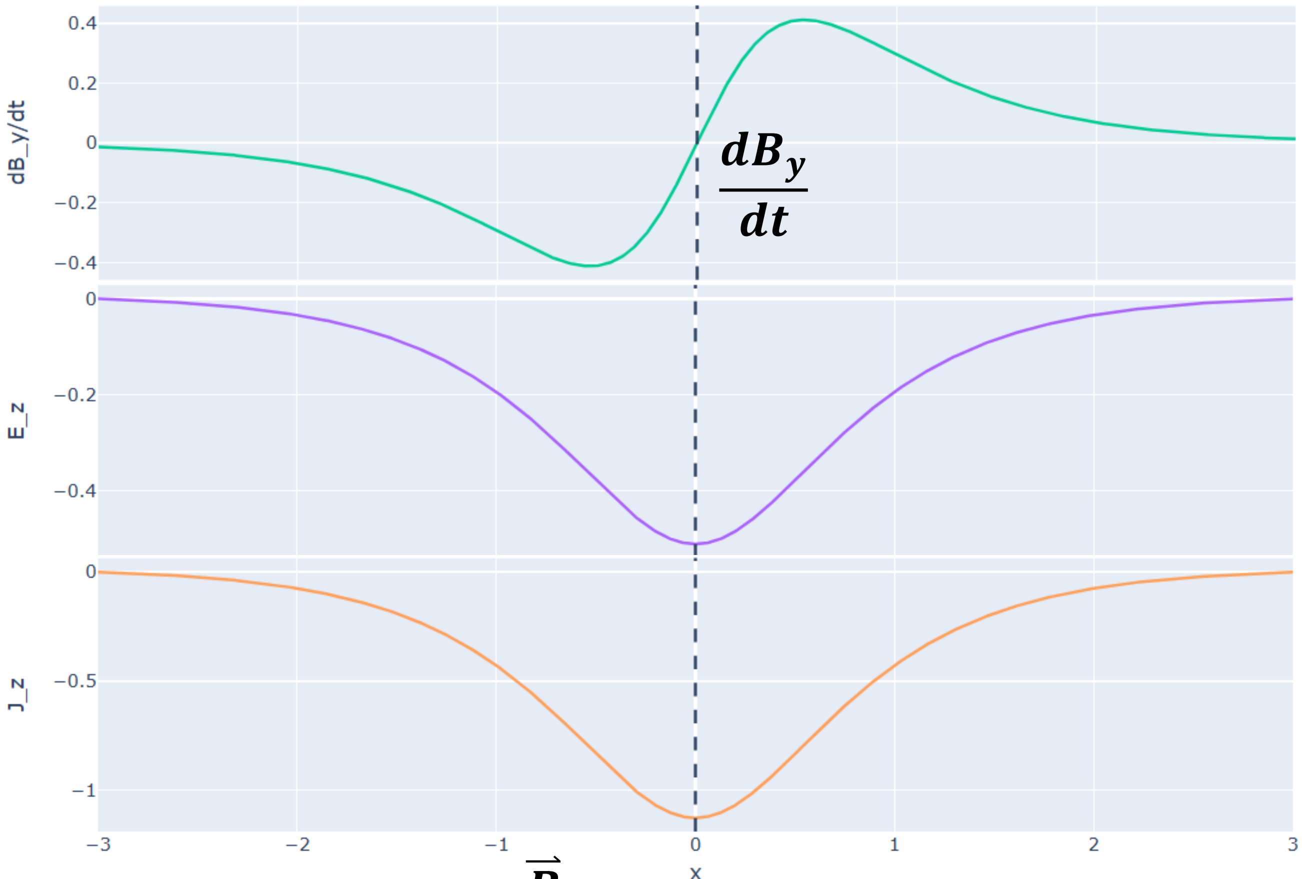
Reconnection



- t0
- t1



Reconnection – electric field is generated due to faraday's law



$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

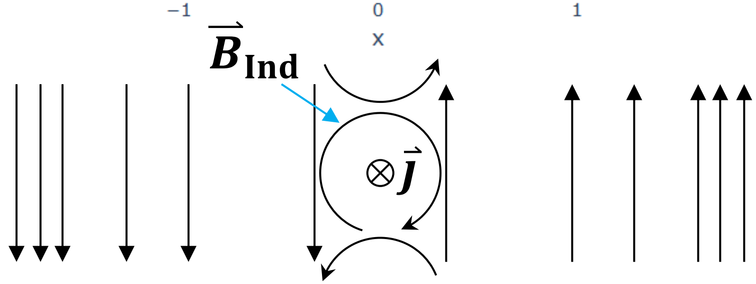
$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}$$

$$E_z = \int \frac{\partial B_y}{\partial t} dx$$

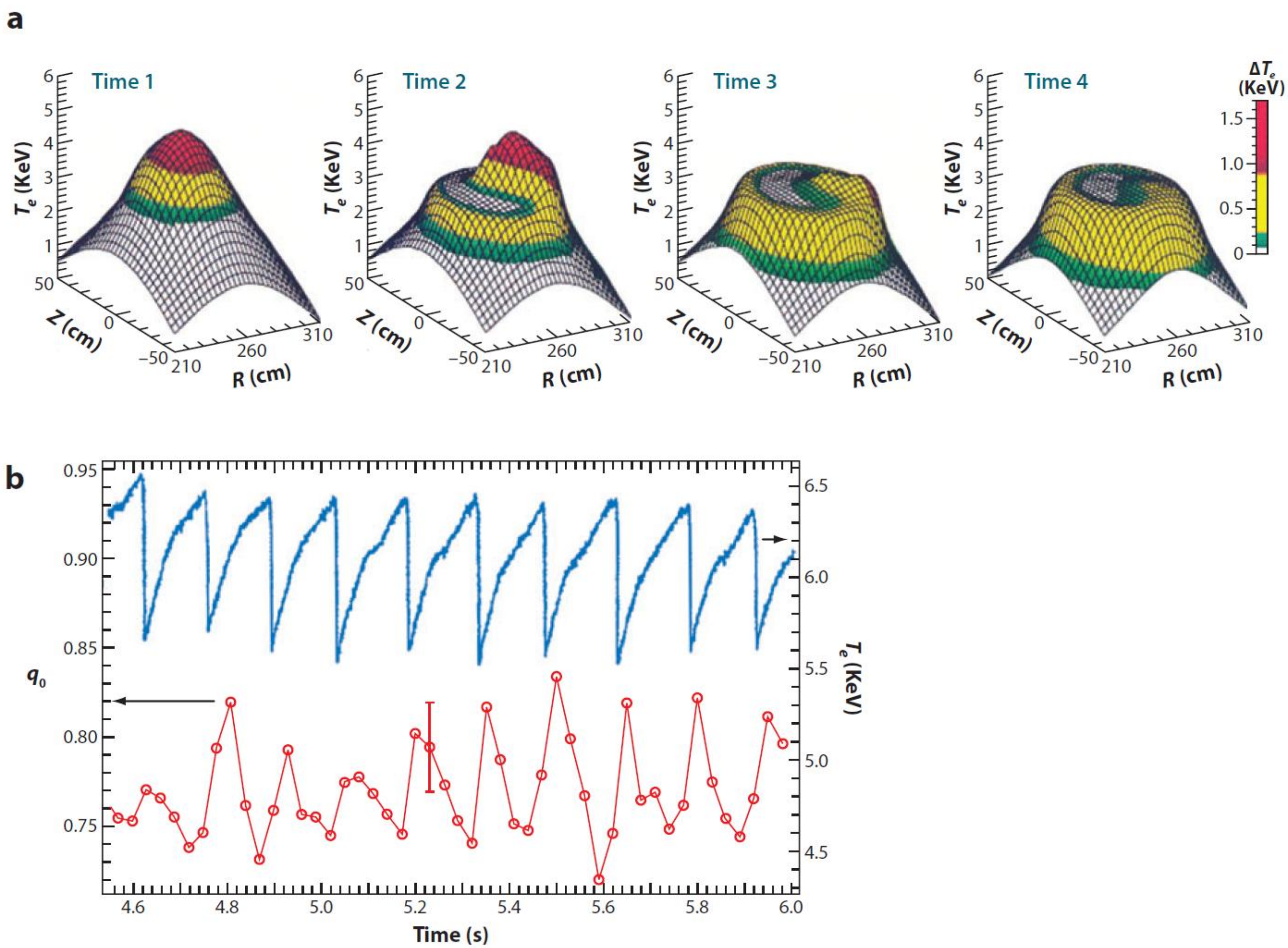
$$\eta j_z = E_z + v_x B_y$$

$$\eta j_z \approx E_z$$

$$B_y \approx 0 \,@\, x = 0$$



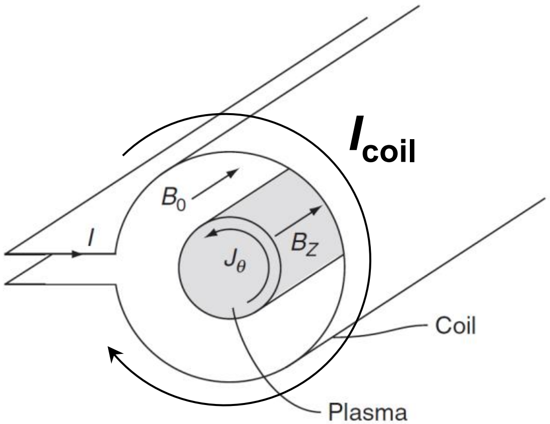
Temperature and safety factor oscillates during the sawtooth crash

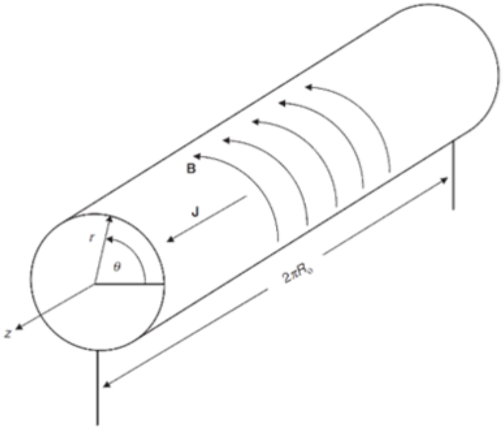


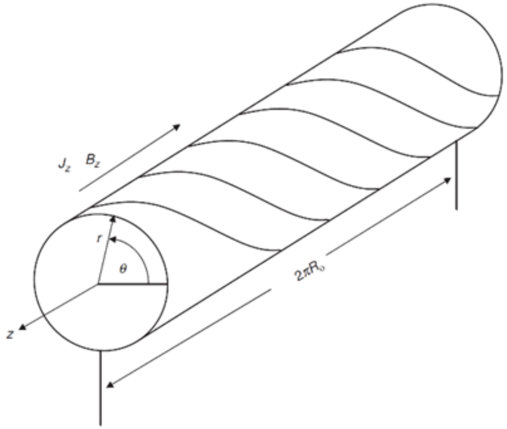
E. G. Zweibel and M. Yamada, Annu. Rev. Astron. Astrophys., 47, 291 (2009)

Theta pinch is stable while z pinch is unstable

- **Theta pinch**
- **Z pinch**
- **Screw pinch**







$\vec{B} = B_z\hat{z}$

$q_\theta = \infty$

Stable

$\vec{B} = B_\theta\hat{\theta}$

$q_z = 0$

Unstable

$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$

q can be controlled.

Stable/Unstable

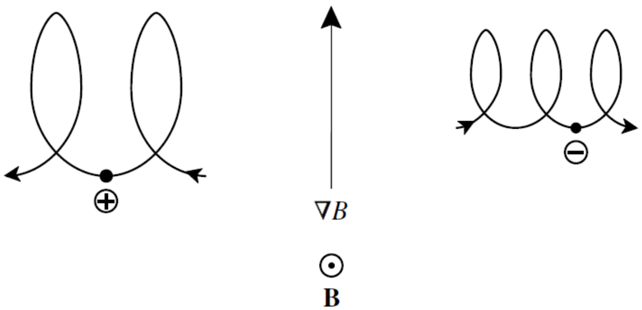
$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

Stellerator

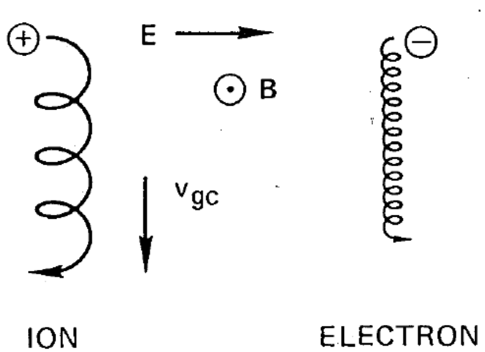
- Figure eight shape
- Oval shape (racetrack)
- Torsatron
- Heliotron
- Heliac (Helical Axis stellarator)
- Helias (W7-x)

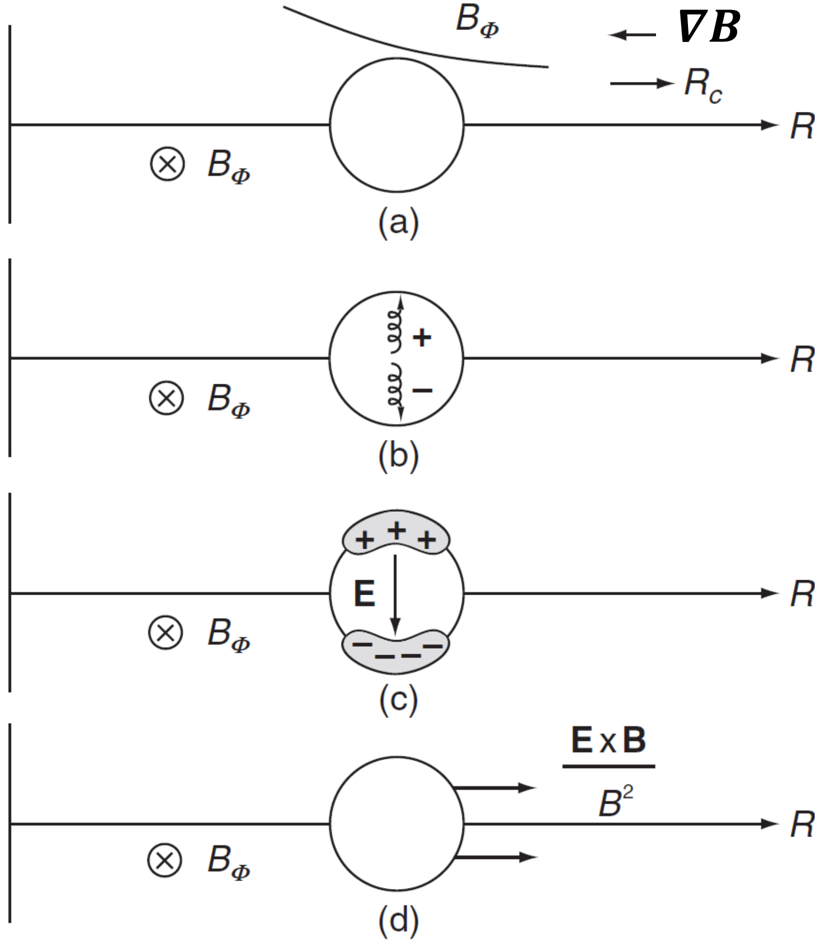
Charged particles drift across field lines

- **Grad-B drift**



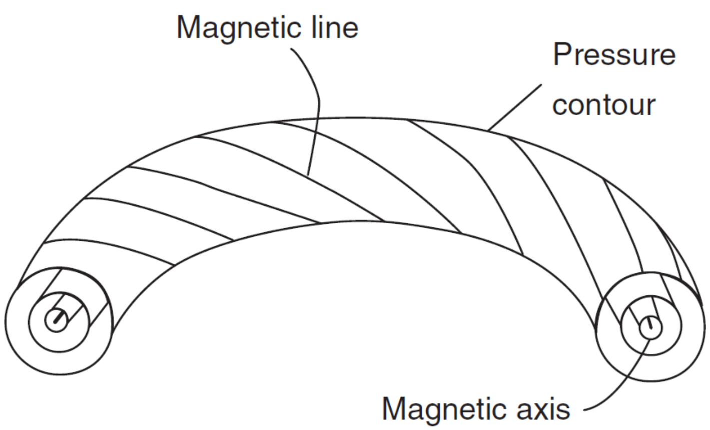
- **ExB drift**

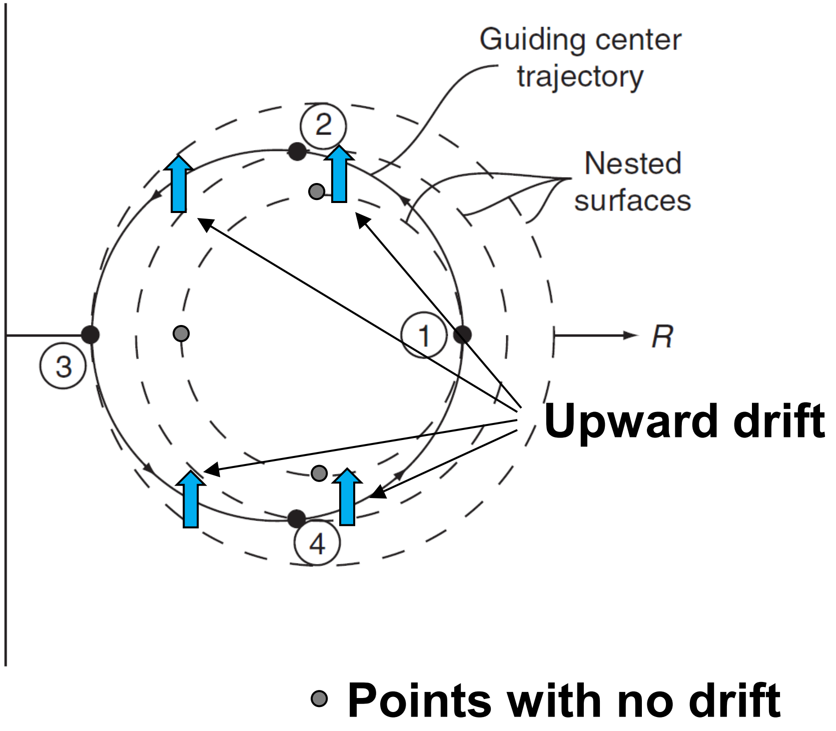




J. P. Freidberg, Ideal Magnetohydrodynamics

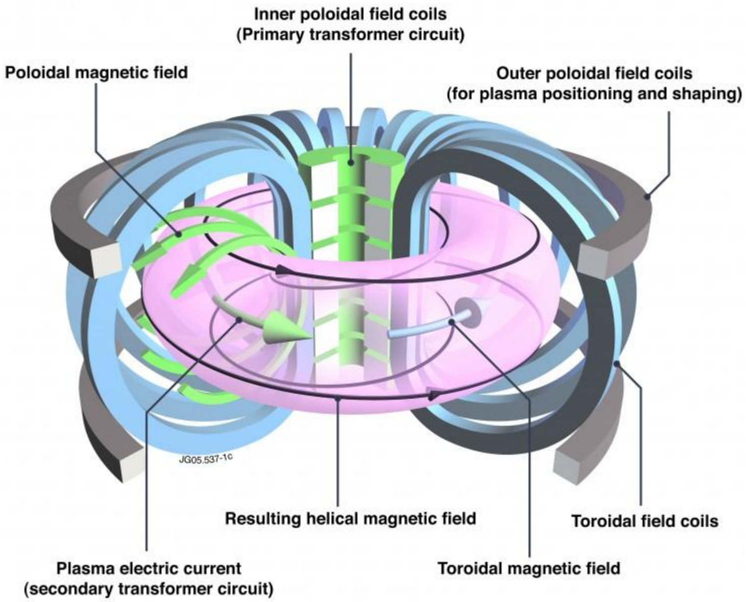
The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field



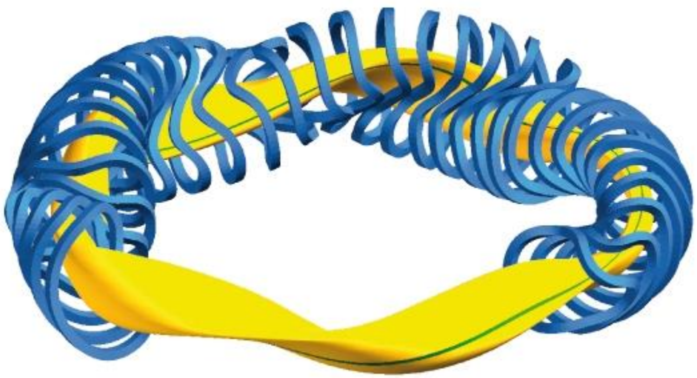


Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



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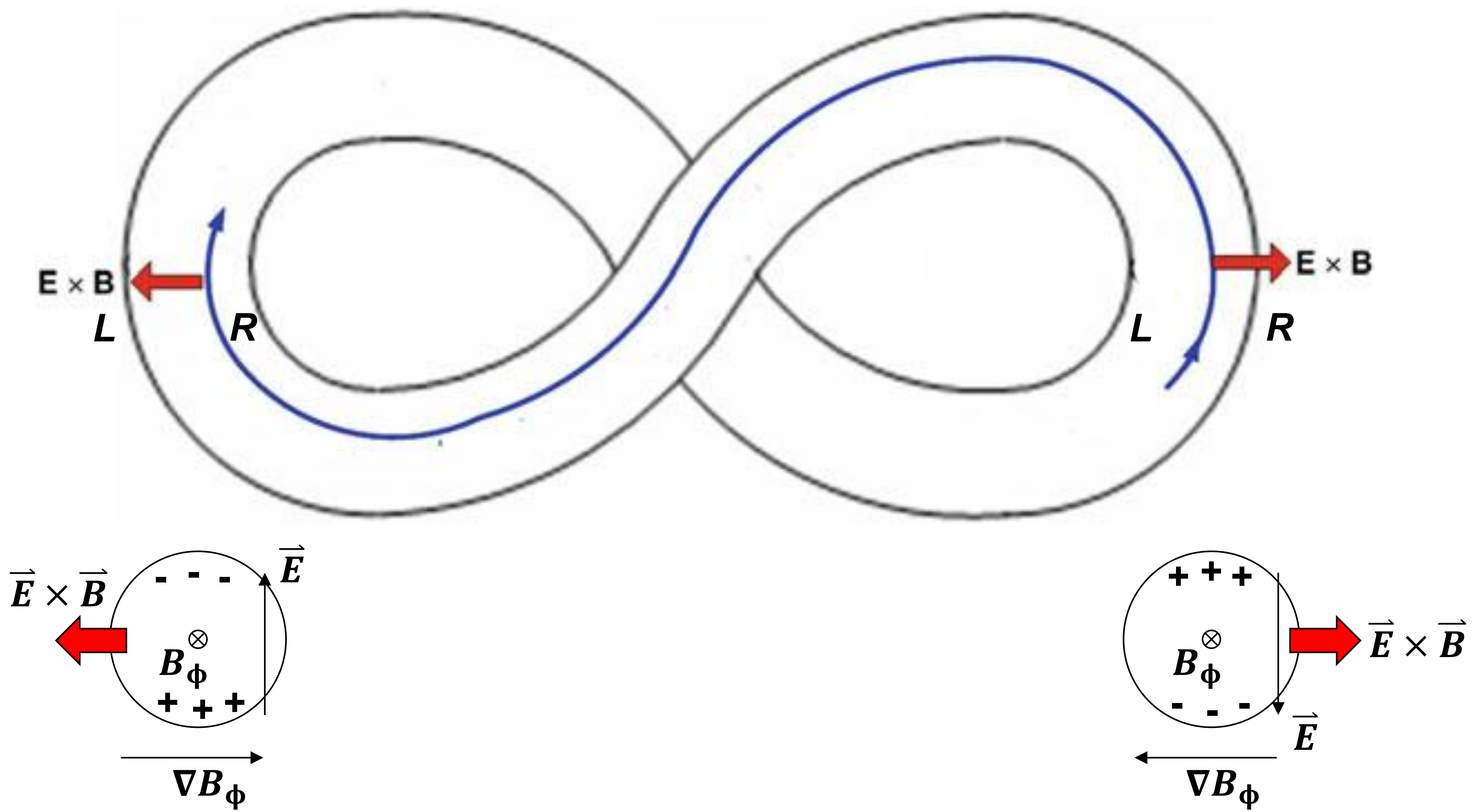
https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

A figure-8 stellarator solved the drift issues



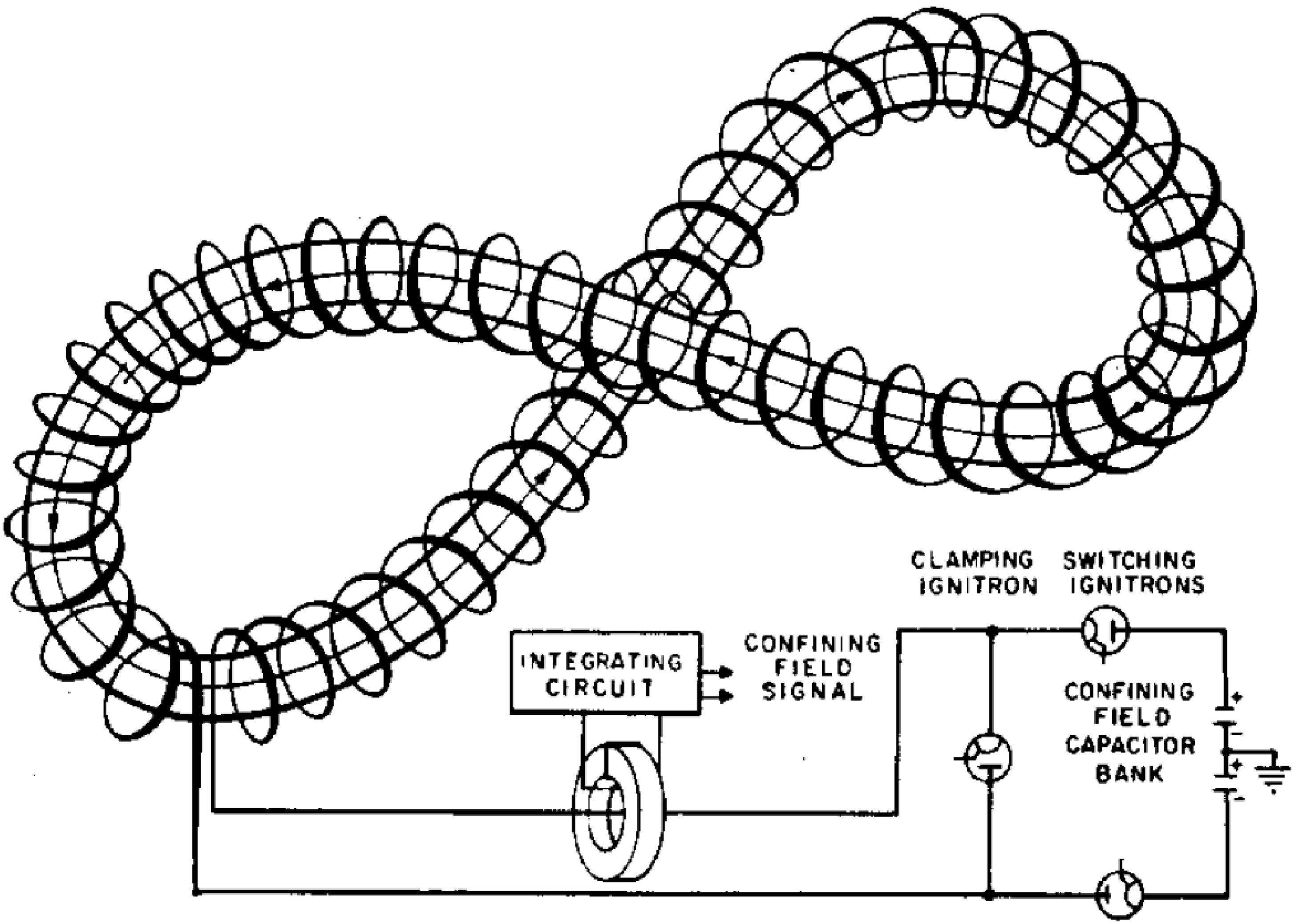
Introduction to Plasma Physics and Controlled Fusion 3rd Edition, by Francis F. Chen

A figure-8 stellarator solved the drift issues

Lyman Spitzer, Jr. came out the idea during a long ride on a ski lift at Garmisch-Partenkirchen

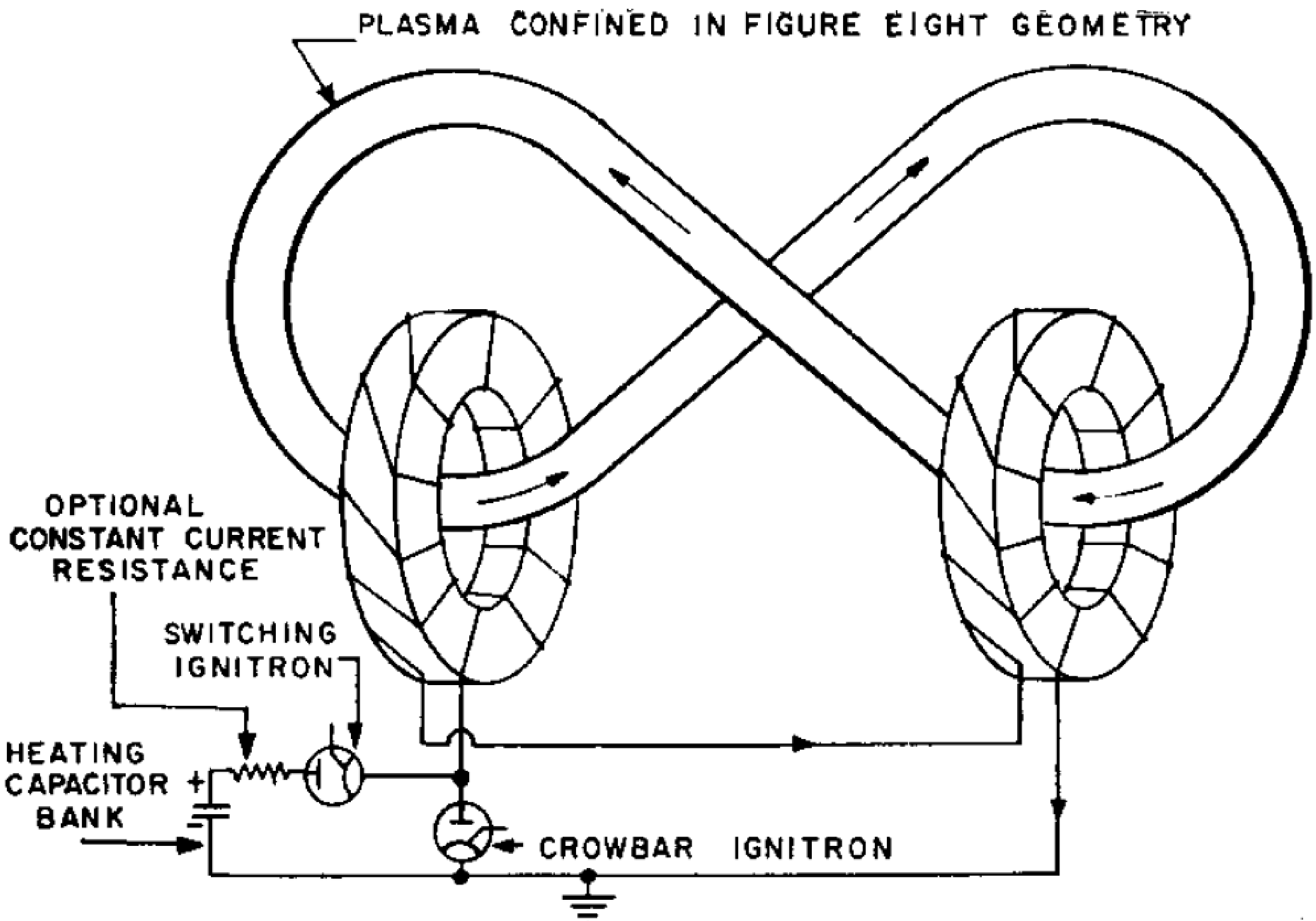
https://www.snowtrex.de/magazin/skigebiete/garmisch-classic-zugspitze/

Concept of figure-8 stellarator



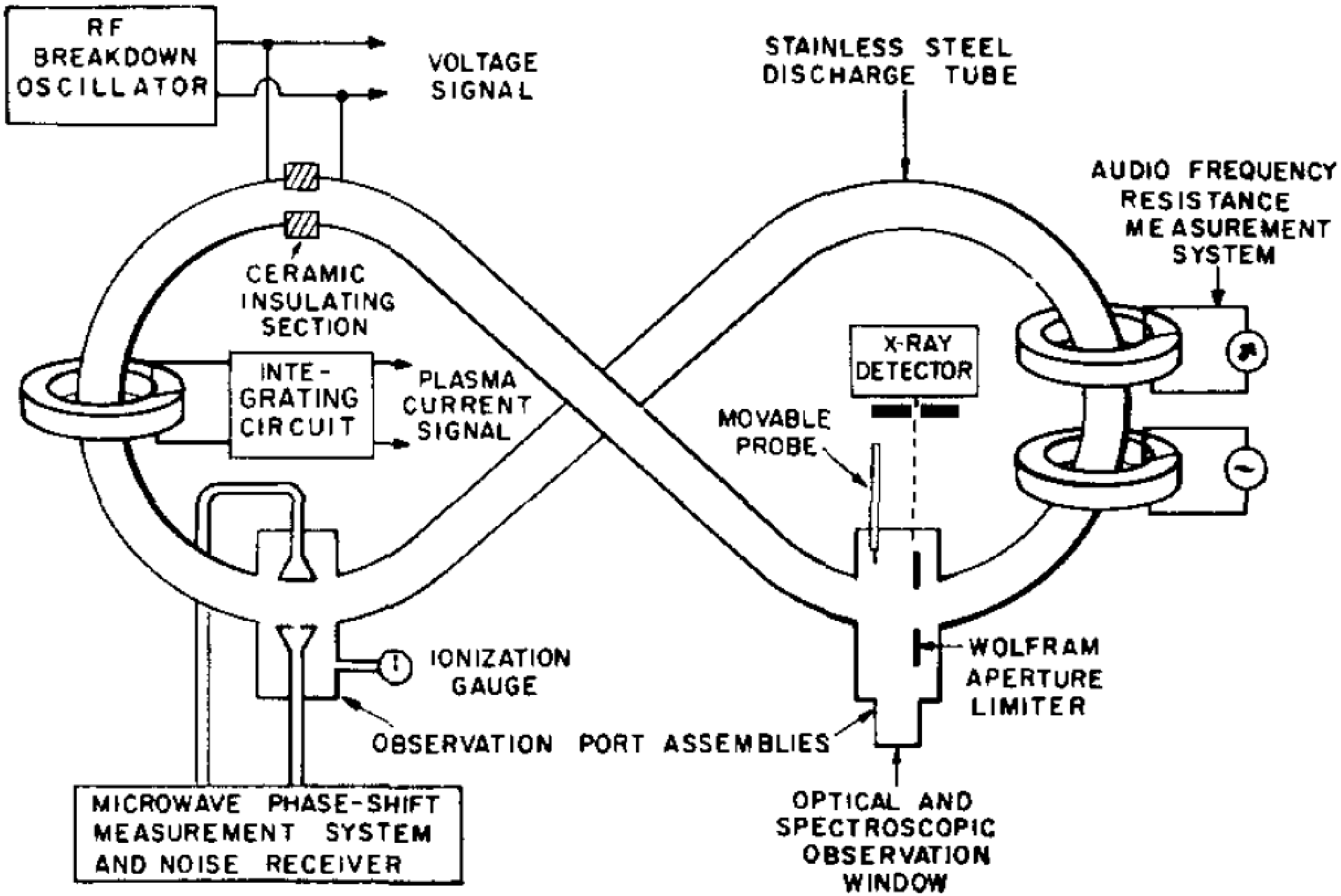
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Figure-8 stellarator with ohmic heating apparatus



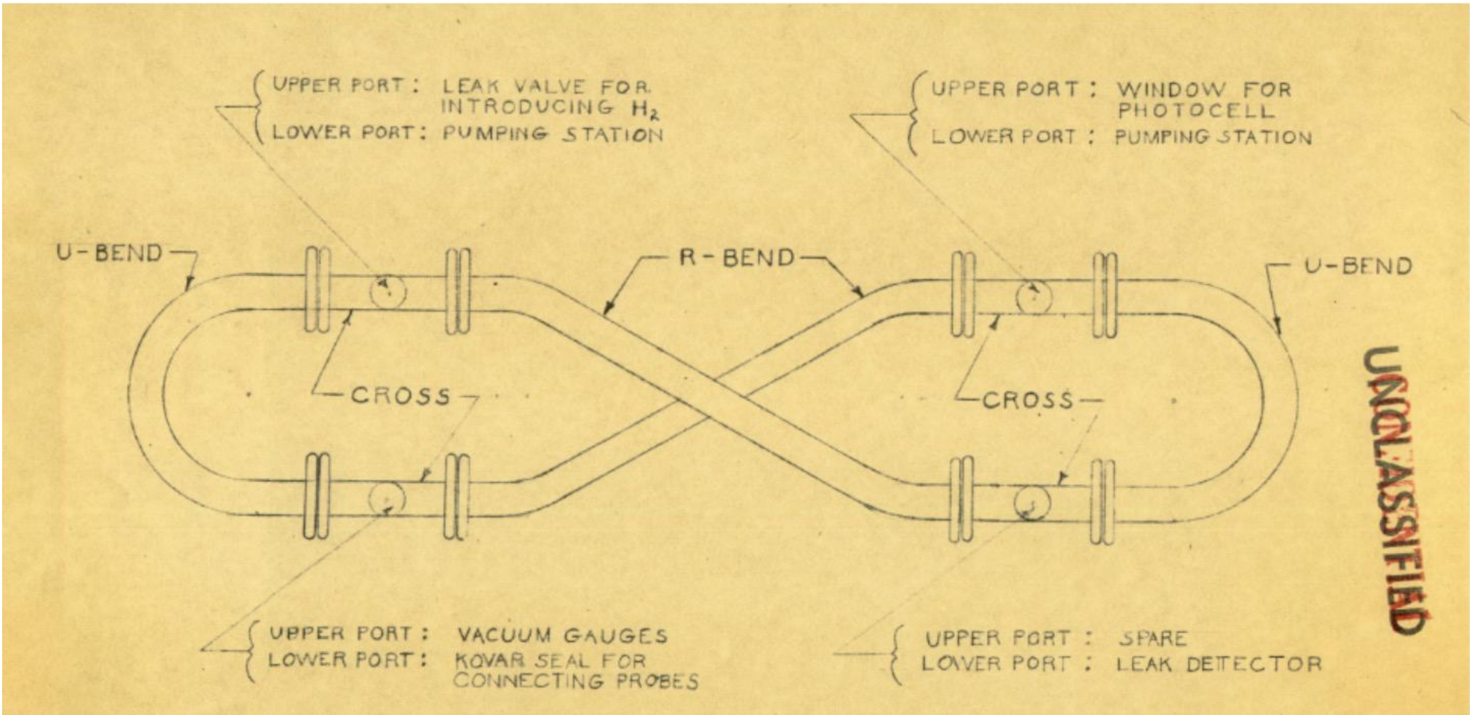
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Schematic diagram of B-1 stellarator

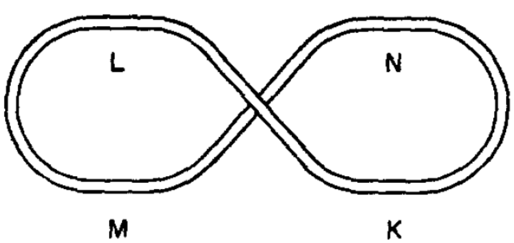


T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Figure-eight (Princeton Model A) – 1953-1958

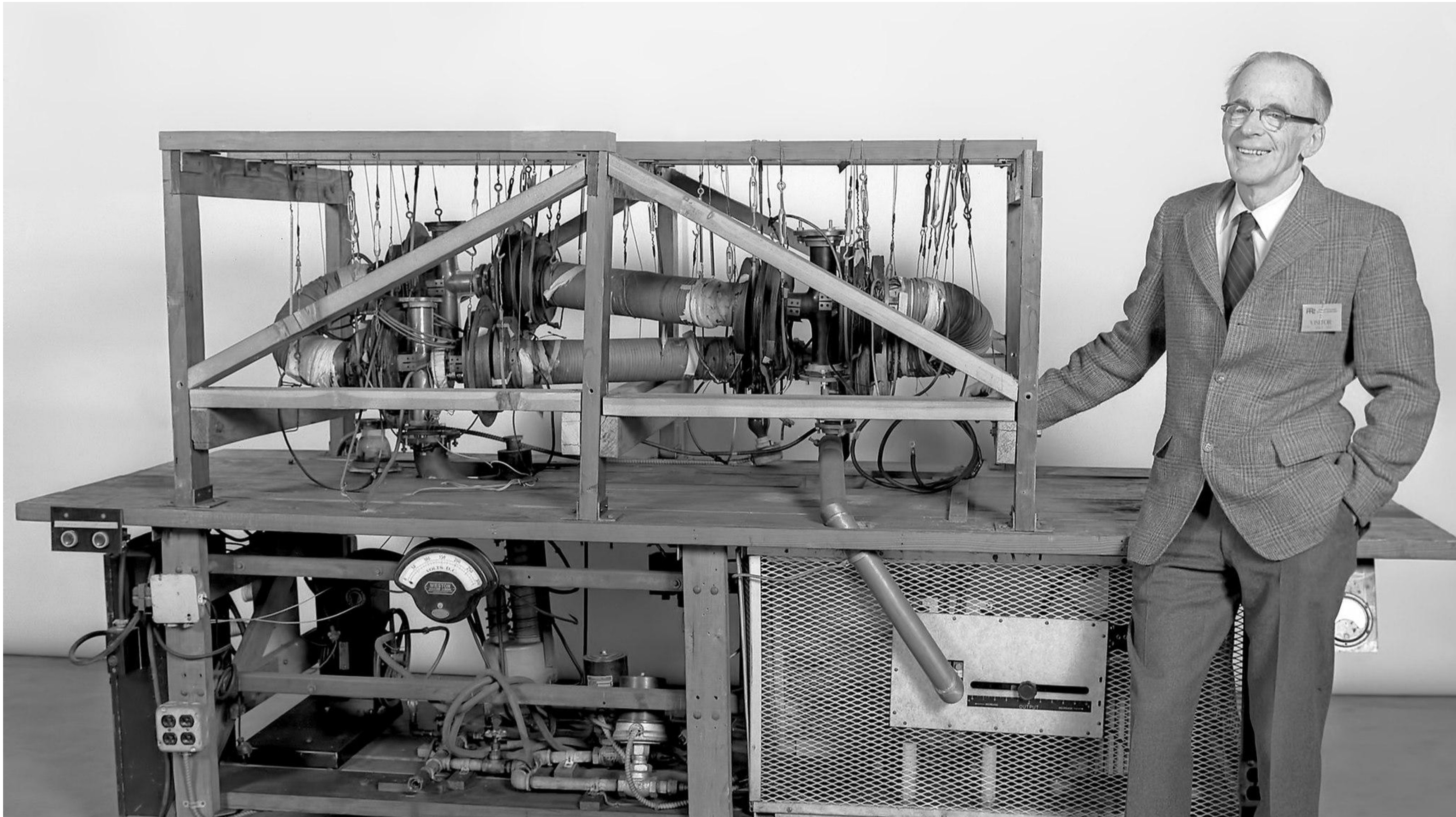


- **Top view**
- **Side view**



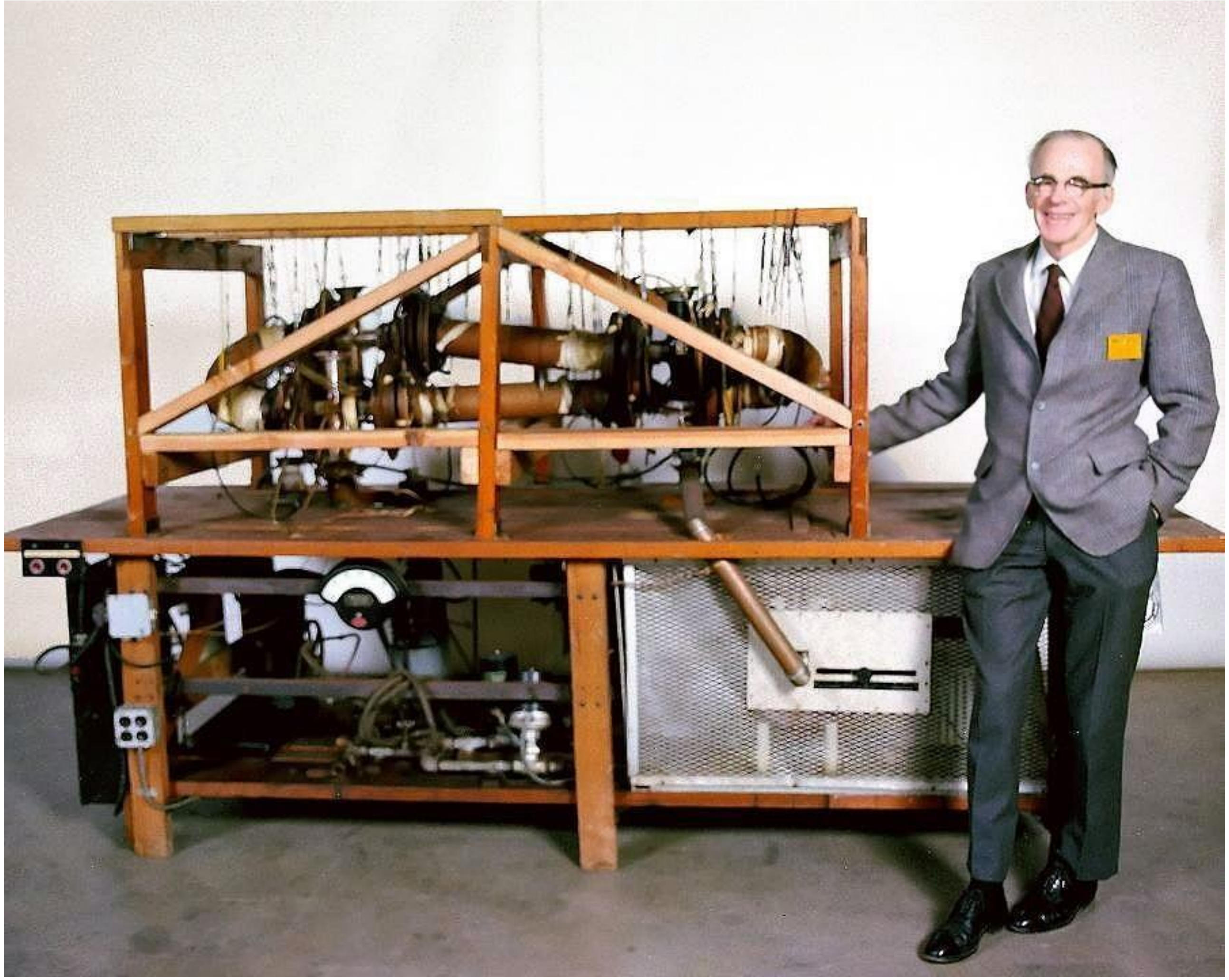
C. H. Willis, NJ Project Matterhorn (1953)
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Model A stellarator



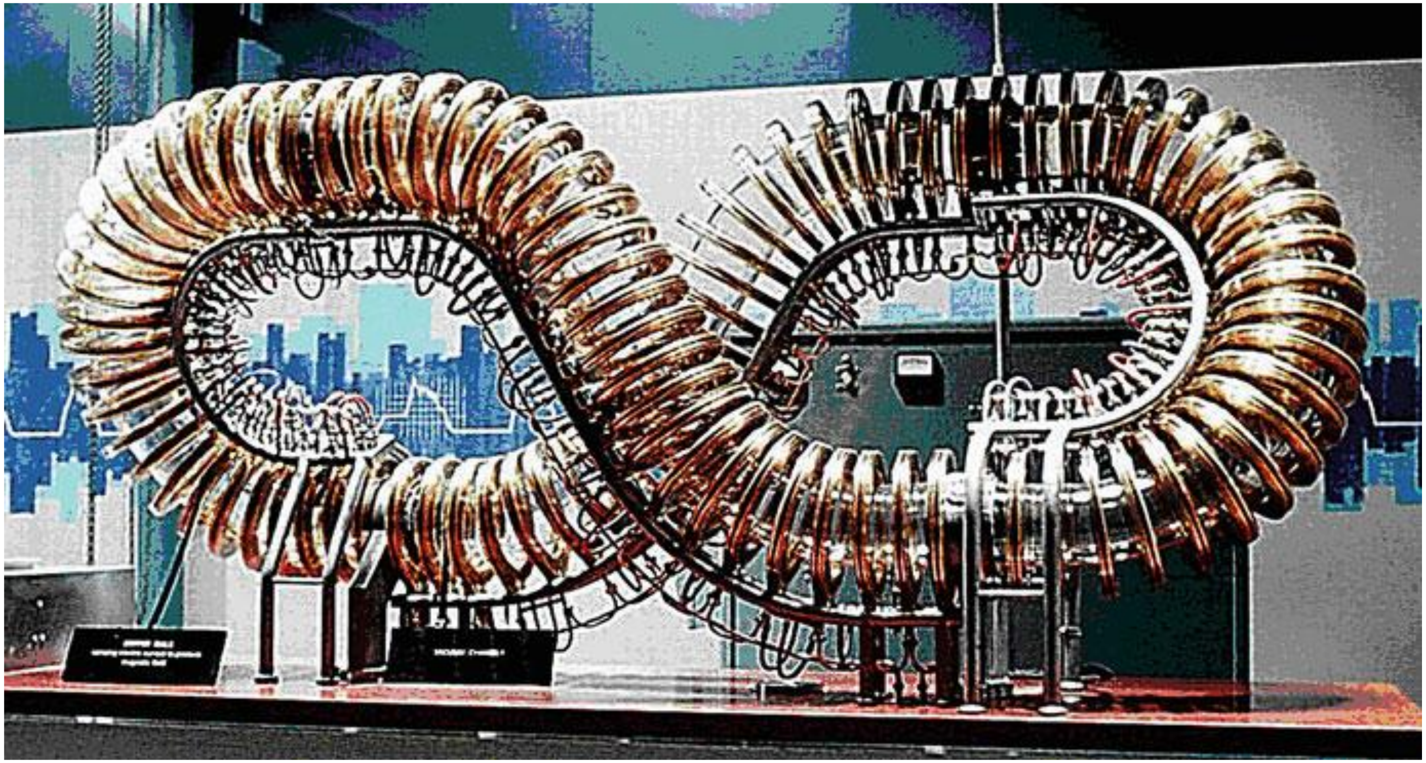
https://www.pppl.gov/timeline

Model A stellarator



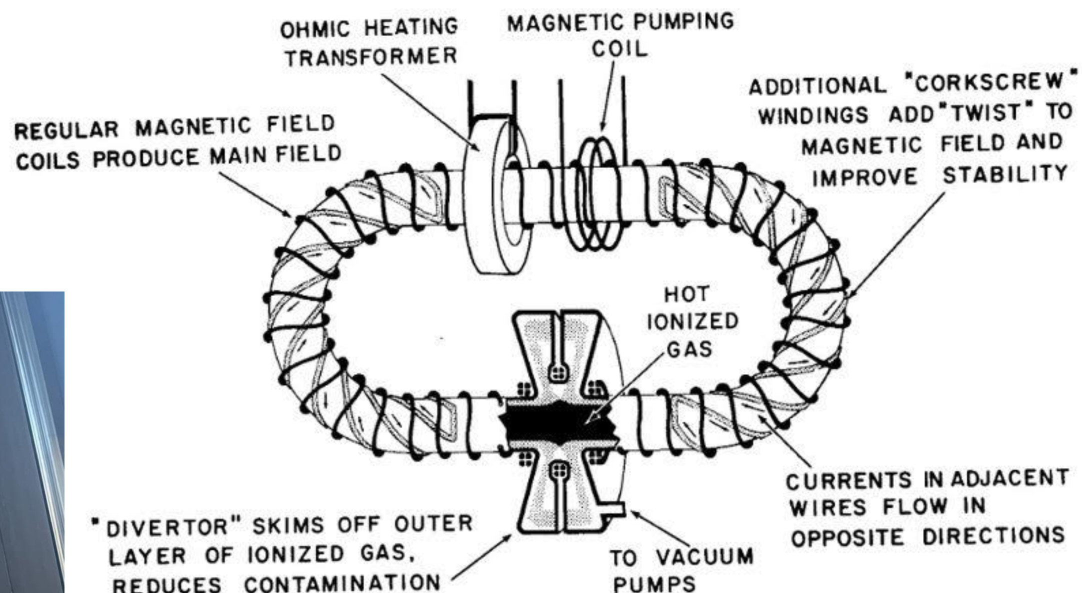
https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

Exhibit model of a figure-8 stellarator for the Atoms for Peace conference in Geneva in 1958



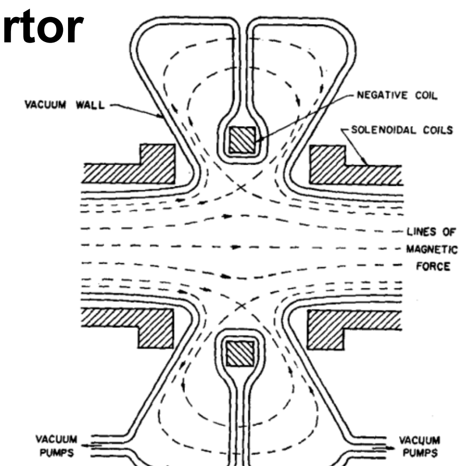
Racetrack Stellarator (Project Matterhorn)

FIG. 4: SCHEMATIC "RACETRACK" STELLARATOR



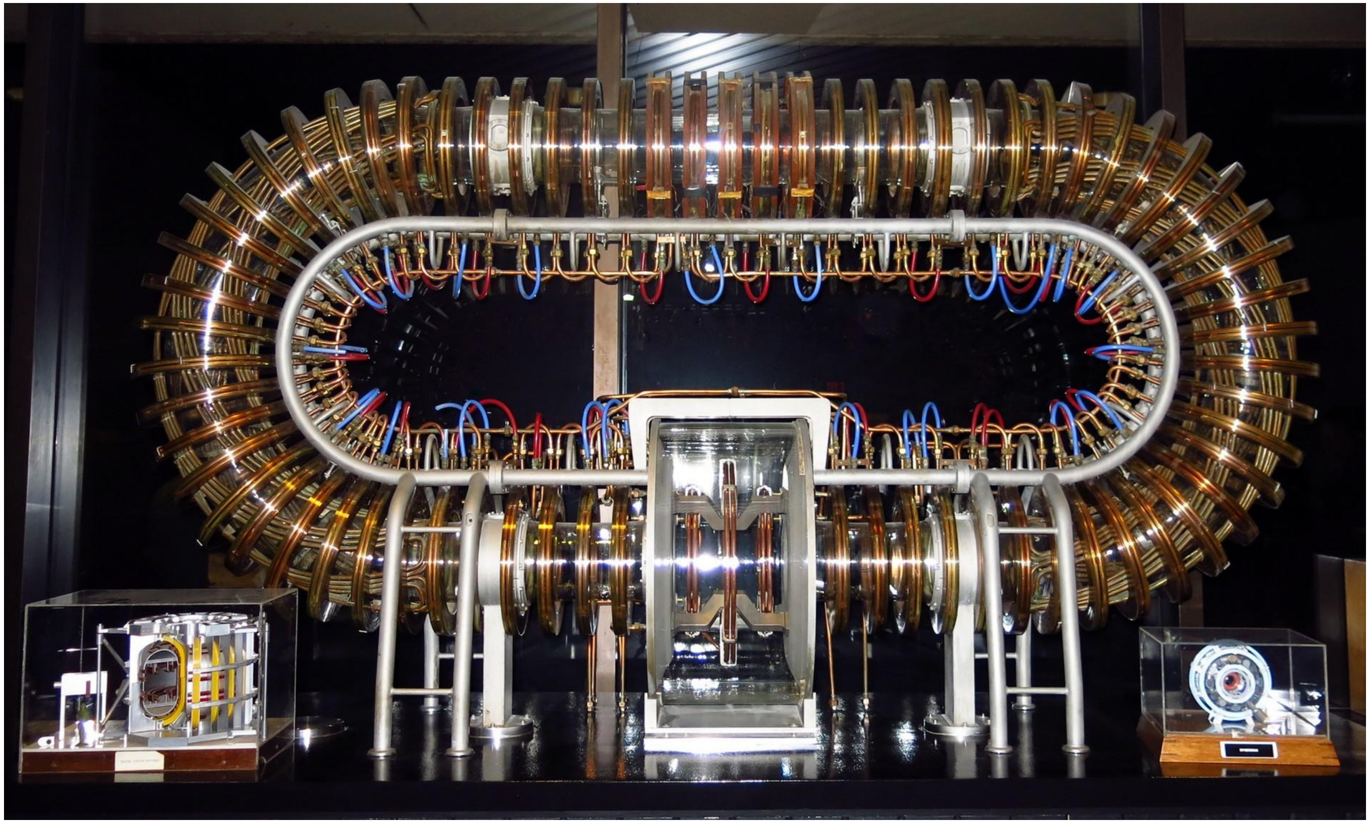
SEPTEMBER 19, 1958 ★ 9

- **Divertor**



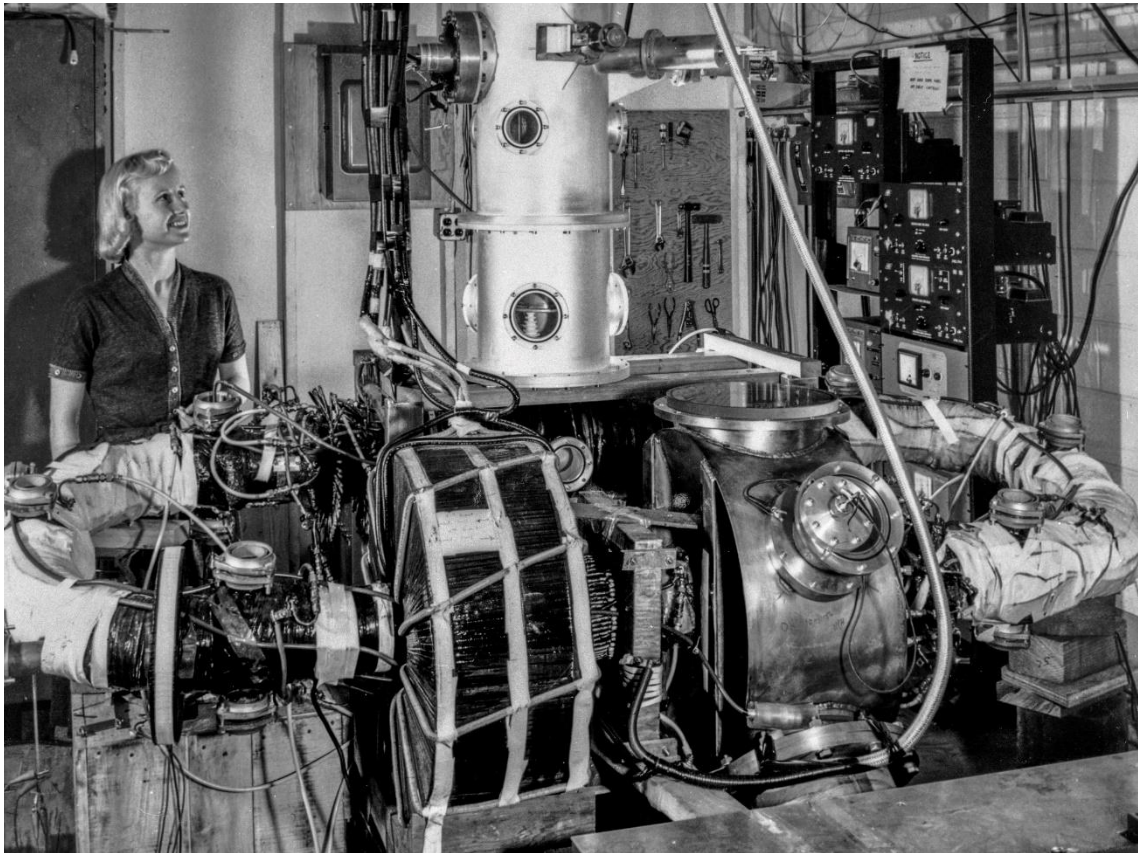
https://twitter.com/quantumwriters/status/1724959216966775270
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Racetrack Stellarator



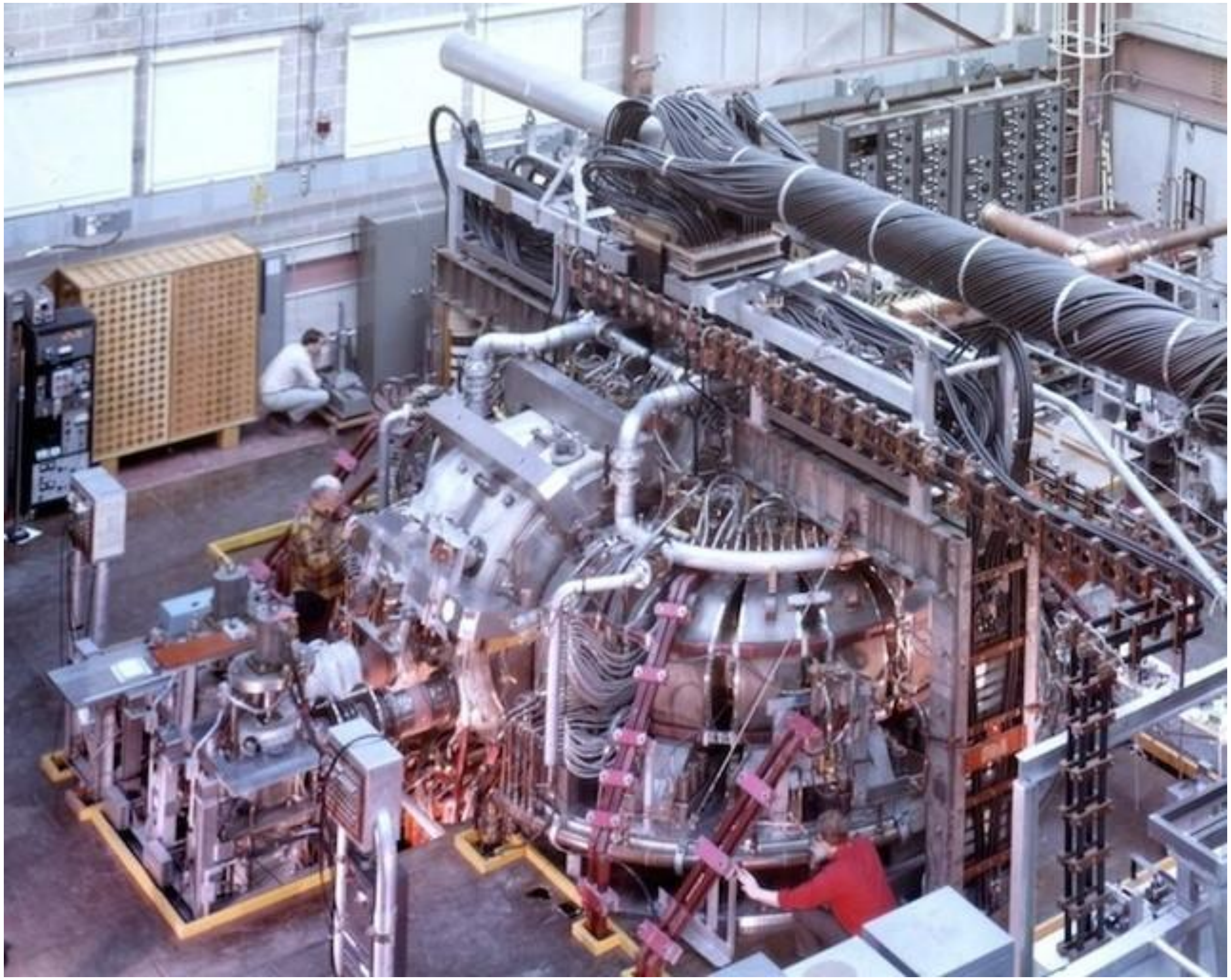
https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

B-65 stellarator



https://www.pppl.gov/timeline
Elizabeth Paul, An introduction to stellarators,
Princeton Alumni Weekly, Sep. 19, 1958

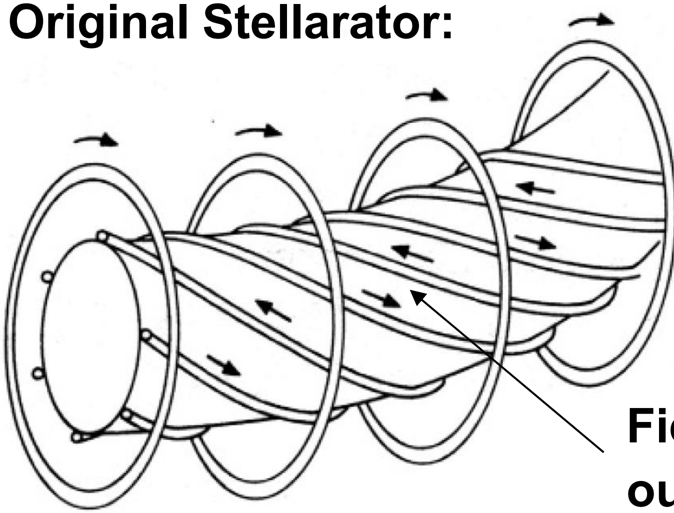
Racetrack (Princeton Model C) – 1962-1969



https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

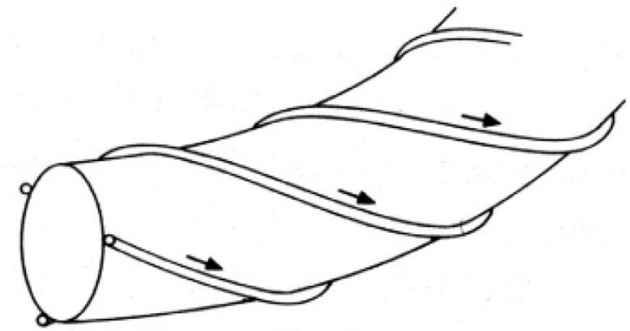
Different types of stellarators

- **Original Stellarator:**

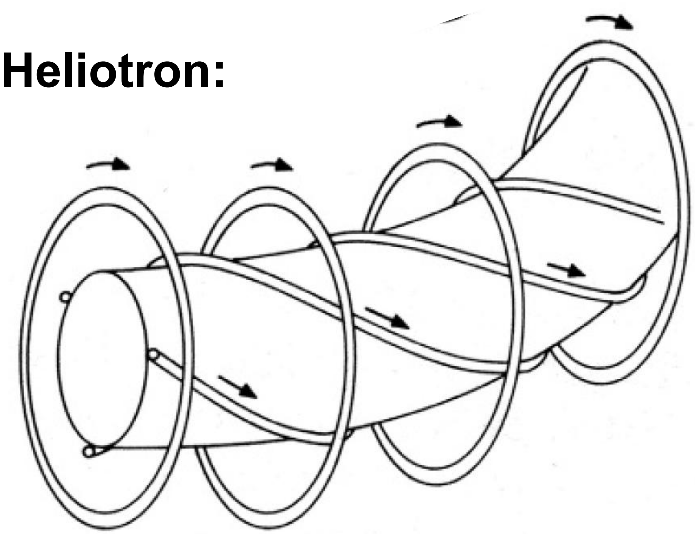


Field cancels out on axis

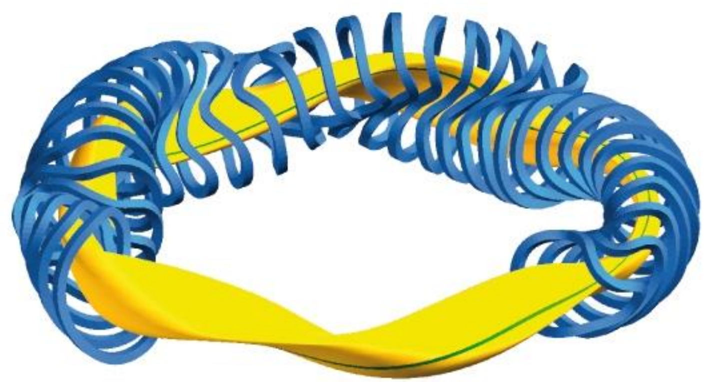
- **Torsatron:**



- **Heliotron:**

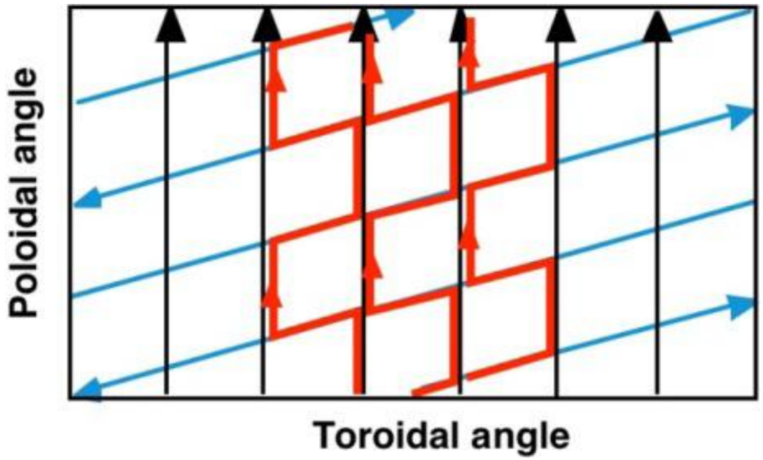


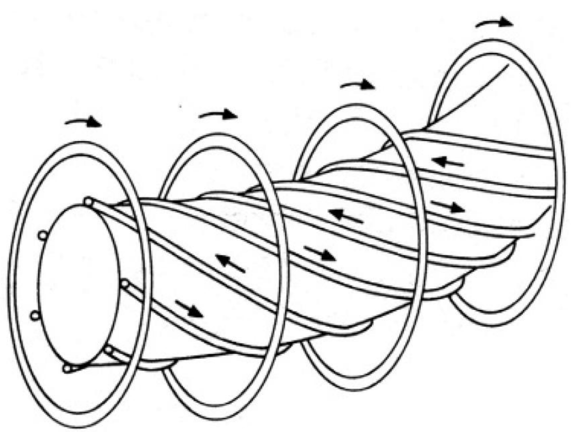
- **Helias:**



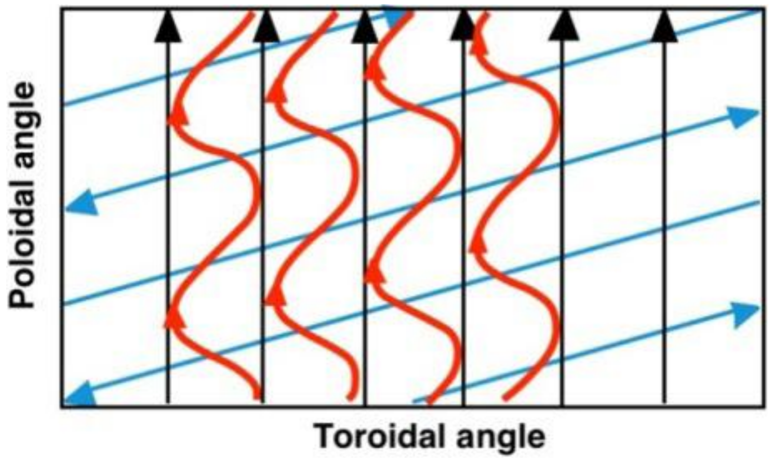
Helical coils with toroidal field coils can be replaced by smoothed twisting coils

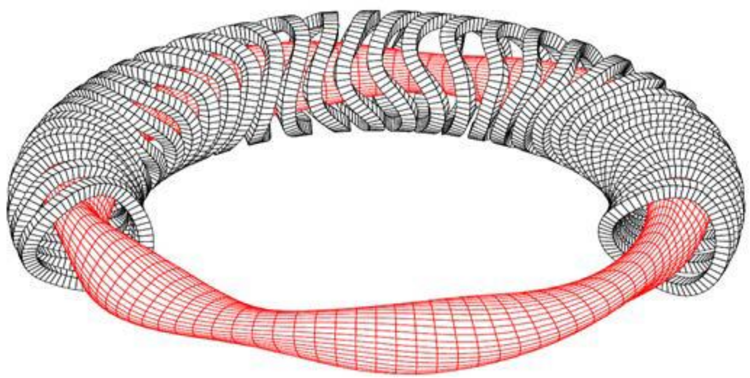
- **Superposition of helical windings (blue) and the TF-coils (black) and mapped into the *θ-Φ* plane.**





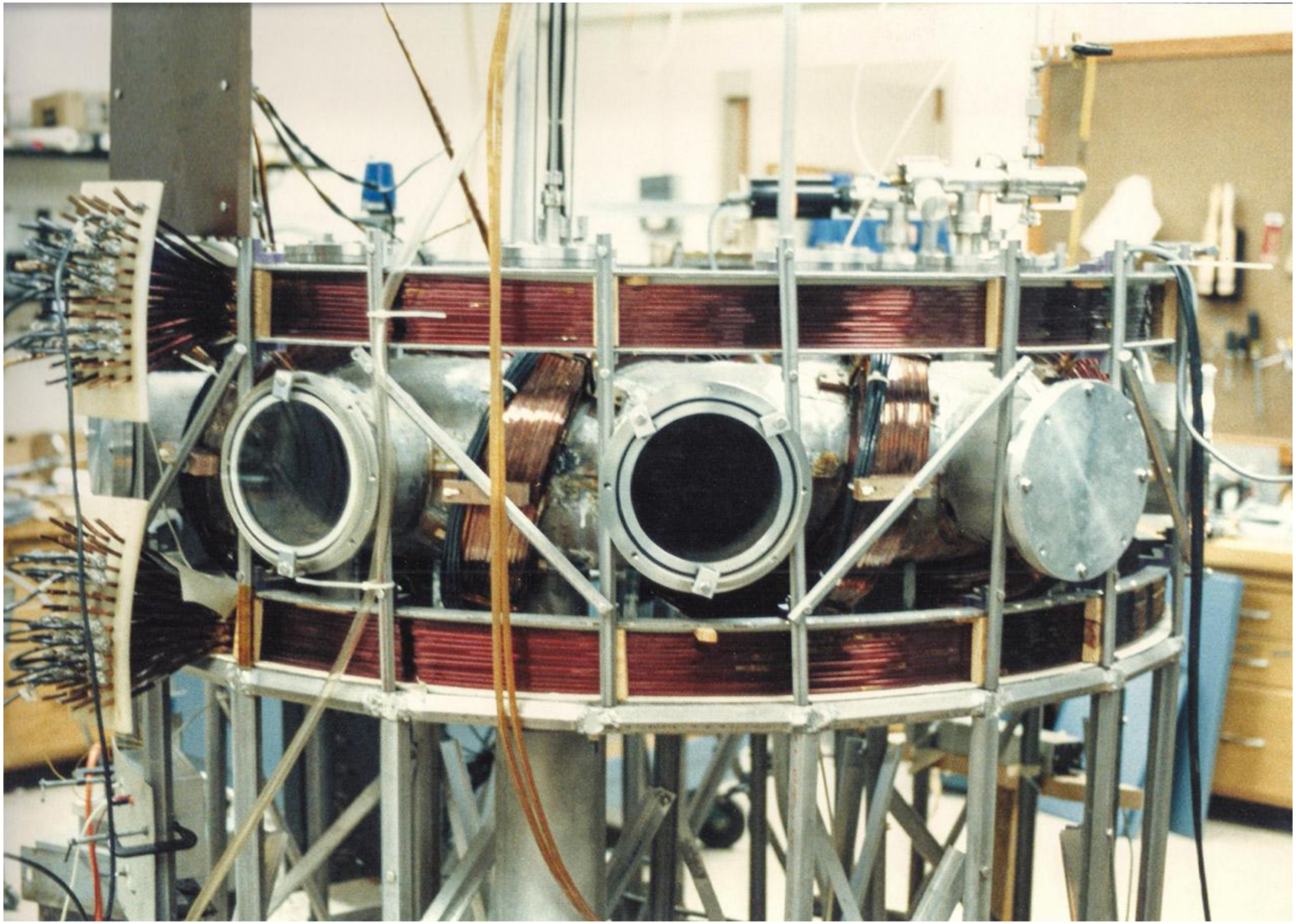
- **Realization of the smoothed twisting coils**





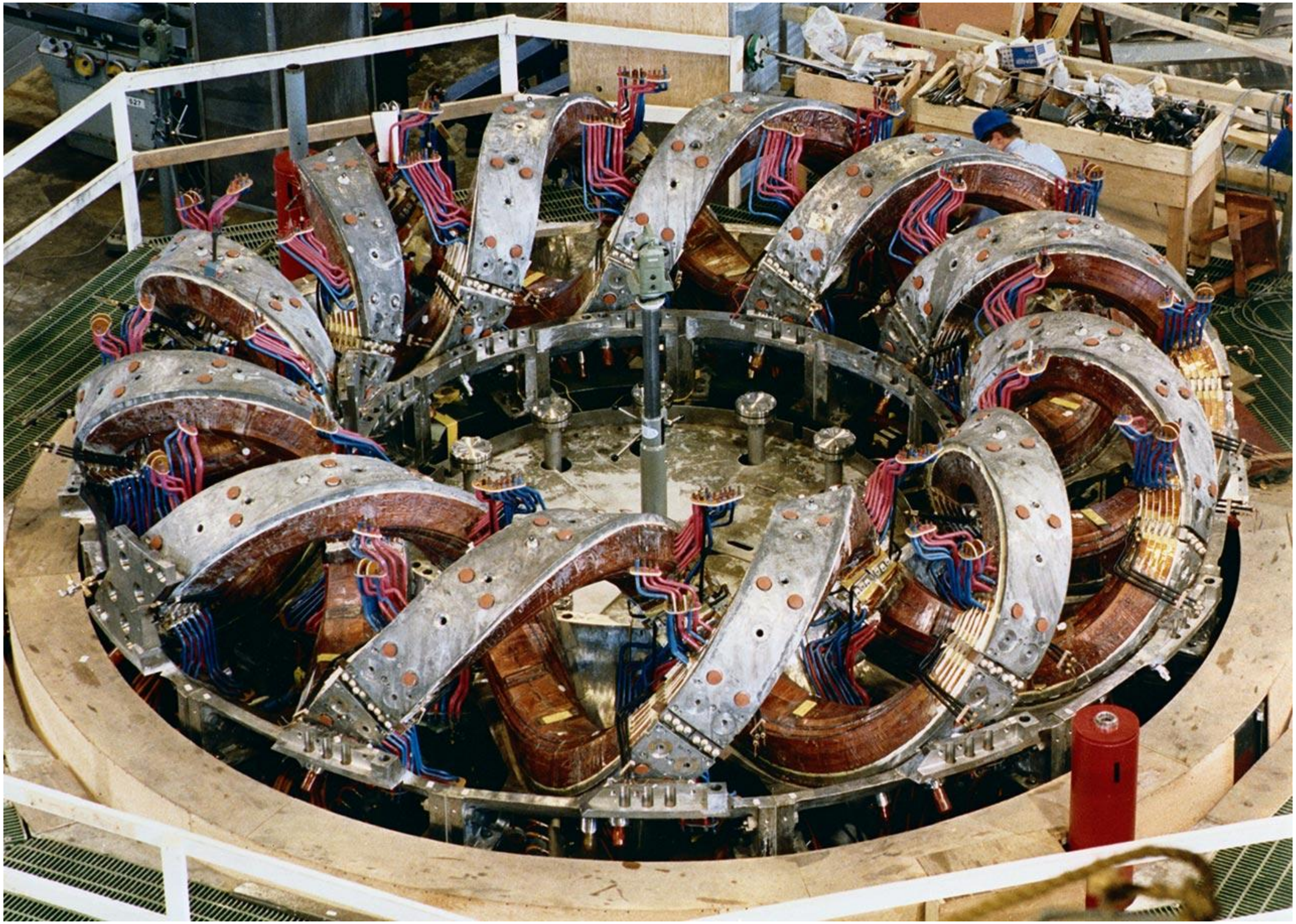
Fusion Physics, by IAEA

Auburn torsatron — winding of both helical and poloidal coils can be seen



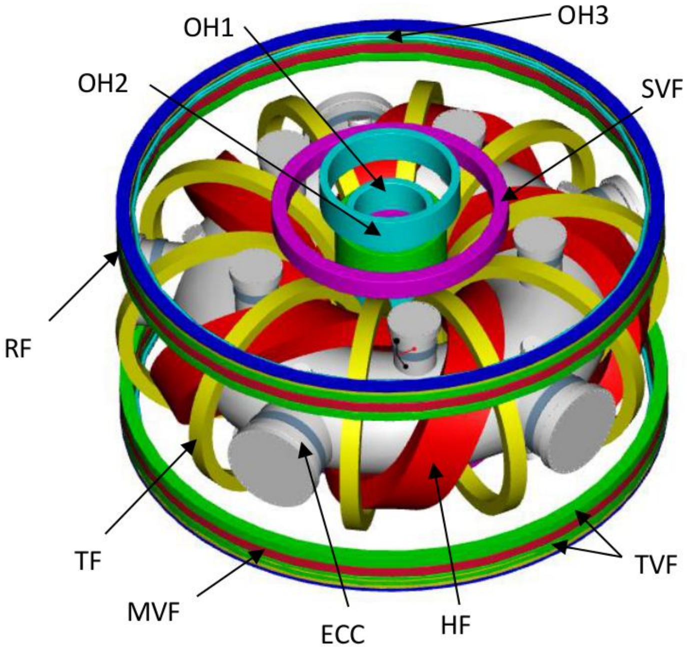
https://www.energyencyclopedia.com/en/glossary/torsatron

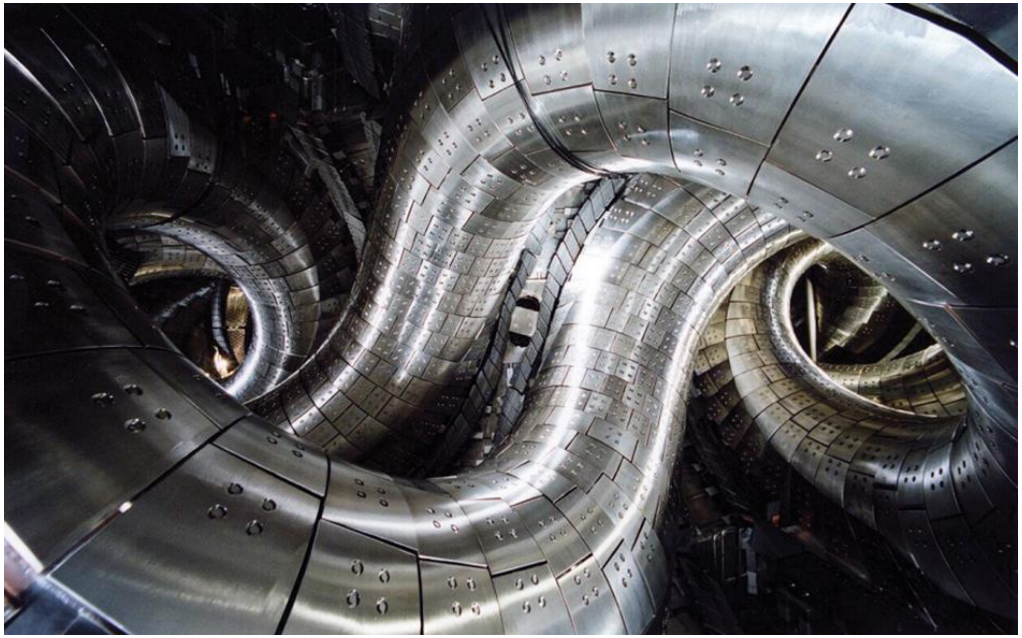
Construction of a pair of helical magnetic coils for the Advanced Toroidal Facility torsatron

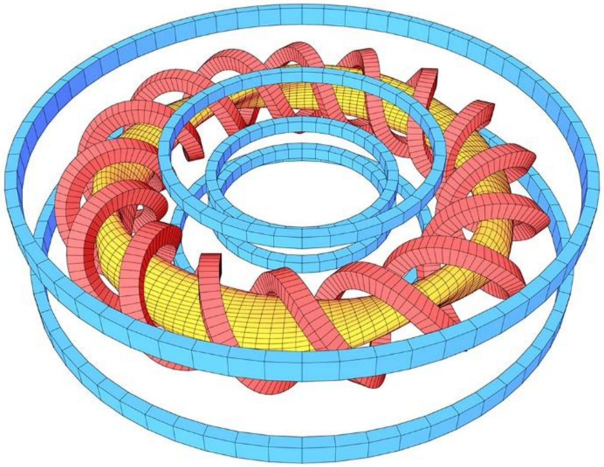


https://www.energyencyclopedia.com/en/glossary/torsatron

LHD stellarator in Japan (Heliotron)

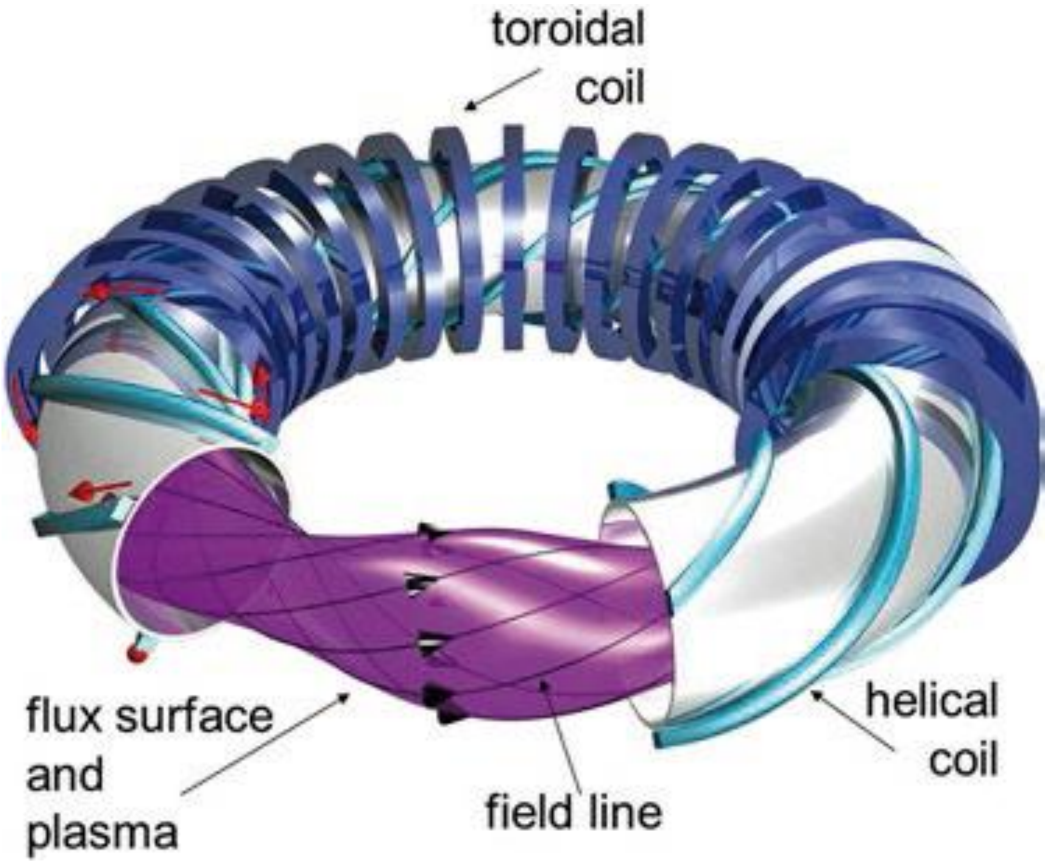






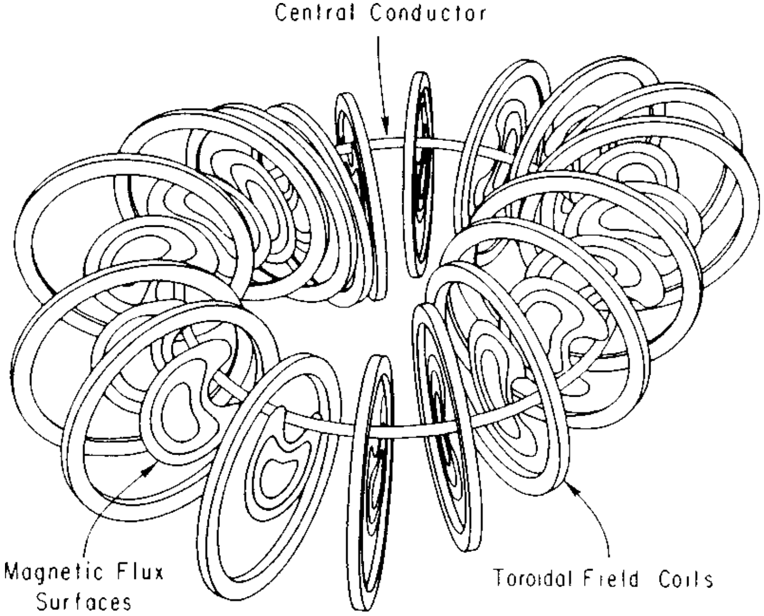
https://en.wikipedia.org/wiki/Compact_Toroidal_Hybrid
https://www.energyencyclopedia.com/en/glossary/heliotron
https://en.wikipedia.org/wiki/Large_Helical_Device

Twisted magnetic field lines can be provided by toroidal coils with helical coils

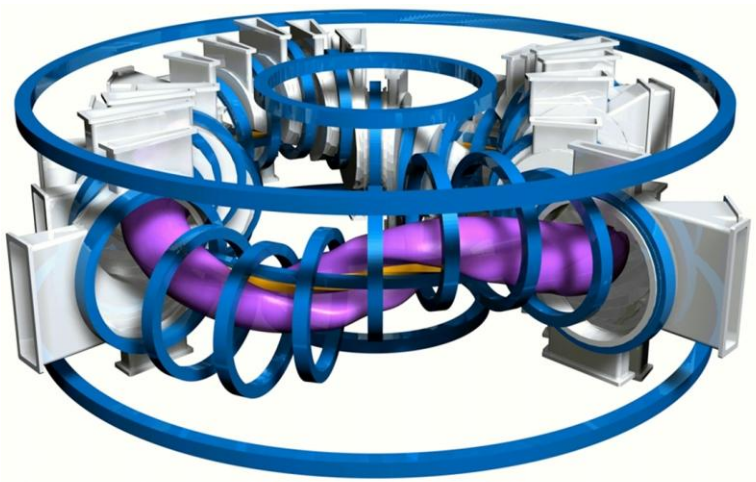


Wagner, F., Fusion energy. MRS Energy & Sustainability, 5, E8 (2018)

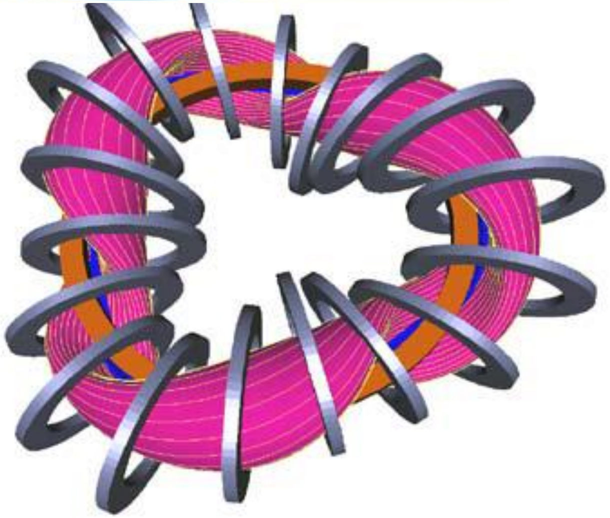
Heliac (Helical Axis stellarator)



- **TJ-II (Spain's National Fusion Laboratory):**

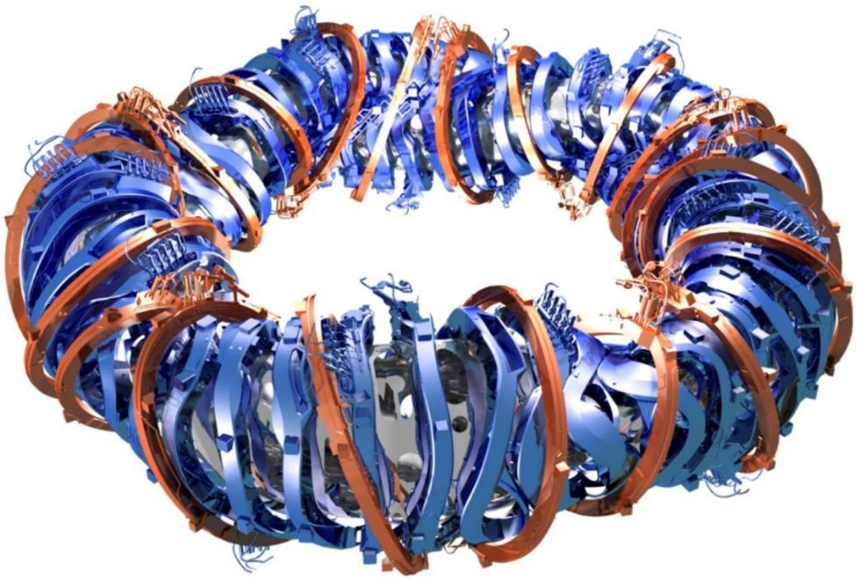


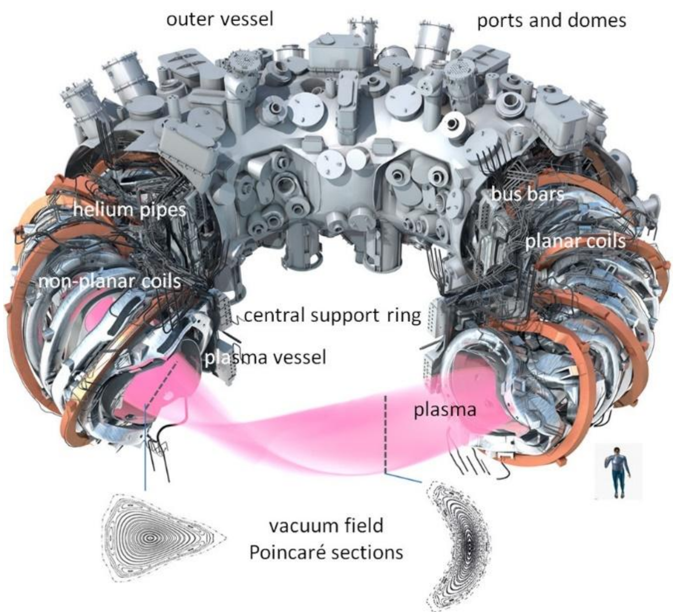
- **H-1 (Australian Plasma Fusion Research Facility):**



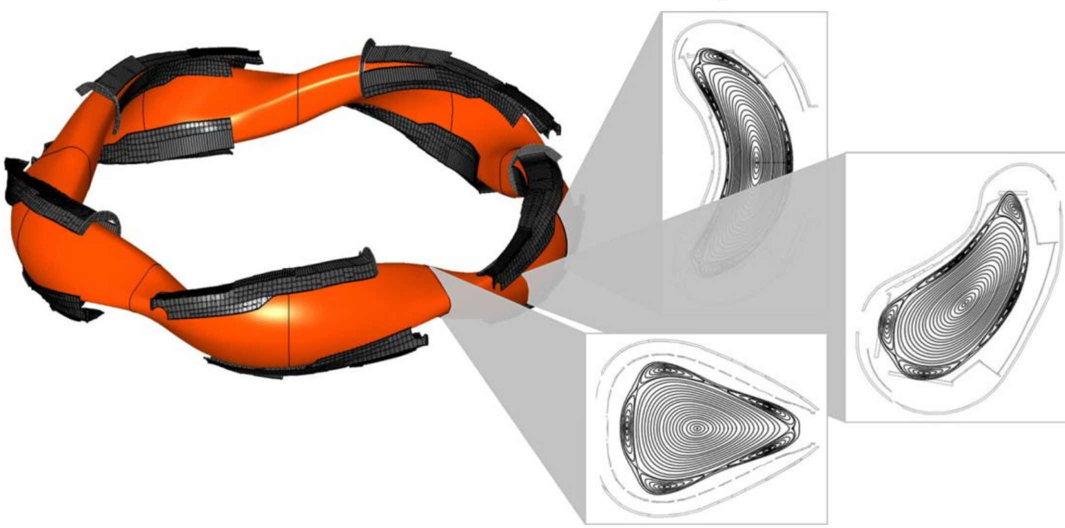
A. H. Boozer, Phys. Plasmas, 5, 1647 (1998)
https://wiki.fusion.ciemat.es/wiki/TJ-II
B. D. Blackwell, et. al, 23rd IAEA Fusion Energy Conference, 2010

Wendelstein 7-X is a (Helias) stellarator built by Max Planck Institute for Plasma Physics (IPP)





- **Wemdelstein 7-x is now installing new diverters.**



Advantages of Stellarator

- No need to drive plasma current. It is intrinsically steady state.
- With zero net current, one potentially dangerous class of MHD instabilities, the current-driven kink modes, is eliminated.
- Magnetic configuration is set by external coils, not by currents in the plasma. Stellarators do not suffer violent disruptions.
- Potential for greater range of designs and optimization of fusion performance.

Disadvantages of Stellarator

- Complicated coil configurations. It's difficult to design. The precision requirement is high. It is expensive to build coils for stellarators.
- Achieving good particle confinement in stellarators is more difficult than that in tokamaks.
- Divertors and heat load geometry in stellarators is more complicated than those in tokamaks.

Course Outline

- Magnetic confinement fusion (MCF)
  - Gyro motion, MHD
  - 1D equilibrium (z pinch, theta pinch)
  - Drift: ExB drift, grad B drift, and curvature B drift
  - Tokamak, Stellarator (toroidal field, poloidal field)
  - Magnetic flux surface
  - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
  - Stability (Kink instability, sausage instability, Safety factor Q)
  - Central-solenoid (CS) start-up (discharge) and current drive
  - CS-free current drive: electron cyclotron current drive, bootstrap current.
  - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt} \left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

J. P. Freidberg, Ideal Magnetohydrodynamics

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"

- **Equilibrium state:**

$$\rho_\mathrm{m}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] = \vec{j}\times\vec{B} - \nabla p \equiv 0$$

$$\vec{j}\times\vec{B} = \nabla p$$

$$\nabla\times\vec{B} = \mu_\mathrm{o}\,\vec{j}$$

$$\vec{j}\times\vec{B} = \frac{1}{\mu_\mathrm{o}}(\vec{\nabla}\times\vec{B})\times\vec{B} = \frac{1}{\mu_\mathrm{o}}\left[(\vec{B}\cdot\vec{\nabla})\,\vec{B} - \frac{1}{2}\,\vec{\nabla}\,B^2\right] = \nabla p$$

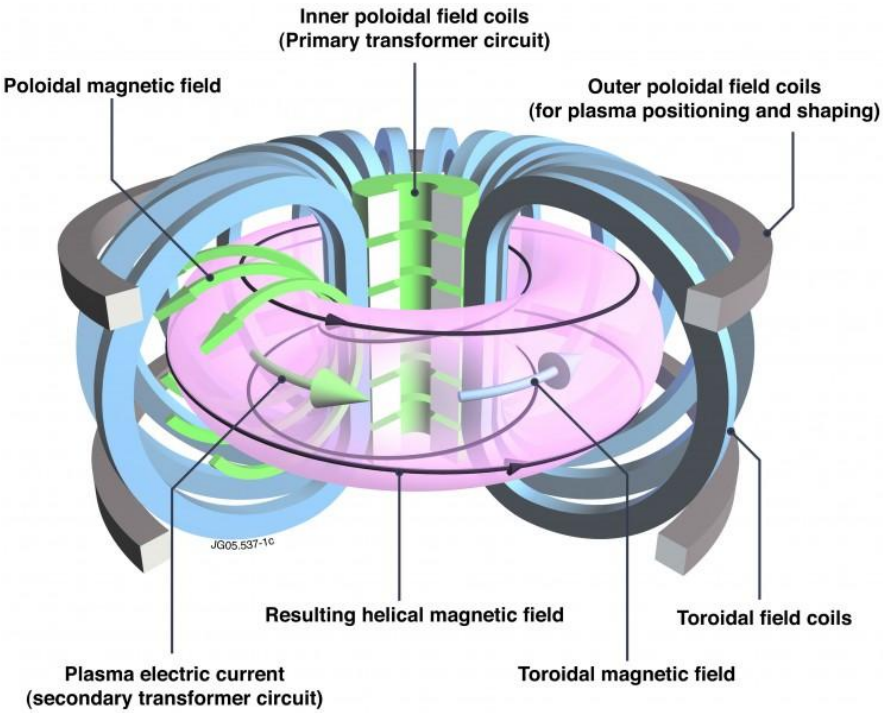
$$\nabla\left(P + \frac{B^2}{2\mu_\mathrm{o}}\right) = \frac{1}{\mu_o}(\vec{B}\cdot\vec{\nabla})\,\vec{B}$$

Magnetic pressure **Magnetic tension** ← **Forces caused by curvature of the field lines**

$$\vec{j}\perp\nabla p \qquad \vec{B}\perp\nabla p \quad \Rightarrow \quad \vec{j}\cdot\nabla p = 0 \qquad \vec{B}\cdot\nabla p = 0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p$$

$$\Downarrow$$

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p$$

$$\Downarrow$$

$$\vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j} \Rightarrow \nabla \cdot \vec{j} = 0$$

$$\nabla \cdot \vec{B} = 0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

https://www.euro-fusion.org/2011/09/tokamak-principle-2/

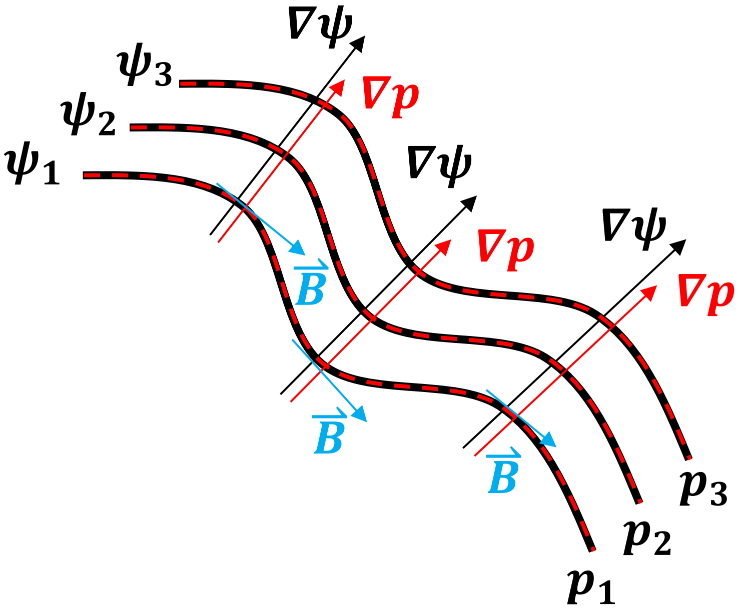
Pressure can be written as a function of flux

$\vec{B} \cdot \nabla\psi = 0$

$\vec{B} \cdot \nabla p = 0$

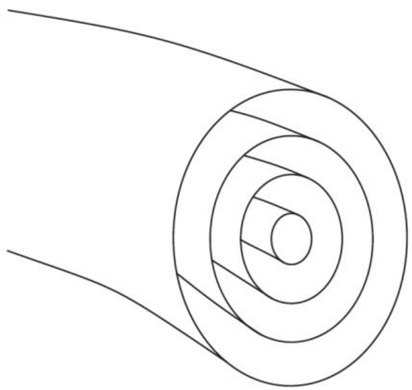
for $\nabla p \neq 0$:

$p = p(\psi)$

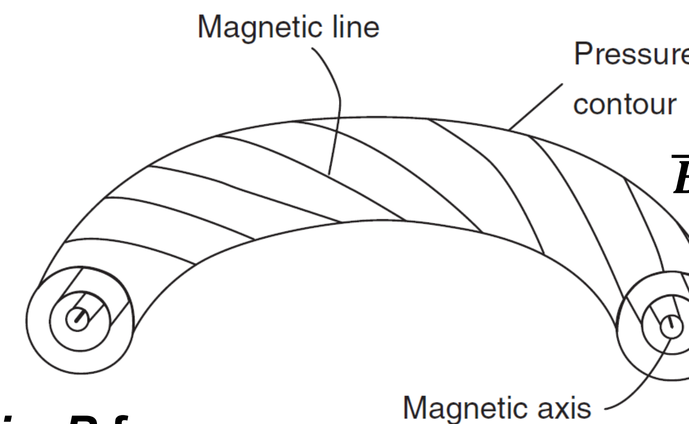


Magnetic lines lying on pressure contour

- **Contours of constant pressure**



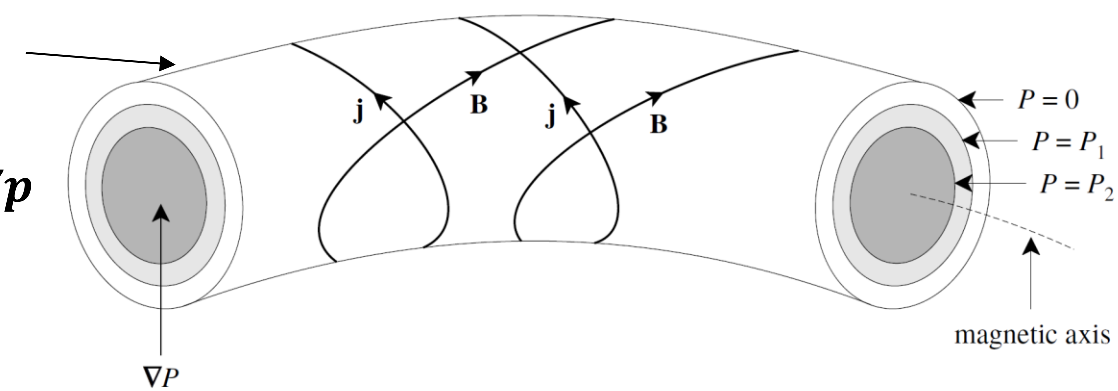
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Derivation of Grad-Shafranov equation

$$\vec{j} \times \vec{B} = \nabla p \qquad \nabla \times \vec{B} = \mu_o \vec{j}$$

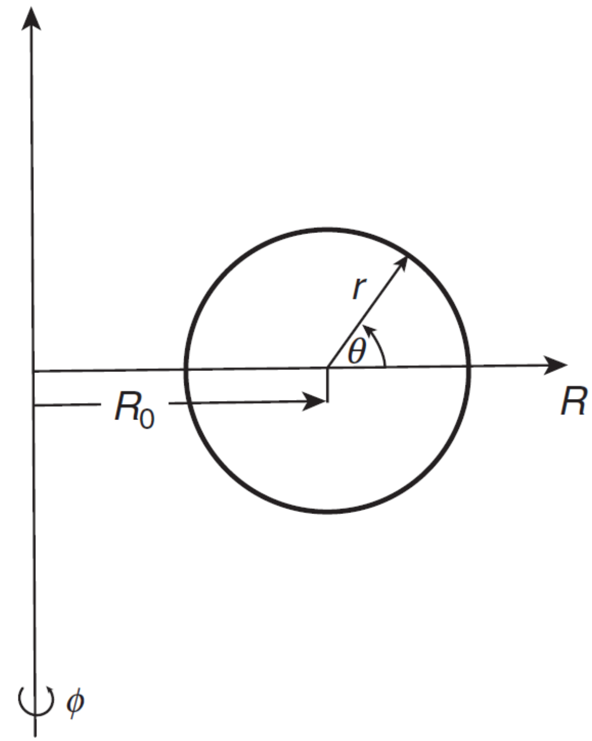
$$\nabla \cdot \vec{j} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = (B_R, B_\phi, B_z) \qquad \textbf{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\cancel{\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$



- **Represent the magnetic field using a vector potential *A*:**

$$\vec{B} = \nabla \times \vec{A} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\cancel{\frac{\partial A_R}{\partial \phi}}\right)$$

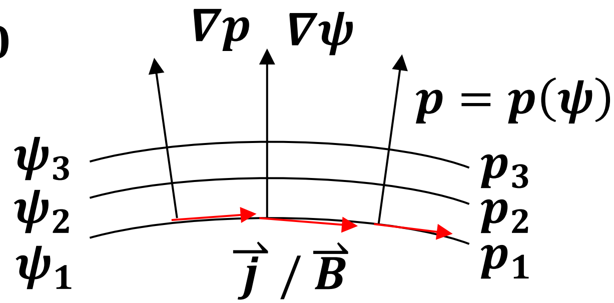
$$= \hat{R}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\right)$$

$$\equiv \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z$$

$$B_R = -\frac{\partial A_\phi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

Pressure can be written as a function of flux

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R)+\frac{\partial B_z}{\partial z}=0$$

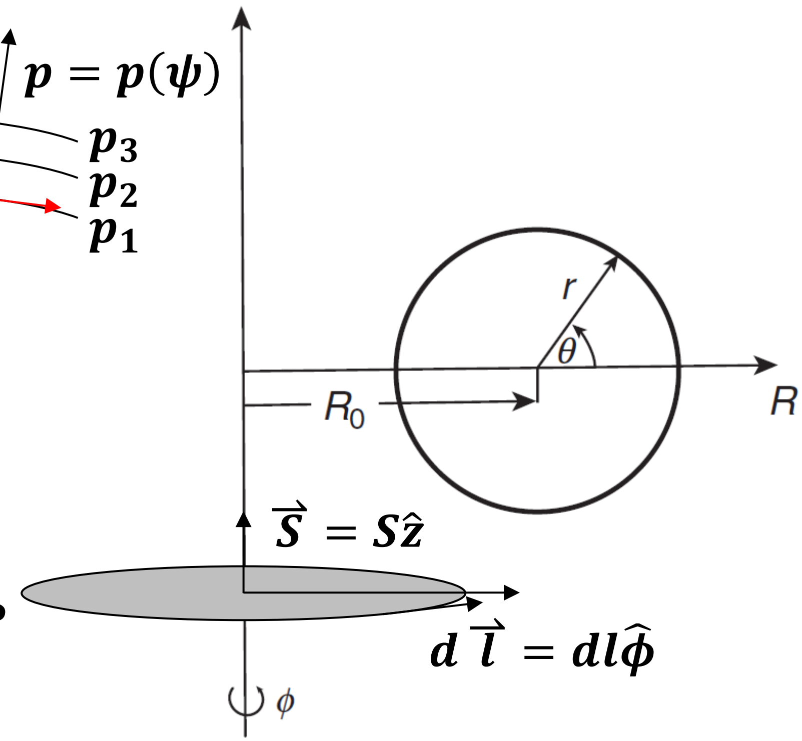


$$B_R=-\frac{\partial A_\phi}{\partial z}$$

$$B_z=\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

$$\psi\equiv\frac{1}{2\pi}\int\vec{B}\cdot d\,\vec{S}=\frac{1}{2\pi}\int\left(\nabla\times\vec{A}\right)\cdot d\,\vec{S}$$

$$=\frac{1}{2\pi}\int\vec{A}\,2\pi R\cdot d\,\vec{l}=\int\vec{A}\cdot\hat{\phi}R\,dl=RA_\phi$$



$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial z}\qquad B_z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}\cdot\nabla\psi=B_R\frac{\partial\psi}{\partial R}+B_\phi\cancel{\frac{1}{R}\frac{\partial\psi}{\partial\phi}}+B_z\frac{\partial\psi}{\partial z}=B_R\frac{\partial\psi}{\partial R}+B_z\frac{\partial\psi}{\partial z}$$

$$=\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)\frac{\partial\psi}{\partial R}+\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\frac{\partial\psi}{\partial z}=0$$

$$\vec{B}\cdot\nabla\psi=0$$

$$\vec{B}\cdot\nabla p=0$$

for $\nabla p\neq 0$:

$$p=p(\psi)$$

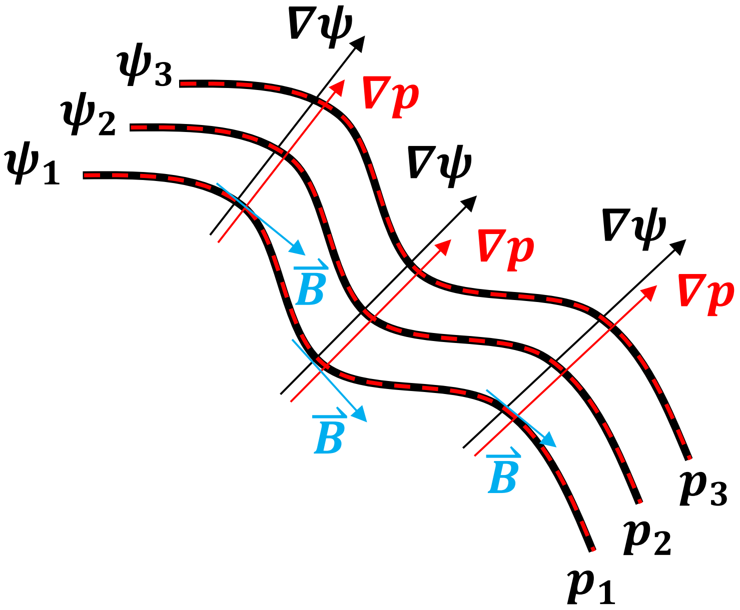
Pressure can be written as a function of flux

$$\vec{B} \cdot \nabla\psi = 0$$

$$\vec{B} \cdot \nabla p = 0$$

for $\nabla p \neq 0$:

$$p = p(\psi)$$



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{\phi}$ component of the force-balance equation:**

$$\left(\vec{j}\times\vec{B}=\nabla p\right)_{\phi} \qquad j_z B_R - j_R B_z = \frac{1}{R}\frac{\partial p}{\partial \phi} \equiv 0$$

- **Ampére's law:**

$$\nabla\times\vec{B}=\mu_o\,\vec{j}$$

$$\nabla\times\vec{B}=\hat{R}\left(\frac{1}{R}\cancel{\frac{\partial B_z}{\partial\phi}}-\frac{\partial B_\phi}{\partial z}\right)+\hat{\phi}\left(\frac{\partial B_R}{\partial z}-\frac{\partial B_z}{\partial R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)-\frac{1}{R}\cancel{\frac{\partial B_R}{\partial\phi}}\right)$$

$$=\hat{R}\left(-\frac{\partial B_\phi}{\partial z}\right)+\hat{\phi}\left(\frac{\partial B_R}{\partial z}-\frac{\partial B_z}{\partial R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)\right)$$

$$=\hat{R}\mu_o j_R+\hat{\phi}\mu_o j_\phi+\hat{z}\mu_o j_z$$

$$j_R=-\frac{1}{\mu_o}\frac{\partial B_\phi}{\partial z} \qquad j_z=\frac{1}{\mu_o}\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)$$

$$\frac{B_R}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)+B_z\frac{\partial B_\phi}{\partial z}=0$$

Magnetic field can be decomposed into the poloidal component and the toroidal component

$$\frac{B_\mathrm{R}}{R}\frac{\partial}{\partial R}(RB_\phi)+B_\mathrm{z}\frac{\partial B_\phi}{\partial z}=0 \quad \Rightarrow \quad B_\mathrm{R}\frac{\partial}{\partial R}(RB_\phi)+B_\mathrm{z}\frac{\partial}{\partial z}(RB_\phi)=0$$

$$F\equiv RB_\phi \quad \Rightarrow \quad B_\mathrm{R}\frac{\partial F}{\partial R}+B_\mathrm{z}\frac{\partial F}{\partial z}=0 \quad \Rightarrow \quad \vec{B}\cdot\nabla F=0$$

$$\left(\frac{\partial}{\partial\phi}=0\right)$$

$$\Downarrow$$

$$F=F(\psi)$$

$$\vec{B}\cdot\nabla p=0$$

$$p=p(\psi)$$

$$B_\mathrm{R}=-\frac{\partial A_\phi}{\partial z}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

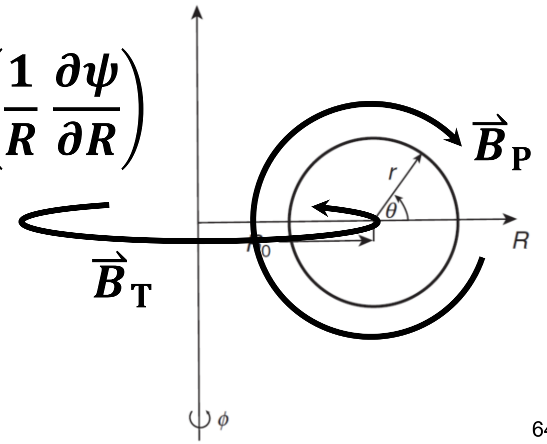
$$B_\mathrm{z}=\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)=\frac{1}{R}\frac{\partial\psi}{\partial R} \qquad (\psi=RA_\phi)$$

$$B_\phi=\frac{F(\psi)}{R}$$

$$\vec{B}=\hat{R}B_\mathrm{R}+\hat{\phi}B_\phi+\hat{z}B_\mathrm{z}=\hat{R}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)+\hat{\phi}\left(\frac{F(\psi)}{R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

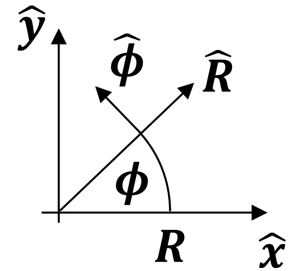
$$\equiv\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}+\frac{F(\psi)}{R}\hat{\phi}$$

Poloidal component $\vec{B}_\mathrm{P}$ **Toroidal component** $\vec{B}_\mathrm{T}$



Alternative way to express magnetic field

$$\vec{B} = \underbrace{\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}}_{\vec{B}_\mathrm{P}} + \underbrace{\frac{F(\psi)}{R}\hat{\phi}}_{\vec{B}_\mathrm{T}} = \underbrace{\nabla\psi\times\nabla\hat{\phi}}_{\text{Poloidal component}} + \underbrace{F(\psi)\nabla\hat{\phi}}_{\text{Toroidal component}}$$



$$\nabla\hat{\phi} = \frac{1}{R}\hat{\phi}$$

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad R^2 = x^2 + y^2$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\cos\phi = \frac{x}{R} \qquad \sin\phi = \frac{y}{R}$$

$$\frac{\partial\phi}{\partial z} = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z} = -\frac{y}{R^2}\hat{x} + \frac{x}{R^2}\hat{y} \quad \Longleftrightarrow \quad \frac{1}{R}\hat{\phi} = -\frac{y}{R^2}\hat{x} + \frac{x}{R^2}\hat{y}$$

Arbitrary integration constant associated with flux can be chosen such that flux equals to zero on the field axis

- **The poloidal flux of the area of a washer-shaped surface lying in the $z = 0$ plane from $R = R_a$ to an arbitrary ψ contour defined by $\psi = \psi(R_b, 0)$:**

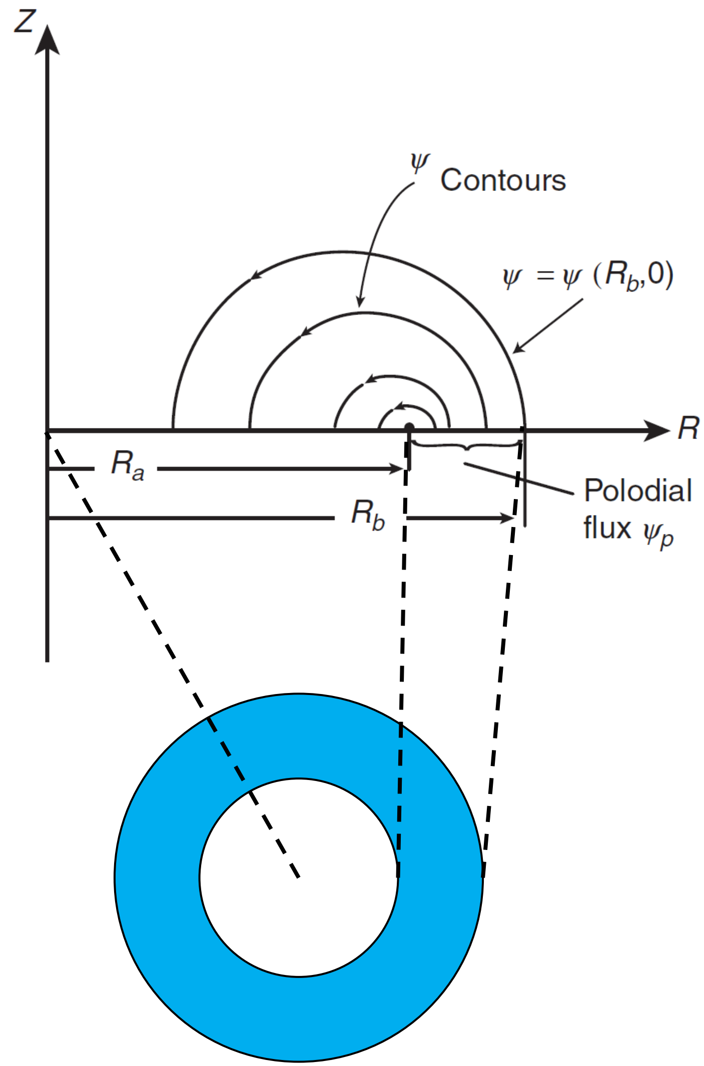
$$\psi_P \equiv \frac{1}{2\pi}\int \vec{B}_P \cdot d\vec{S}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{R_a}^{R_b} dR R B_z(R,0)$$

$$= \psi(R_b,0) - \psi(R_a,0)$$

$$\equiv \psi(R_b,0)$$

where $\psi(R_a, 0) \equiv 0$ is chosen.



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{R}$ component of the force-balance equation:**

$$\left(\vec{j}\times\vec{B}=\nabla p\right)_{\mathrm{R}} \qquad j_{\phi}B_{\mathrm{z}}-j_{\mathrm{z}}B_{\phi}=\frac{\partial p}{\partial R}$$

$$B_{\mathrm{R}}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

$$B_{\mathrm{z}}=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$B_{\phi}=\frac{F(\psi)}{R}$$

- **Ampére's law:**

$$\nabla\times\vec{B}=\mu_{\mathrm{o}}\,\vec{j}$$

$$\nabla\times\vec{B}=\hat{R}\left(-\frac{\partial B_{\phi}}{\partial z}\right)+\hat{\phi}\left(\frac{\partial B_{\mathrm{R}}}{\partial z}-\frac{\partial B_{\mathrm{z}}}{\partial R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(RB_{\phi}\right)\right)=\hat{R}\mu_{\mathrm{o}}j_{\mathrm{R}}+\hat{\phi}\mu_{\mathrm{o}}j_{\phi}+\hat{z}\mu_{\mathrm{o}}j_{z}$$

$$\mu_{\mathrm{o}}j_{\phi}=\frac{\partial B_{\mathrm{R}}}{\partial z}-\frac{\partial B_{\mathrm{z}}}{\partial R}=\frac{\partial}{\partial z}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)-\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)=-\frac{1}{R}\frac{\partial^2\psi}{\partial z^2}-\frac{1}{R}\frac{\partial^2\psi}{\partial R^2}+\frac{1}{R^2}\frac{\partial\psi}{\partial R}$$

$$\equiv-\frac{1}{R}\Delta^{*}\psi \qquad \text{where } \Delta^{*}\psi\equiv\frac{\partial^2\psi}{\partial z^2}+R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)=R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

$$\mu_{\mathrm{o}}j_{\mathrm{z}}=\frac{1}{R}\frac{\partial}{\partial R}\left(RB_{\phi}\right)=\frac{1}{R}\frac{\partial F}{\partial R}=\frac{1}{R}\frac{dF}{d\psi}\frac{\partial\psi}{\partial R}$$

Derivation of Grad-Shafranov equation

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial R}$$

$$B_\phi = \frac{F}{R} \qquad j_\phi = -\frac{1}{\mu_o R}\Delta^*\psi$$

$$\frac{\partial p}{\partial R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$B_z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad j_z = \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\frac{1}{\mu_o R}\Delta^*\psi \frac{1}{R}\cancel{\frac{\partial \psi}{\partial R}} - \frac{1}{\mu_o R}\frac{dF}{d\psi}\cancel{\frac{\partial \psi}{\partial R}}\frac{F}{R} = \frac{dp}{d\psi}\cancel{\frac{\partial \psi}{\partial R}}$$

$$-\Delta^*\psi\frac{1}{\mu_o}\frac{1}{R^2} - \frac{1}{\mu_o}\frac{F}{R^2}\frac{dF}{d\psi} = \frac{dp}{d\psi}$$

$$\boxed{\text{Grad} - \text{Shafranov equation: } \Delta^*\psi = -\mu_o R^2\frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}}$$

$$F \equiv RB_\phi$$

$$\text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right)\times\hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Derivation of Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \quad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right) \quad \vec{B} = \left(\frac{\nabla\psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

$$\mu_o j_\phi = -\frac{1}{R}\Delta^*\psi \qquad \mu_o j_z = \frac{1}{R}\frac{\partial F}{\partial R} \qquad F \equiv RB_\phi$$

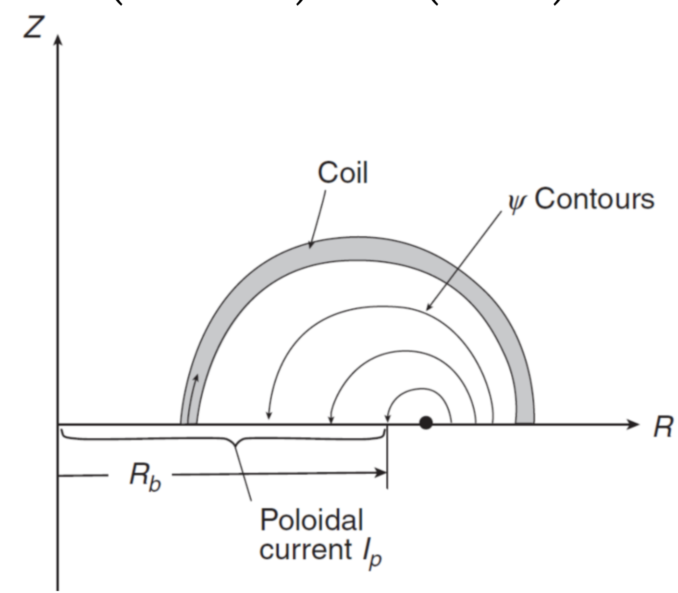
$$\mu_o j_R = -\frac{\partial B_\phi}{\partial z} = -\frac{1}{R}\frac{\partial}{\partial z}(RB_\phi) = -\frac{1}{R}\frac{\partial F}{\partial z}$$

$$\mu_o \vec{j} = \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z = \hat{R}\left(-\frac{1}{R}\frac{\partial F}{\partial z}\right) + \hat{\phi}\left(-\frac{1}{R}\Delta^*\psi\right) + \hat{z}\left(\frac{1}{R}\frac{\partial F}{\partial R}\right)$$

$$\equiv \left(\frac{\nabla F}{R}\right) \times \hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi}$$

$$I_P = \int \vec{j}_P \cdot d\vec{S} = -\int_0^{2\pi} d\phi \int_0^{R_b} dR R j_z(R,0)$$

$$= -2\pi \int_0^{R_b} dR R \frac{1}{R}\frac{\partial F(R,0)}{\partial R} = -2\pi F(\psi)$$



Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \qquad \mu_o j_\phi = -\frac{1}{R}\Delta^*\psi$$

$$j_\phi = R\frac{dp}{d\psi} + \frac{1}{2\mu_o R}\frac{dF^2}{d\psi}$$

- **The usual strategy to solve the Grad-Shafranov equation:**
  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Application of solving Grad-Shafranov equation for designing a tokamak

- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, perfect conductor as the chamber.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, insulator chamber.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, chamber with eddy current.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, fixed boundary of plasma. Then, use I_{coils}, free boundary of plasma and match the plasma shape calculated in the fixed boundary condition.

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

$$j_\phi = R\frac{dp}{d\psi} + \frac{1}{2\mu_0 R}\frac{dF^2}{d\psi} \qquad I_P = -2\pi F(\psi)$$

$$\mu_0 \vec{j} = \left(\frac{\nabla F}{R}\right)\times\hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi} \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right)\times\hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Example of the analytical solution of the Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$

- For $\mu_o \frac{dp}{d\psi} = -C_2$ $\quad \frac{1}{2}\frac{dF^2}{d\psi} = C_1$

$$\psi(R,z) = -\frac{C_1}{2}z^2 + \frac{C_2}{8}R^4 + C_3 + C_4R^2 + C_5\left(R^4 - 4R^2z^2\right)$$

$$B_R(R,z) = -\frac{1}{R}\left(-C_1 z - 8C_5R^2z\right) \quad B_z(R,z) = \frac{1}{R}\left(\frac{C_2}{2}R^3 + 2C_4R + C_5\left(4R^3 - 8Rz^2\right)\right)$$

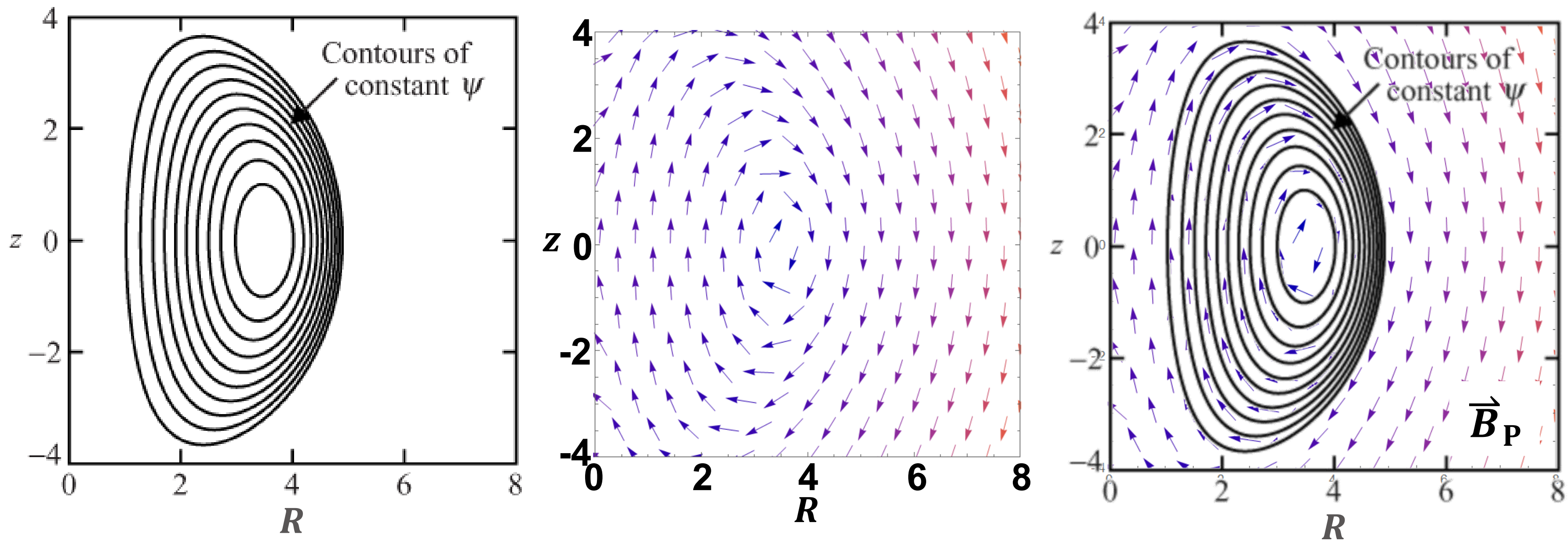
$C_1 = 1$

$C_2 = -8$

$C_3 = -20$

$C_4 = 20$

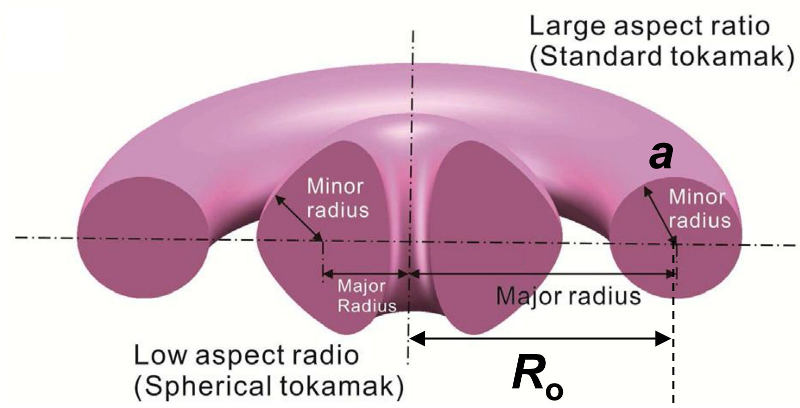
$C_5 = 0.2$



Examples: Large Aspect Ratio Tokamak

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$

$$j_\phi = R\frac{dp}{d\psi} + \frac{1}{2\mu_o R}\frac{dF^2}{d\psi}$$



- **Assuming:**
  - ***R_o >> a***
  - ***j_Φ= j_o= constant***
  - ***F(ψ)= B_oR_o = constant***

$$\Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = \frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

$$\Rightarrow \frac{dp}{d\psi} = \text{const} \qquad \mu_o j_\phi = -\frac{1}{R}\Delta^*\psi \qquad \Rightarrow \quad -\mu_o R j_o = \frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

$$\psi(R,Z) = \frac{1}{2}A\left(R^2 - {R_o}^2\right) + \frac{1}{2}BZ^2 \qquad \Delta^*\psi = A + B = -\mu_o R j_o$$

$$\text{Let } A = B \equiv -\frac{\mu_o j_o}{2} \qquad \psi(R,Z) = -\frac{\mu_o j_o}{4}\left[\left(R^2 - {R_o}^2\right) + Z^2\right]$$

Equilibrium reconstruction

- **Magnetic Diagnostics (Essential for the Grad-Shafranov equation)**
  - **Magnetic flux loops: Measure poloidal flux ψ at different locations.**
  - **Poloidal field (PF) probes: Measure B_p.**
  - **Toroidal field (TF) coils: Provide information on B_ϕ.**
  - **Rogowski coils: Measure total plasma current I_p.**
  - **Hall probes: Measure local magnetic fields.**
- **Interferometry & Thomson Scattering - Provide electron density profile $n_e(r)$.**
- **X-ray & Spectroscopy - Measure soft X-ray emission, used to infer internal plasma profiles.**
- **Motional Stark Effect (MSE) & Polarimetry - Measure the pitch angle of the magnetic field, used to infer the safety factor profile $q(\psi)$.**
- **Soft X-ray and ECE Imaging - Can give additional constraints on plasma shape and profiles.**

Equilibrium reconstruction – conti.

$$\Delta^{*}\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad j_\phi = R\frac{dp}{d\psi} + \frac{1}{2\mu_o R}\frac{dF^2}{d\psi}$$

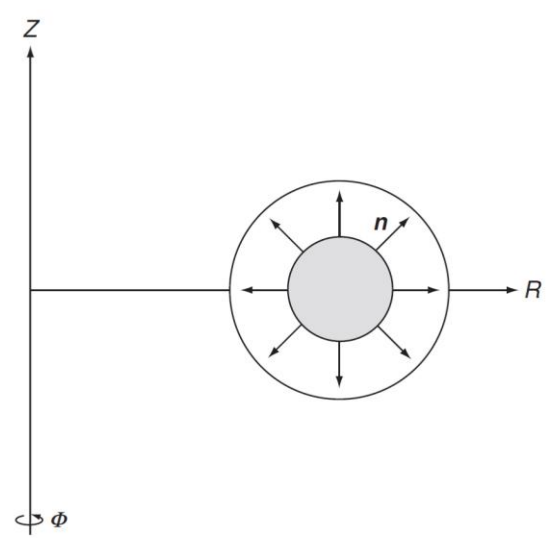
$j_\phi(\psi)$ $p(\psi)$ $F(\psi)$ **<= They are constrained with each other. When two are given, the third can be calculated.**

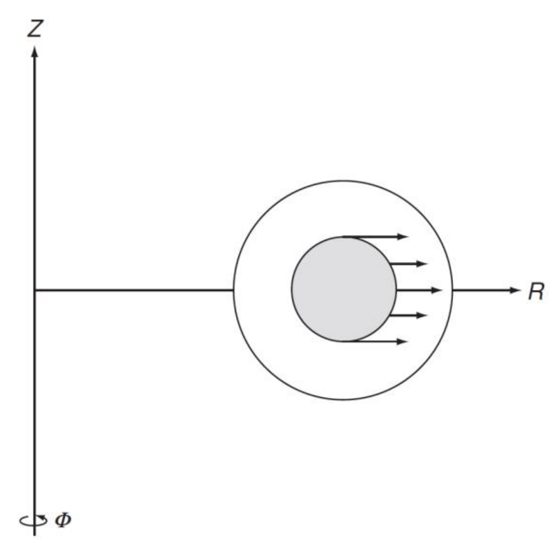
- **Adjustable functions:** $F(\psi)$ **and** $p(\psi)$ **are written in analytical forms.**
- **Solve Grad-Shafranov for** $\psi(R,Z)$
- **Compute derived quantities (flux loops, magnetic probes, pitch angles)**
- **Compare to diagnostics**
- **Adjust profile parameters**
- **Repeat until residuals are minimized**
- **Calculate** $j_\phi(\psi)$

Magnetically confined toroidal equilibrium

1. Radial pressure balance in the poloidal plan needs to be provided so that the pressure contours form closed nested surfaces. Both toroidal and poloidal fields can readily accomplish this task.
2. The radially outward expansion force inherent in all toroidal geometries needs to be balanced without sacrificing stability.

- Forces associated with toroidal force balance are usually than those corresponding to radial pressure balance. However, they are more difficult to compensate.





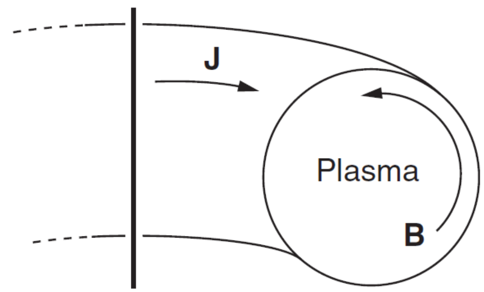
Toroidal configuration with a purely poloidal magnetic field

- **Hoop force:**

$$\psi_1 = \psi_2 \equiv \psi$$

$$S_1 < S_2 \qquad B_1 > B_2$$

$$\vec{F}_{\mathrm{H,R}} \propto \hat{e}_{\mathrm{R}} \frac{{B_1}^2 S_1 - {B_2}^2 S_2}{2\mu_o} > 0$$

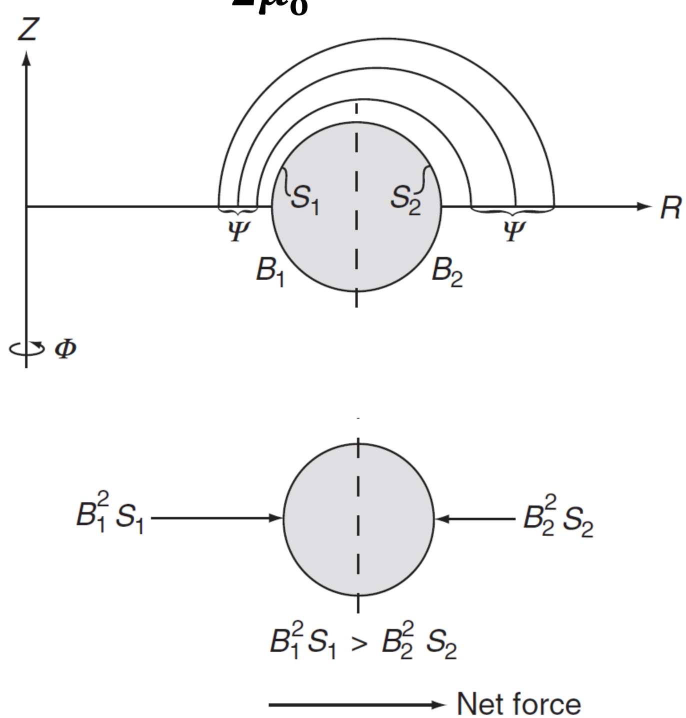


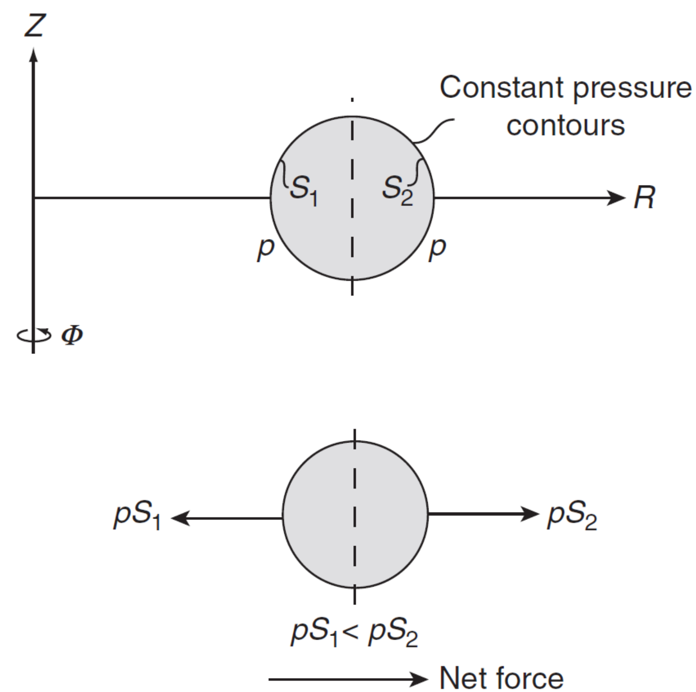
- **Tire tube force**

$$p_1 = p_2 \equiv p$$

$$S_1 < S_2$$

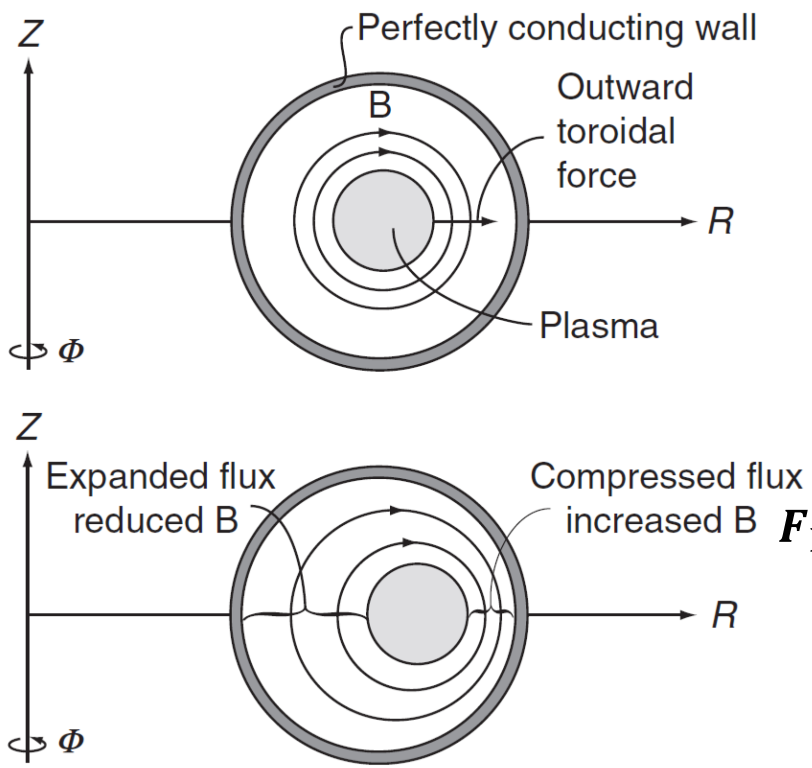
$$\vec{F}_{\mathrm{T,R}} \propto -\hat{e}_{\mathrm{R}}(pS_1 - \mathrm{p}S_2) > 0$$



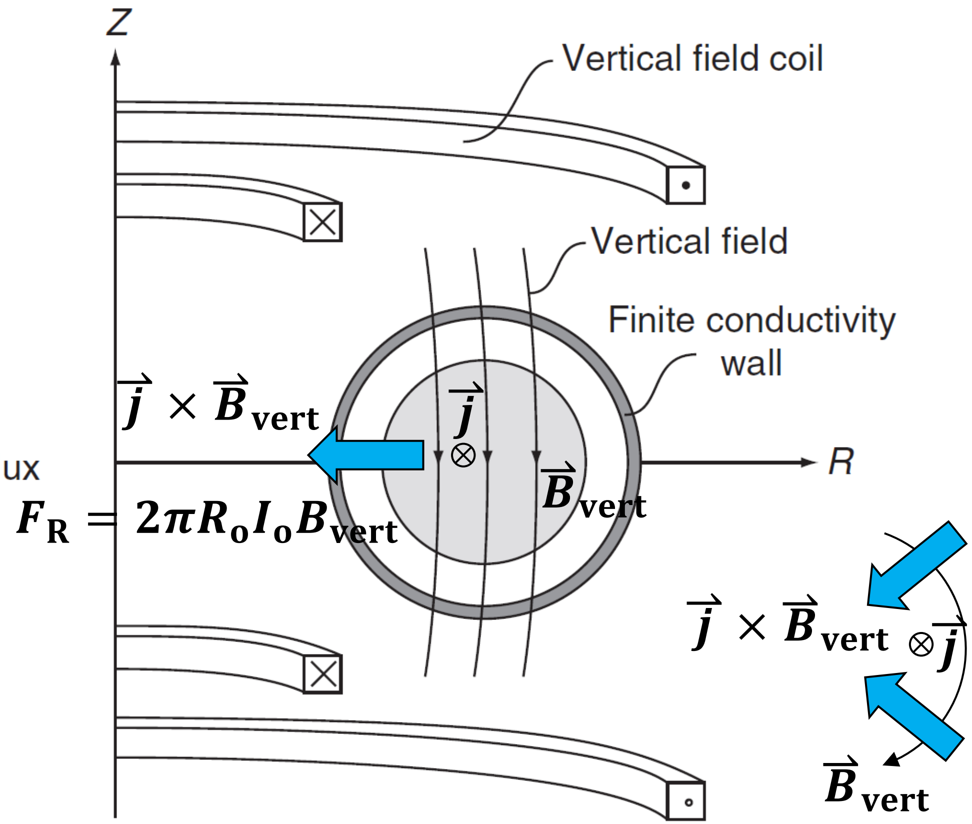


The outward force can be compensated by either a perfectly conducting shell or externally applied vertical field

- **Perfectly conducting shell**

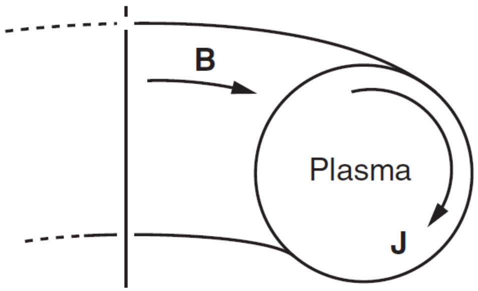


- **Externally applied vertical field**



- **With a finite conductivity wall, flux can only remain compressed for about a skin time.**
- **This configuration develops disastrous MHD instabilities (z pinch).**

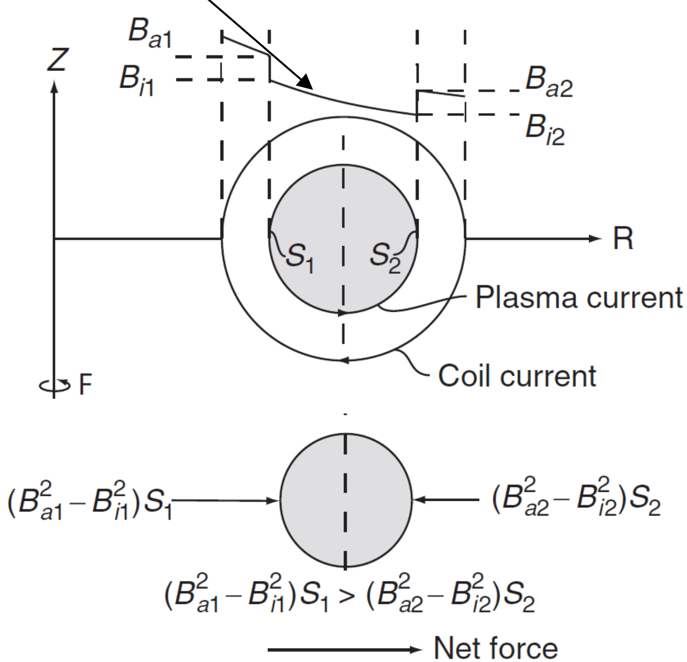
Toroidal configuration with a purely toroidal magnetic field, stable but NOT balanced

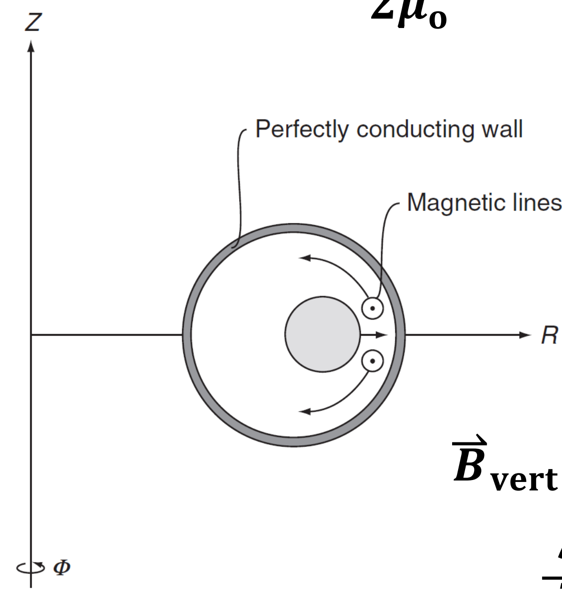


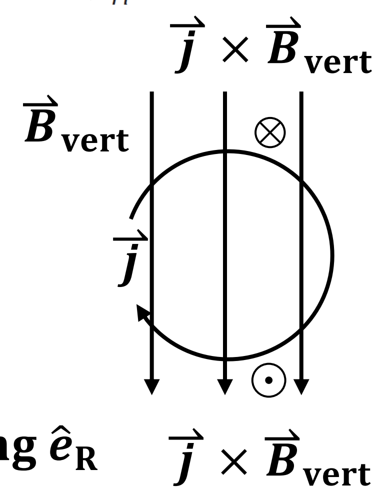
$$\vec{B} = B_\phi \hat{e}_\phi \qquad B_\phi = B_o \frac{R_o}{R} \qquad B_o = \frac{\mu_o I_c}{2\pi R_o}$$

$$\vec{F}_R \propto \hat{e}_R \frac{\left(B_{a1}{}^2 - B_{i1}{}^2\right) S_1 - \left(B_{a2}{}^2 - B_{i2}{}^2\right) S_2}{2\mu_o} > 0$$

Plasma is diamagnetic

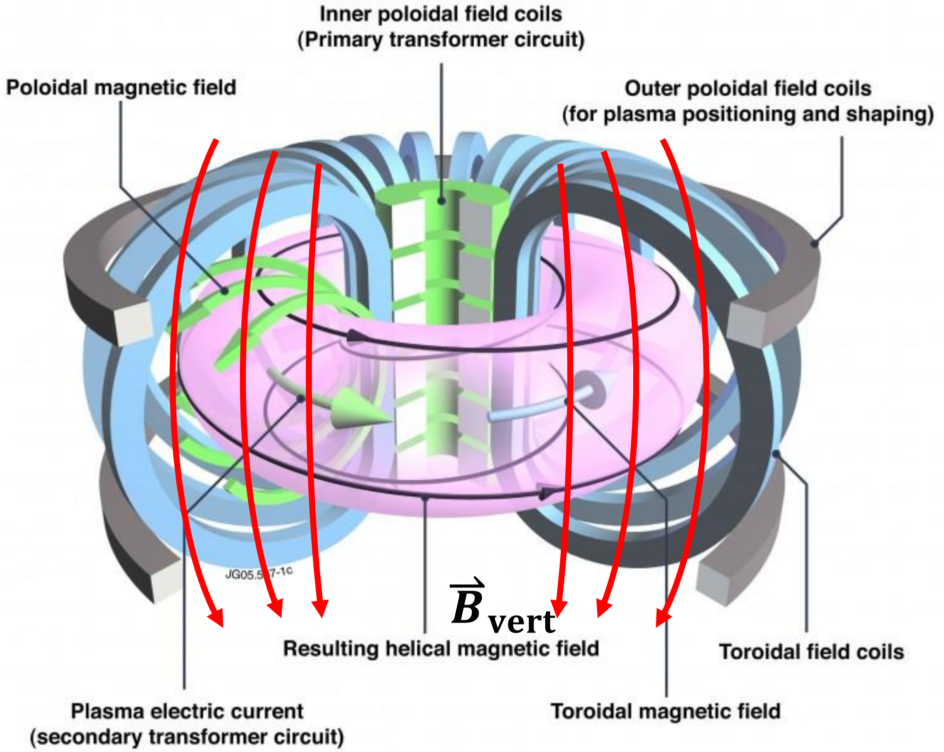






$\vec{j} \times \vec{B}_{\text{vert}}$ is not pointed along $\hat{e}_R$

Coils in a tokamak



- **Toroidal field coils (in poloidal direction) – generate toroidal field for confinement.**
- **Poloidal field coils – generate vertical field for plasma positioning and shaping.**
- **Central solenoid – for breakdown and generating plasma current (in toroidal direction) and thus generating poloidal field for confinement.**

https://www.euro-fusion.org/2011/09/tokamak-principle-2/

Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right) \qquad F = RB_\phi = -\frac{I_P}{2\pi}$$

- **The usual strategy to solve the Grad-Shafranov equation:**
  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

     $$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Application of solving Grad-Shafranov equation for designing a tokamak

- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, perfect conductor as the chamber.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, insulator chamber.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, chamber with eddy current.
- Given I_{plasma}, $p(\psi)$, $I(\psi)$, fixed boundary of plasma. Then, use I_{coils}, free boundary of plasma and match the plasma shape calculated in the fixed boundary condition.

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right)$$

$$I_{\text{P}} = -2\pi F(\psi)$$

$$\mu_0 \vec{j} = \left(\frac{\nabla F}{R}\right) \times \hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi} \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Application of solving Grad-Shafranov equation for reconstructing a tokamak equilibrium state

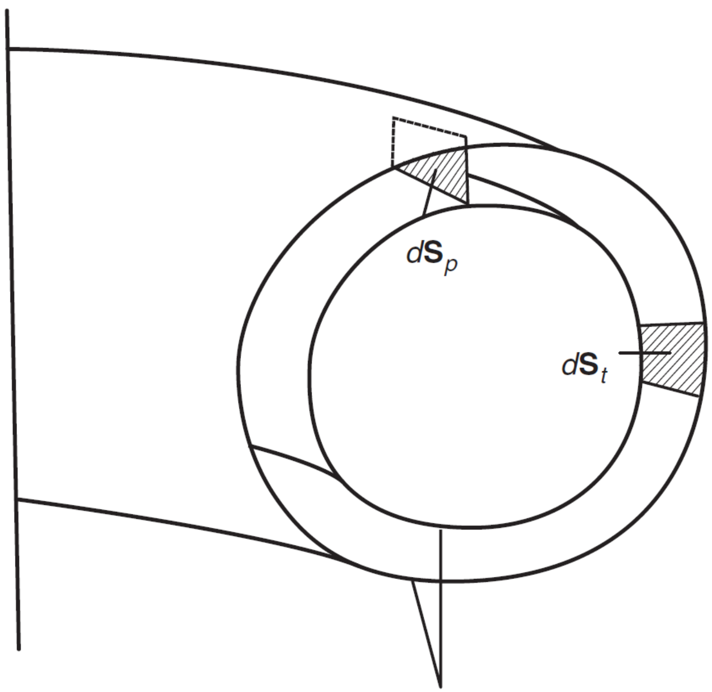
- **Measure**
  - **boundary conditions, including *ψ*, *B*, etc., on the wall (using flux loop and B-dot probe).**
  - **Pressure.**
  - **Plasma current (using Rogowski coil).**
- **Reconstruct *ψ*(*r*,*z*), *j*, *p*(*ψ*), *I*(*ψ*), etc.**

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right)$$

$$I_P = -2\pi F(\psi)$$

$$\mu_o \vec{j} = \left(\frac{\nabla F}{R}\right) \times \hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi} \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Fluxes and currents



- **Poloidal flux:** $\psi_\mathrm{p} = \int \vec{B} \cdot d\,\vec{S}_\mathrm{p}$

  $\psi_\mathrm{p} = \psi_\mathrm{p}(p)$

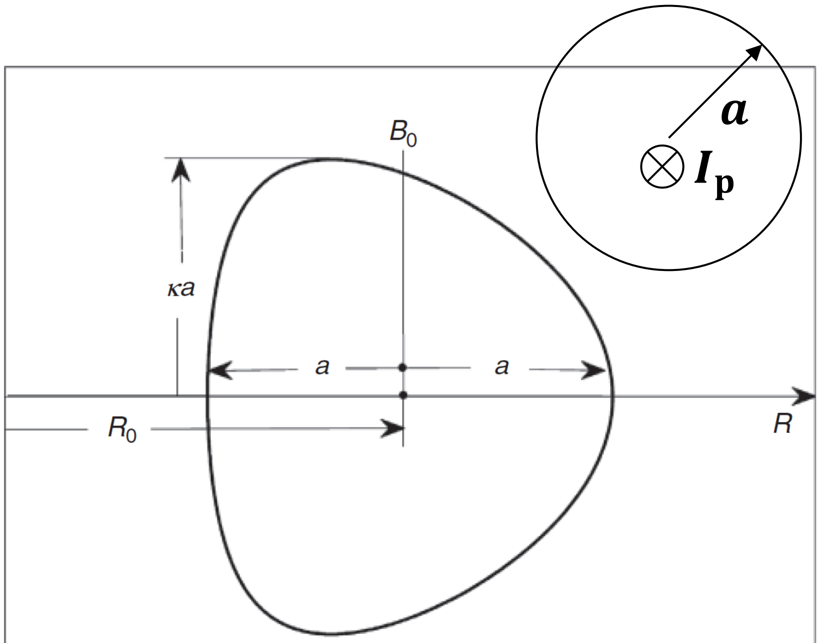
- **Toroidal flux:** $\psi_\mathrm{t} = \int \vec{B} \cdot d\,\vec{S}_\mathrm{t}$

- **Poloidal current:** $I_\mathrm{p} = \int \vec{j} \cdot d\,\vec{S}_\mathrm{p}$

- **Toroidal current:** $I_\mathrm{t} = \int \vec{j} \cdot d\,\vec{S}_\mathrm{t}$

Normalized plasma pressure, β

$$\beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{2\mu_o\langle p\rangle}{B^2}$$



- **Plasma pressure:** $\langle p\rangle = \dfrac{1}{V_p}\displaystyle\int p\, d\vec{r}$
- **Magnetic pressure:** $P_B = \dfrac{B^2}{2\mu_o}$

$$B^2 = {B_t}^2 + {B_p}^2 = {B_o}^2 + \left(\frac{\mu_o I_p}{2\pi a}\right)^2 \frac{2}{1+\kappa^2}$$

$${B_t}^2 = {B_o}^2 \qquad B_o = B\ @R = R_o$$

- **Elongation:** κ

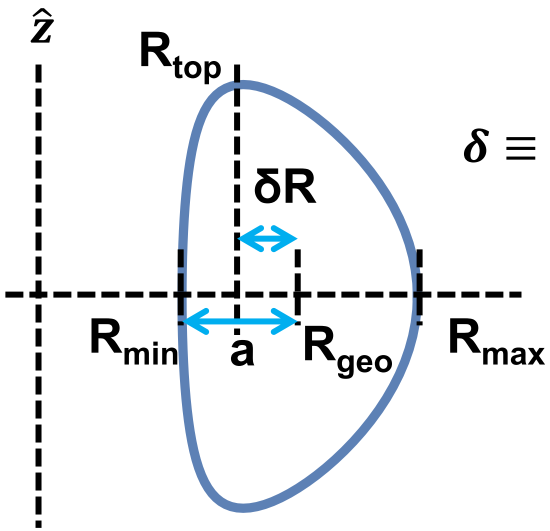
$${B_p}^2 = \left(\frac{\mu_o I_p}{2\pi a}\right)^2 = \left(\frac{\mu_o I_p}{C_p}\right)^2$$

$$\beta_t = \frac{2\mu_o\langle p\rangle}{{B_o}^2} \qquad \beta_p = \frac{4\pi^2 a^2(1+\kappa^2)p}{\mu_o {I_p}^2}$$

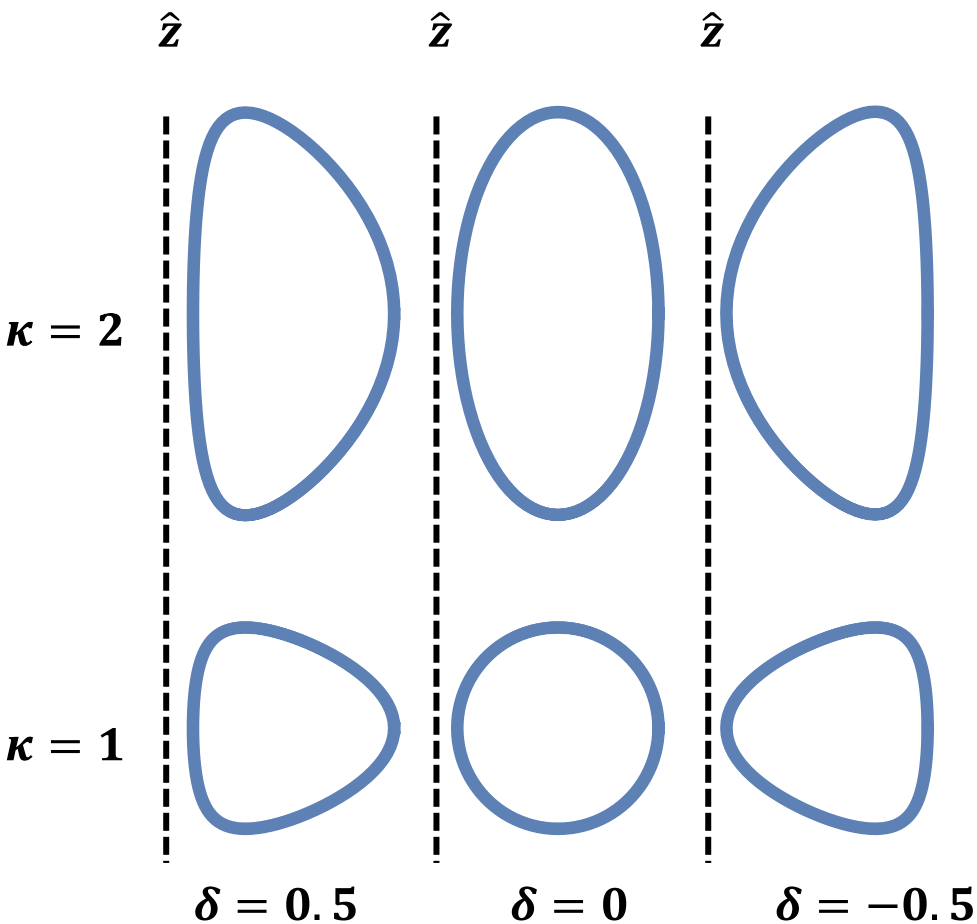
$$C_p \approx 2\pi a\sqrt{\frac{1+\kappa^2}{2}}$$

$$\frac{1}{\beta} = \frac{1}{\beta_t} + \frac{1}{\beta_p}$$

Different poloidal shapes



$$\delta \equiv \frac{\delta R}{a}$$



$$r = R + a\mathrm{Cos}\big(\theta + \delta \mathrm{Sin}(\theta)\big)$$

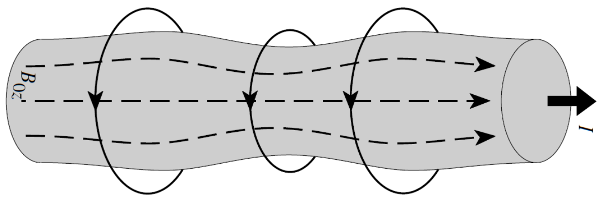
$$z = a\kappa \mathrm{Sin}(\theta)$$

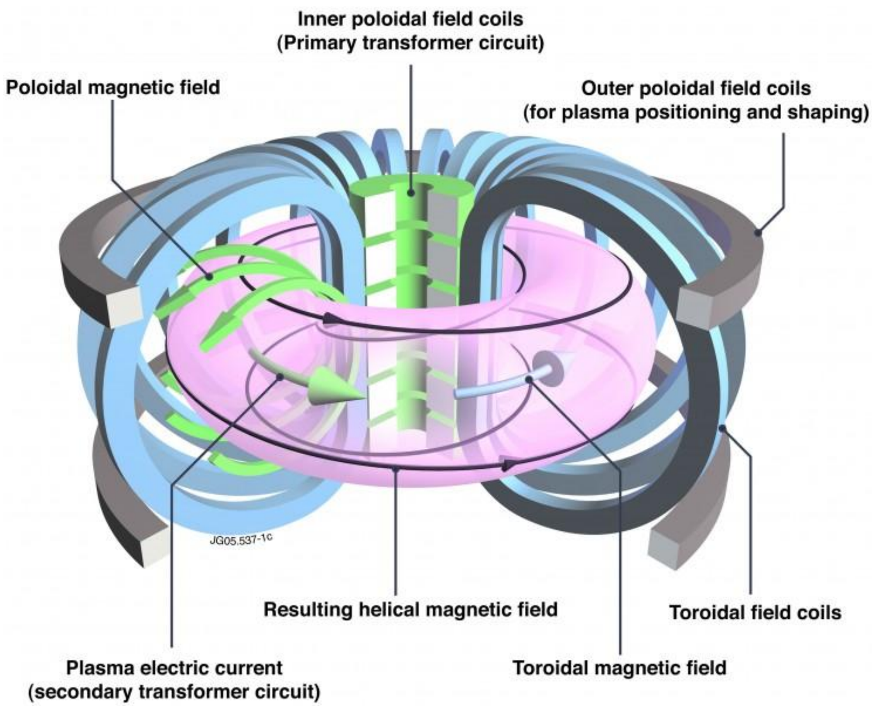
- **Aspect ratio:** $\frac{R}{a}$
- **Elongation:** κ
- **Triangularity:** δ

Safety factor

- **Kink Safety Factor:**

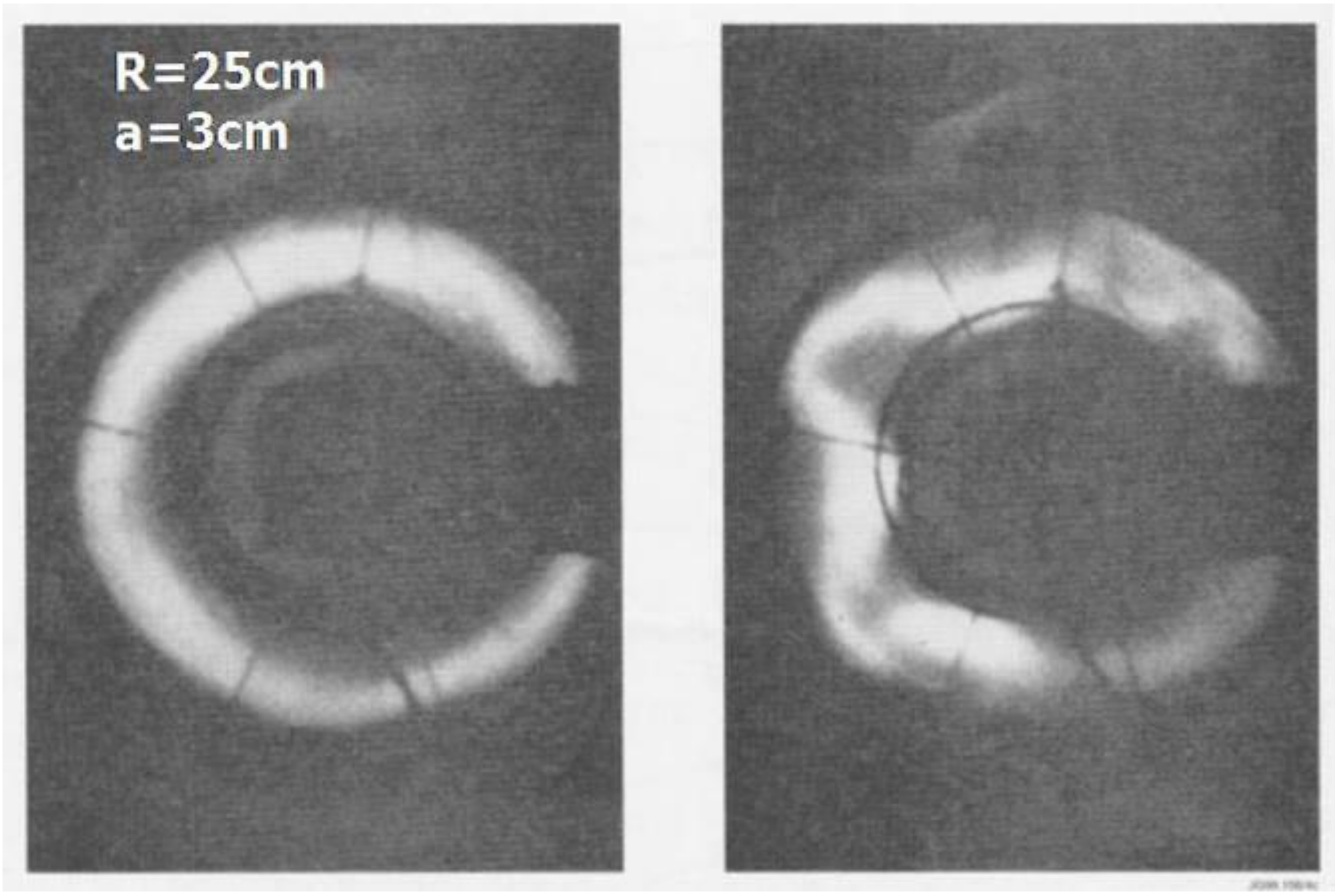
$$q^*(r) = \frac{aB_o}{R_oB_p} = \frac{2\pi a^2\kappa B_o}{\mu_o R_o I_o}$$





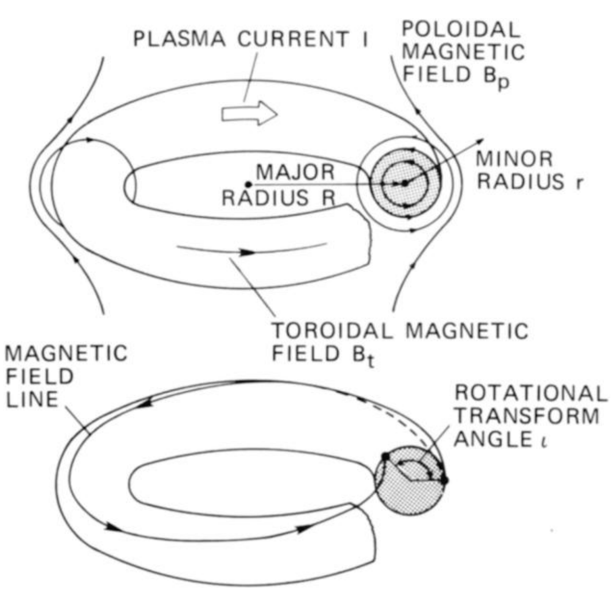
$$q(r) = \frac{rB_z(r)}{R_oB_\theta(r)}$$

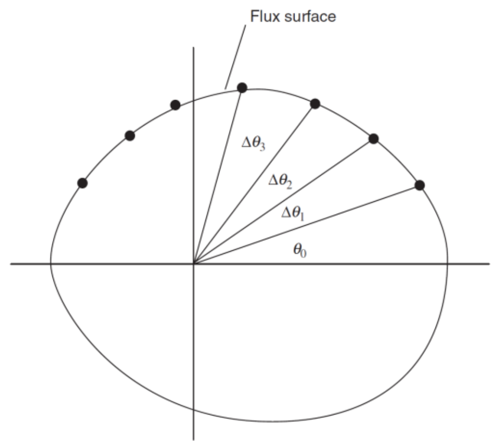
Kink instability in action in a 3 by 25-cm pyrex tube at Aldermaston



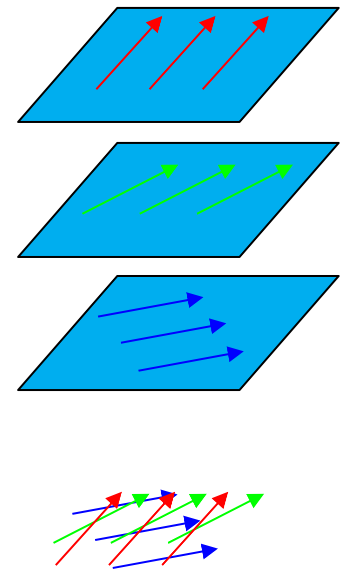
https://en.wikipedia.org/wiki/Kink_instability
R A Bingham et al 2026 Plasma Phys. Control. Fusion 68 030201

Safety factor





- **Shear:**



- **Rotational transform:** $$\iota \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \Delta\theta_{\mathrm{n}}$$
- **MHD safety factor:** $$q(V) \equiv \frac{2\pi}{\iota(V)} = \frac{d\psi_t/dV}{d\psi_p/dV}$$

$$\psi_{\mathrm{t}} = \psi_{\mathrm{t}}(V) \qquad \psi_{\mathrm{p}} = \psi_{\mathrm{p}}(V)$$

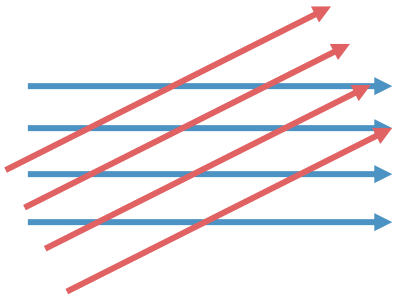
- **Shear:** $$s(V) \equiv 2\frac{V}{q}\frac{dq}{dV}$$

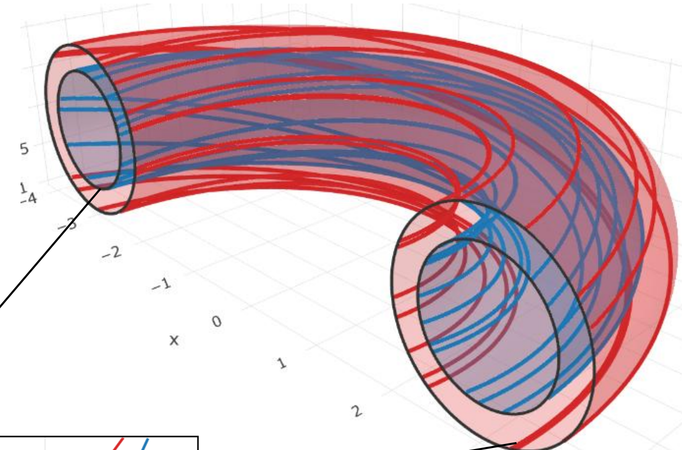
$$\iota(V) \equiv 2\pi\left(\frac{d\psi_p/dV}{d\psi_t/dV}\right)$$

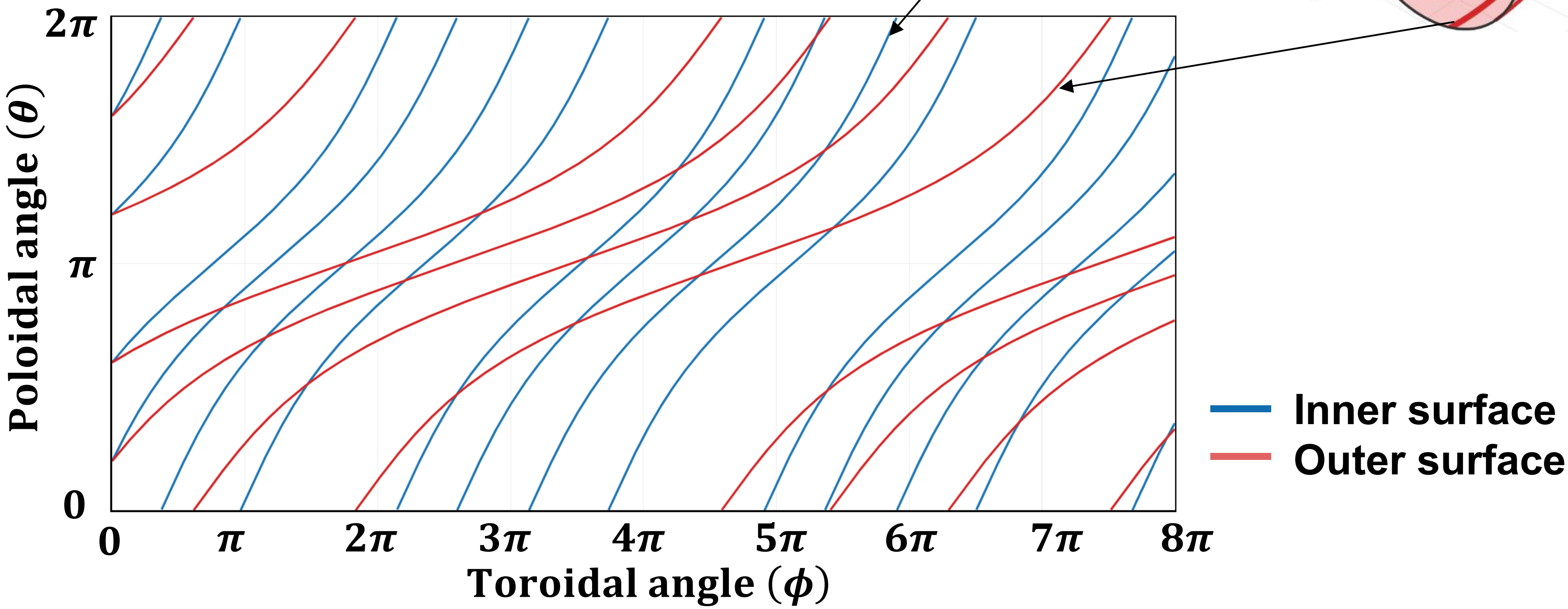
C. C. Baker, *et al*, Nuclear Technology/Fusion, 1, 5 (1981)

Global magnetic shear

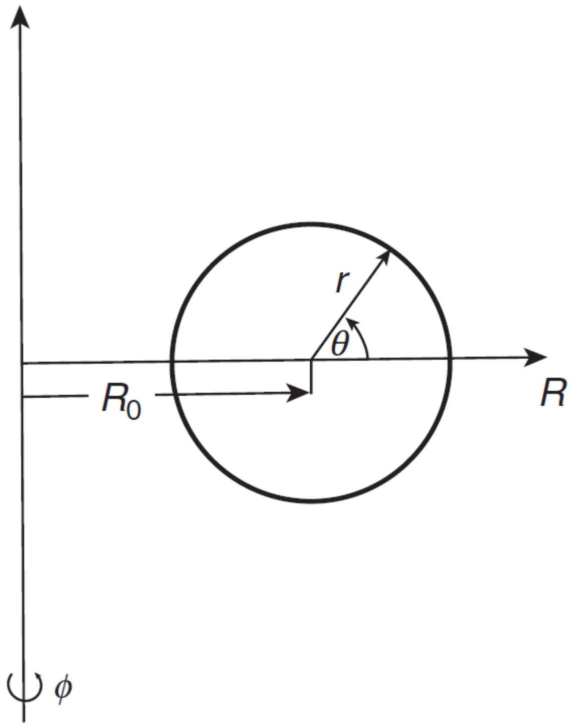
- **Shear**







Volume of the constant flux



$$\boldsymbol{R = R_o + r\cos\theta}$$

$$\boldsymbol{Z = r\sin\theta}$$

$$\boldsymbol{\psi = \psi(r, \theta)}$$

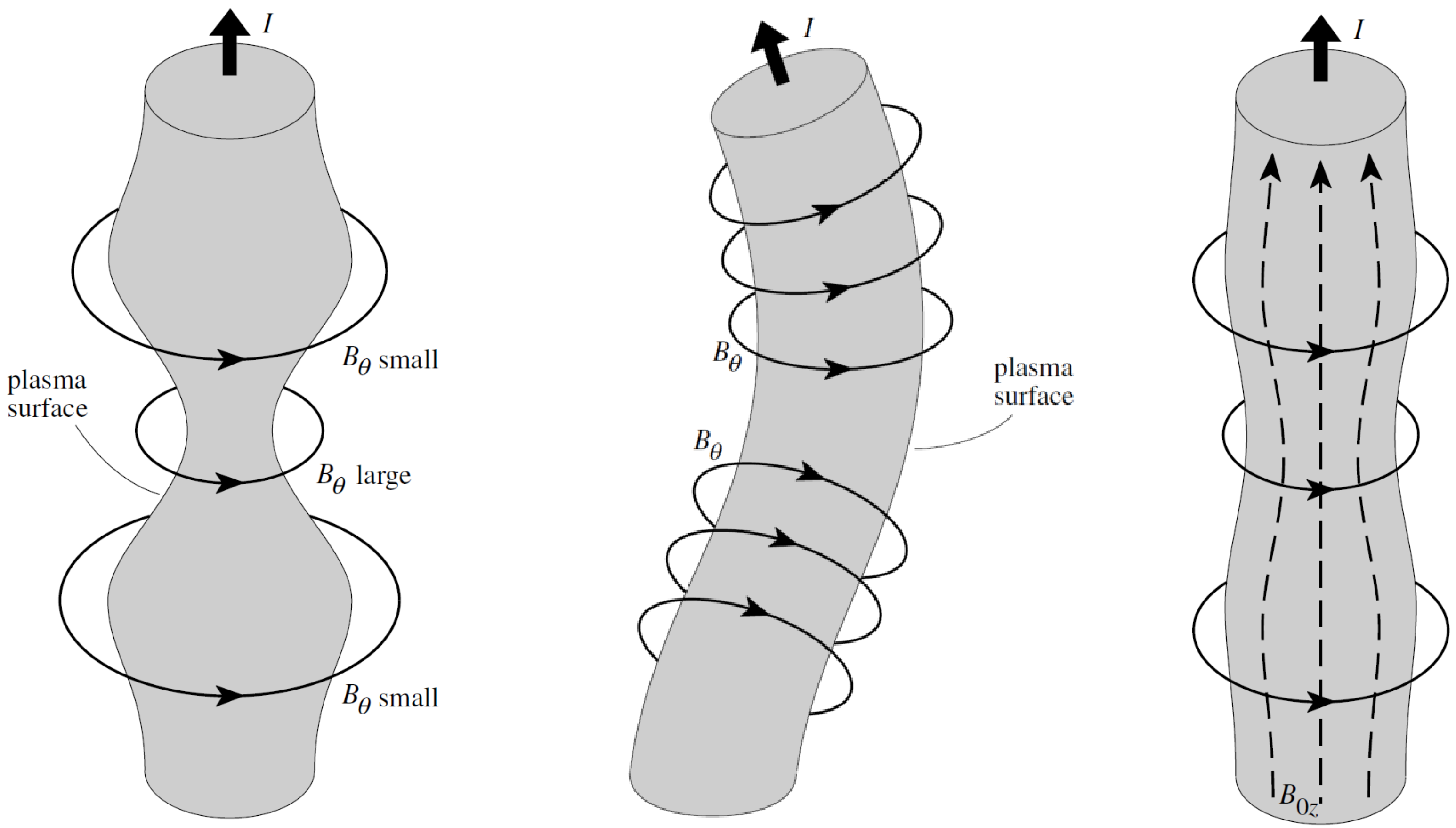
$$\boldsymbol{r = \hat{r}(\theta, \psi)}$$

$$\boldsymbol{V(\psi) = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\hat{r}} R r\, dr\, d\theta\, d\phi}$$

$$\boldsymbol{V(\psi) = \pi R_o \int_0^{2\pi} d\theta\, \hat{r}\left[1 + \frac{2}{3}\left(\frac{\hat{r}}{R_o}\right) cos\theta\right]}$$

A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**
- **Kink instability**



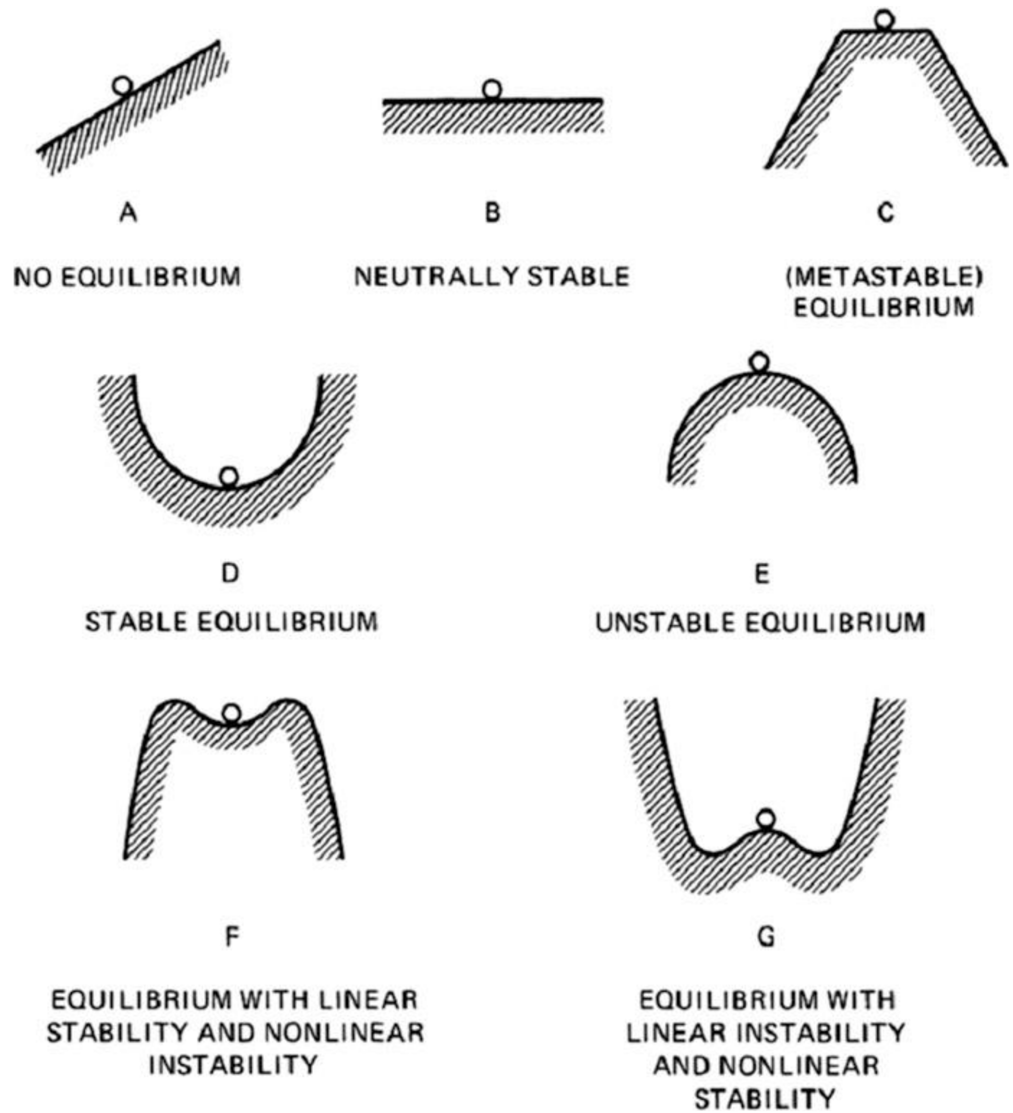
- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_o B_\theta(r)}$ **Kruskal–Shafranov limit**

Magnetic well

$$\widehat{W} = 2\frac{V}{\langle B^2\rangle}\frac{d}{dV}\left\langle\frac{B^2}{2}\right\rangle$$

$$= 2\frac{V}{\langle B^2\rangle}\frac{d}{dV}\left\langle\mu_o p + \frac{B^2}{2}\right\rangle$$

- **A magnetic well is a quantity that measures plasma stability against short perpendicular wavelength modes driven by the plasma pressure gradient.**



Variational formulation for checking stabilization

- **Equilibrium state:** $\vec{j}_o \times \vec{B}_o = \nabla p_o$
- **Momentum eq:** $\rho_m \left[\dfrac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$

$$\rho_m = \rho_o + \tilde{\rho}_1 \qquad p = p_o + \tilde{p}_1 \qquad \vec{v} = \vec{v}_o + \tilde{\vec{v}}_1 = \tilde{\vec{v}}_1 \equiv \frac{\partial \vec{\xi}}{\partial t}$$

$$\vec{j} = \vec{j}_o + \tilde{\vec{j}}_1 \qquad \vec{B} = \vec{B}_o + \tilde{\vec{B}}_1 \equiv \vec{B}_o + \vec{Q}$$

$$\rho_m \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{F}\left(\vec{\xi}\right) \qquad \qquad \vec{F}\left(\vec{\xi}\right) = \vec{j}_o \times \vec{Q} + \tilde{\vec{j}}_1 \times \vec{B}_o - \nabla \tilde{p}_1$$

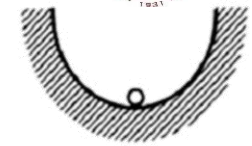
$$\vec{F}\left(\vec{\xi}\right) = \frac{1}{\mu_o}\left(\nabla \times \vec{B}_o\right) \times \vec{Q} + \frac{1}{\mu_o}\left(\nabla \times \vec{Q}\right) \times \vec{B}_o + \nabla\left(\vec{\xi} \cdot \nabla p + \gamma p \nabla \cdot \vec{\xi}\right)$$

- **The change in potential energy associated with the perturbation:**

$$\delta W = -\frac{1}{2} \int \vec{\xi}^{\,*} \cdot \vec{F}\left(\vec{\xi}\right) d\,\vec{r}$$

- **Stable requirement:** $\delta W \geq 0$

$$\delta W_F = \frac{1}{2} \int d\,\vec{r} \left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_o} + \frac{B^2}{\mu_o} \left|\nabla \cdot \vec{\xi}_\perp + 2\,\vec{\xi}_\perp \cdot \vec{\kappa}\right|^2 + \gamma p \left|\nabla \cdot \vec{\xi}\right|^2 - 2\left(\vec{\xi}_\perp \cdot \nabla p\right)\left(\vec{\kappa} \cdot \vec{\xi}_\perp^{\,*}\right) - J_{||}\left(\vec{\xi}_\perp^{\,*} \times \vec{b}\right) \cdot \vec{Q}_\perp \right]$$



D

STABLE EQUILIBRIUM

E

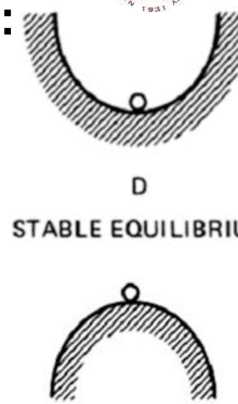
UNSTABLE EQUILIBRIUM

Variational formulation for checking stabilization

- **The change in potential energy associated with the perturbation:**

$$\delta W = -\frac{1}{2}\int \vec{\xi}^{\,*}\cdot\vec{F}\left(\vec{\xi}\right)d\,\vec{r}$$

- **Stable requirement:** $\delta W \geq 0$



D

STABLE EQUILIBRIUM

E

UNSTABLE EQUILIBRIUM

$$\delta W_F = \frac{1}{2}\int d\,\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_o} + \frac{B^2}{\mu_o}\left|\nabla\cdot\vec{\xi}_\perp + 2\,\vec{\xi}_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\vec{\xi}\right|^2 \right.$$
$$\left. -2\left(\vec{\xi}_\perp\cdot\nabla p\right)\left(\vec{\kappa}\cdot\vec{\xi}_\perp^{\,*}\right) - J_{||}\left(\vec{\xi}_\perp^{\,*}\times\vec{b}\right)\cdot\vec{Q}_\perp\right]$$

- **Stabilization terms:**

$\dfrac{\left|\vec{Q}_\perp\right|^2}{\mu_o}$: **For bending magnetic field lines, (shear Alfvén wave).**

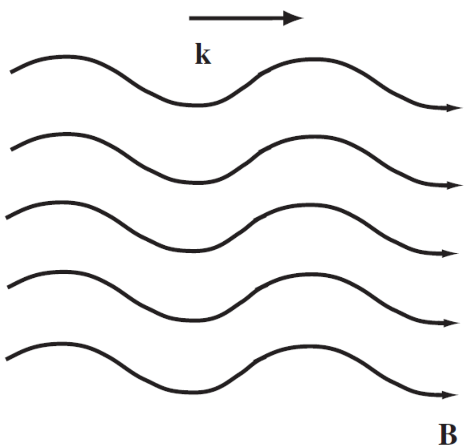
$\dfrac{B^2}{\mu_o}\left(\left|\nabla\cdot\vec{\xi}_\perp + 2\,\vec{\xi}_\perp\cdot\vec{\kappa}\right|\right)^2$: **For compressing the magnetic field, (compressional Alfvén wave).**

$\gamma p\left|\nabla\cdot\vec{\xi}\right|^2$: **For compressing the plasma, (sound wave).**

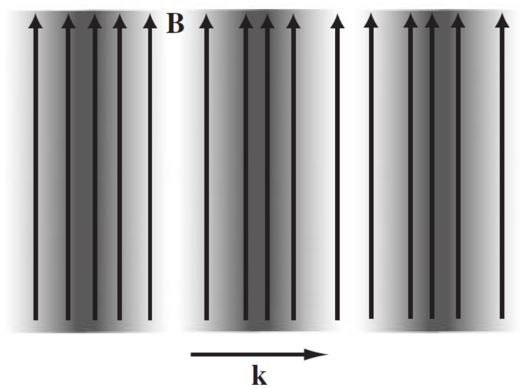
- **Destabilization terms:** $-2\left(\vec{\xi}_\perp\cdot\nabla p\right)\left(\vec{\kappa}\cdot\vec{\xi}_\perp^{\,*}\right) - J_{||}\left(\vec{\xi}_\perp^{\,*}\times\vec{b}\right)\cdot\vec{Q}_\perp$

Alfvén waves

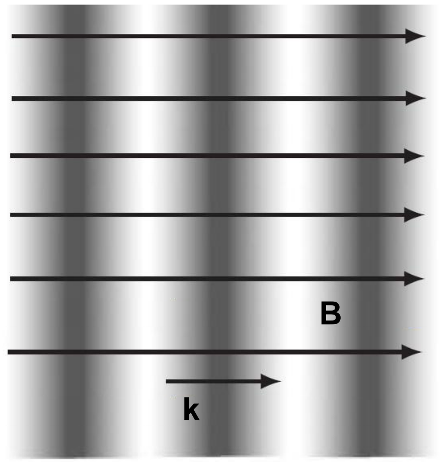
- **Shear Alfvén wave:**



- **The fast magnetosonic wave (Compressional Alfvén wave):**

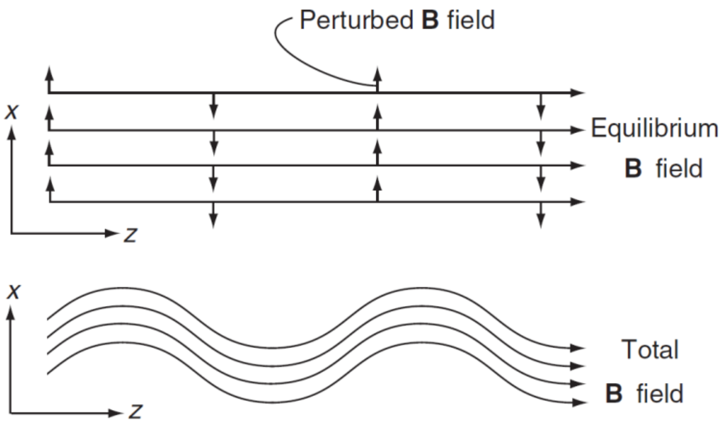


- **Longitudinal sound wave (the slow magnetosonic wave)**

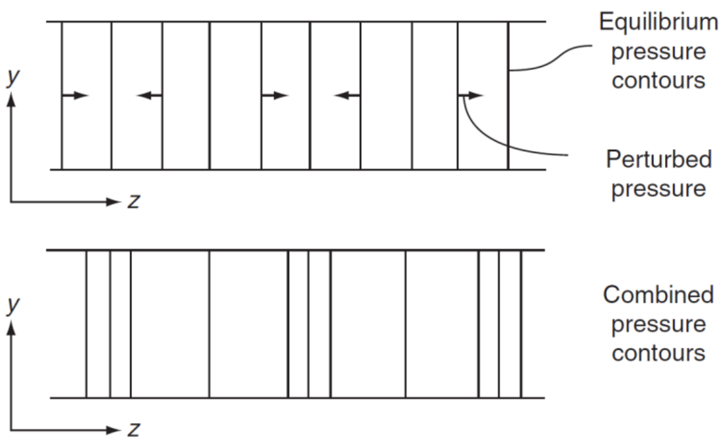


Alfvén waves

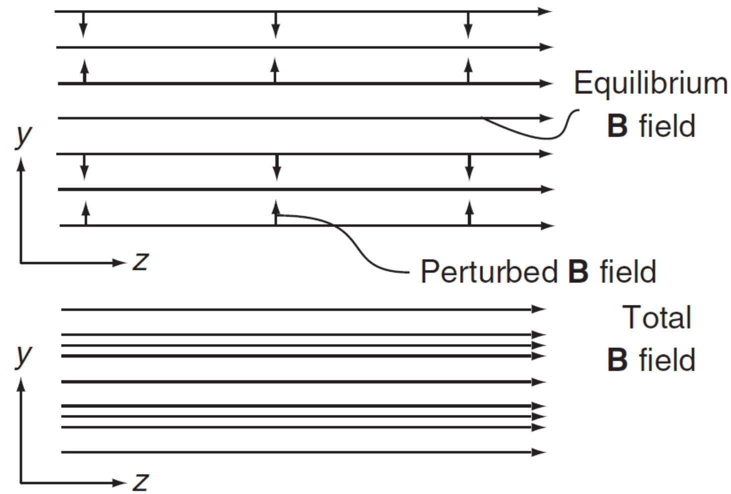
- **Shear Alfvén wave:**



- **The slow magnetosonic wave (Shear + Compressional Alfvén wave):**



- **The fast magnetosonic wave (Compressional Alfvén wave):**



Classification of MHD instabilities

- **Locations:**
  - **Internal/Fixed boundary modes: mode structure does not require any motion of the plasma-vacuum interface away from its equilibrium position.**
  - **External/Free-boundary modes: the plasma-vacuum interface moves from its equilibrium position during an unstable MHD perturbation.**
- **Dominant destabilizing term**
  - **Pressure-driven modes: the dominant destabilizing term is the one proportional to ∇p.**
  - **Current-driven modes: the dominant destabilizing term is the one proportional to $J_{||}$.**

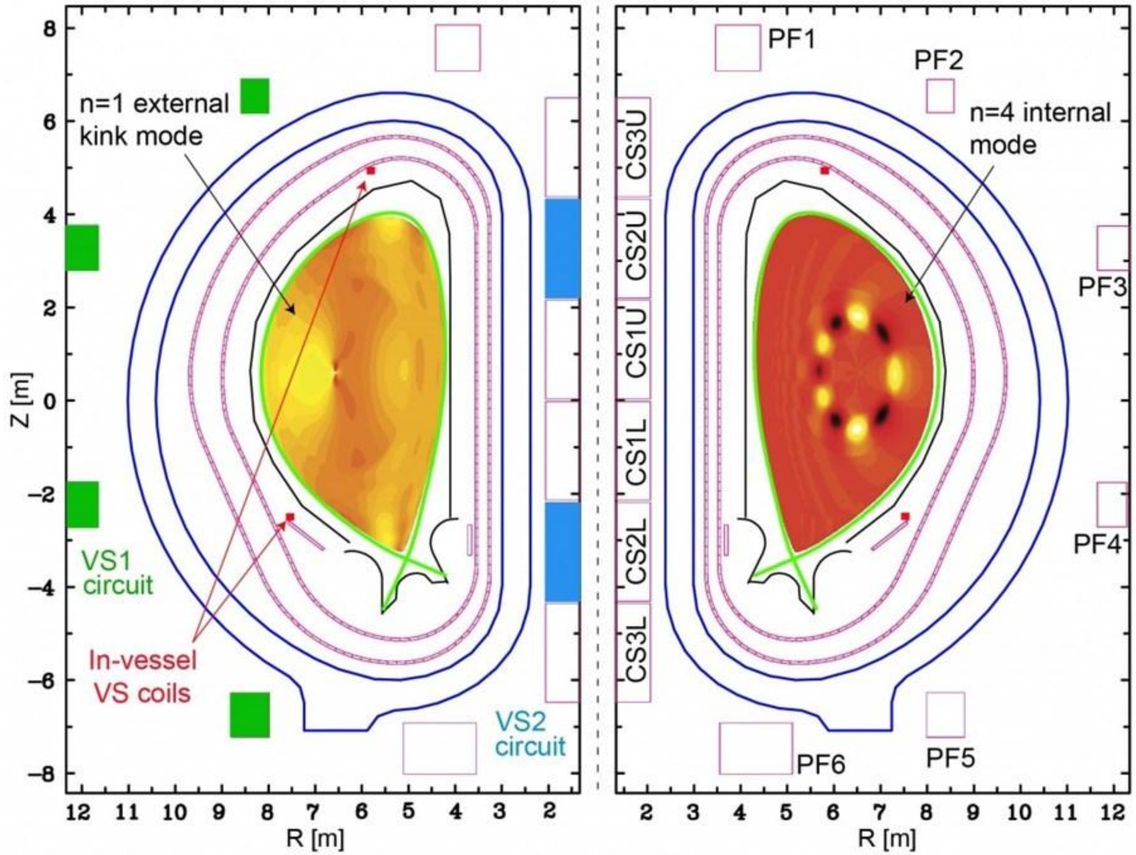
$$\delta W_F = \frac{1}{2}\int d\vec{r}\left[\frac{|\vec{Q}_\perp|^2}{\mu_o} + \frac{B^2}{\mu_o}\left|\nabla\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\vec{\xi}\right|^2 \right.$$
$$\left. -2(\vec{\xi}_\perp\cdot\nabla p)\left(\vec{\kappa}\cdot\vec{\xi}_\perp^{\,*}\right) - J_{||}\left(\vec{\xi}_\perp^{\,*}\times\vec{b}\right)\cdot\vec{Q}_\perp\right]$$

Classification of MHD instabilities

- **Locations:**
  - **Internal/Fixed boundary modes: mode structure does not require any motion of the plasma-vacuum interface away from its equilibrium position.**
  - **External/Free-boundary modes: the plasma-vacuum interface moves from its equilibrium position during an unstable MHD perturbation.**
- **Dominant destabilizing term**
  - **Current-driven modes, e.g., kink instability, sausage instability.**
  - **Pressure-driven modes, e.g., interchange mode, ballooning mode.**

External mode vs internal mode

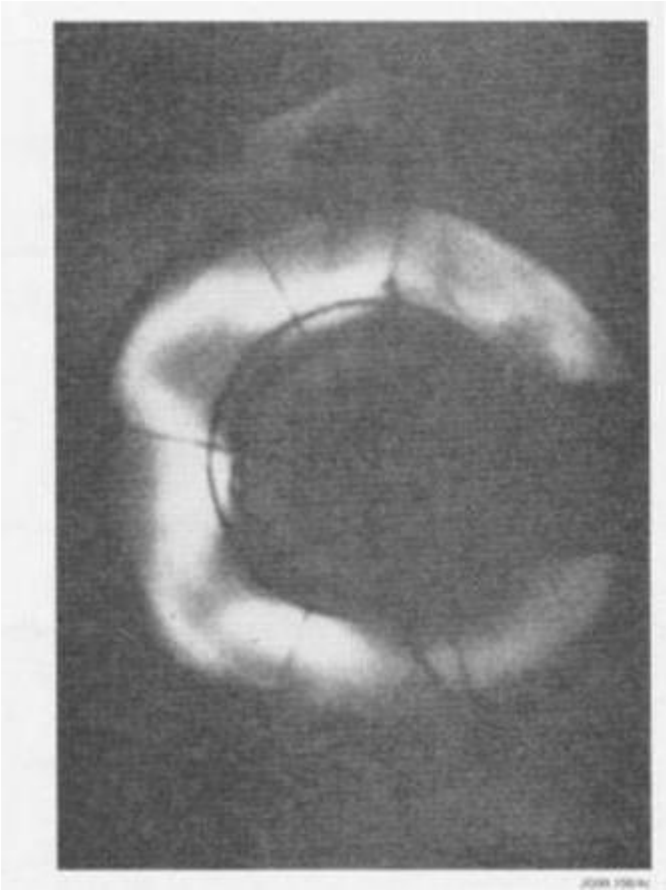
- **Predicted behaviors of the plasma in ITER**



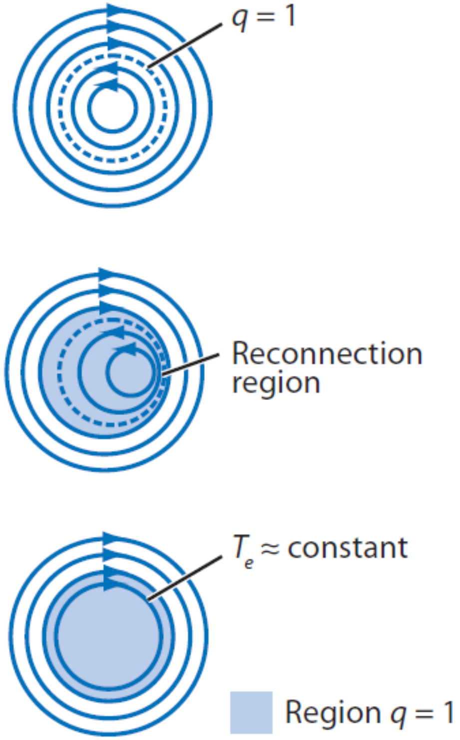
https://www.iter.org/newsline/-/3401

External vs internal kink instability

- **External kink instability**



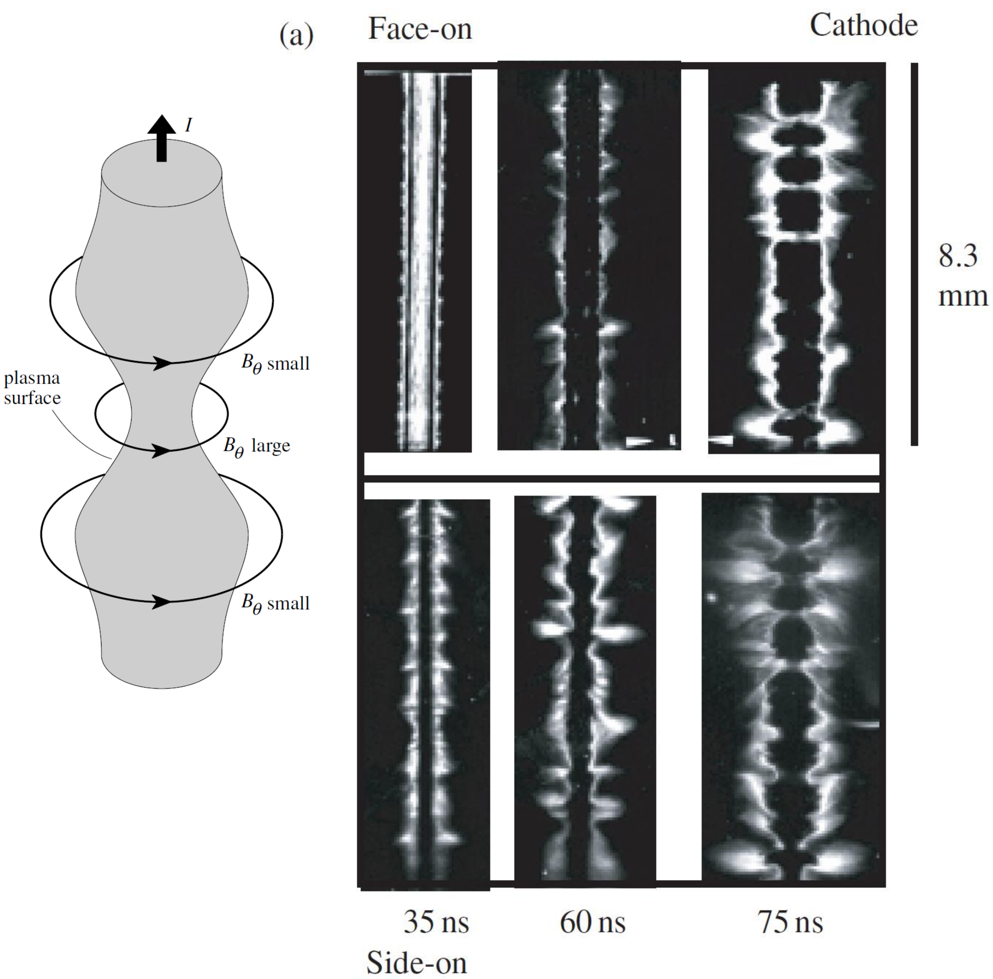
- **Internal kink instability**



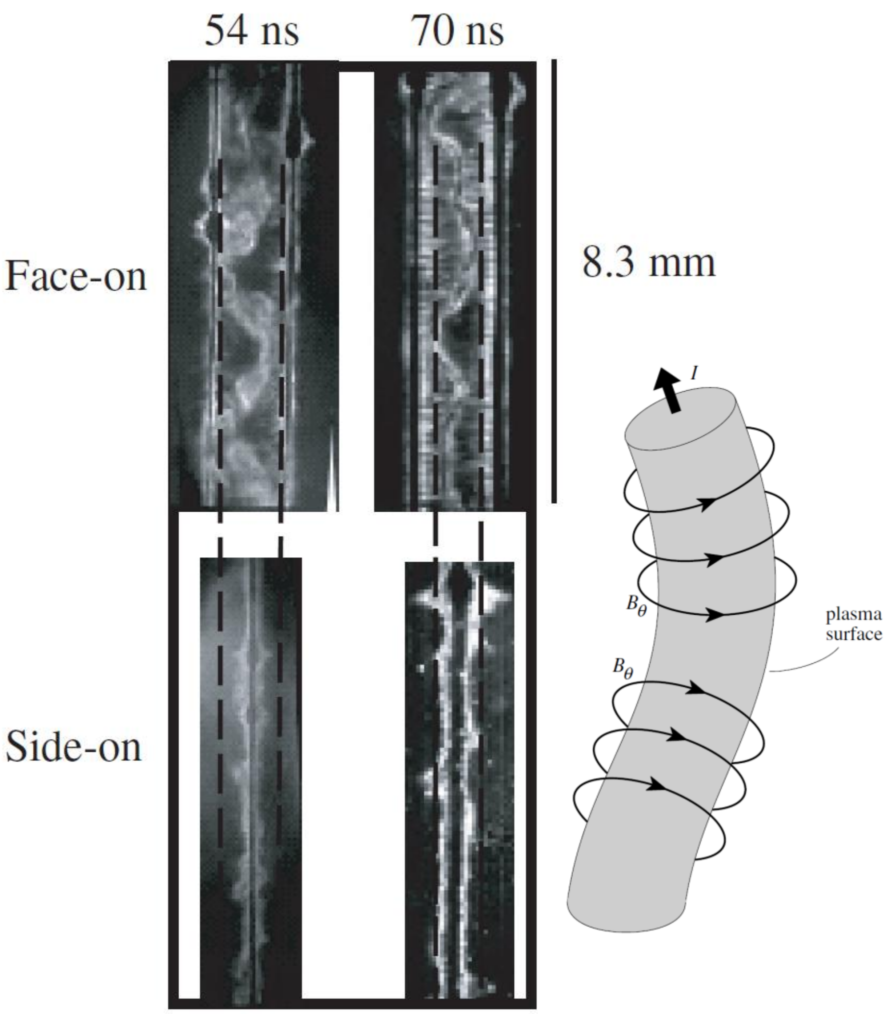
https://en.wikipedia.org/wiki/Kink_instability
R A Bingham et al 2026 Plasma Phys. Control. Fusion 68 030201
E. G. Zweibel and M. Yamada, Annu. Rev. Astron. Astrophys., 47, 291 (2009)

Current-driven instability

- **Sausage instability (m=0)**

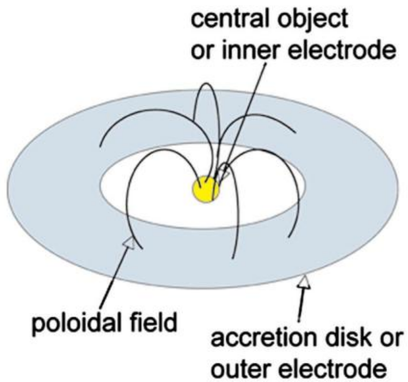


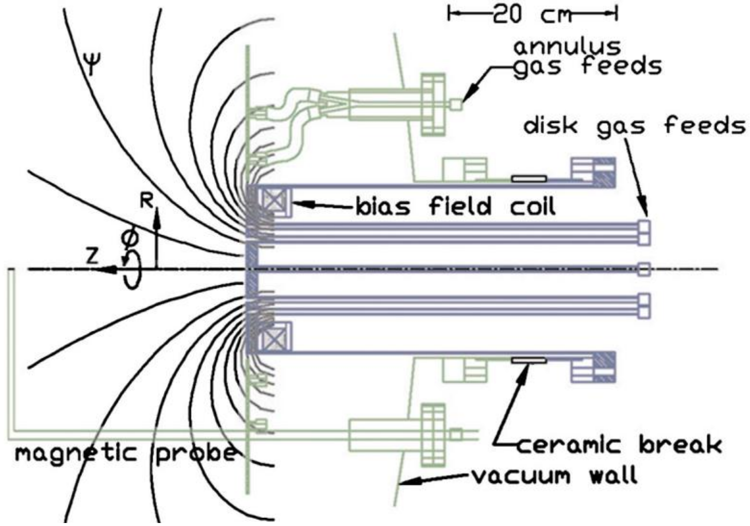
- **Kink instability (m=1)**

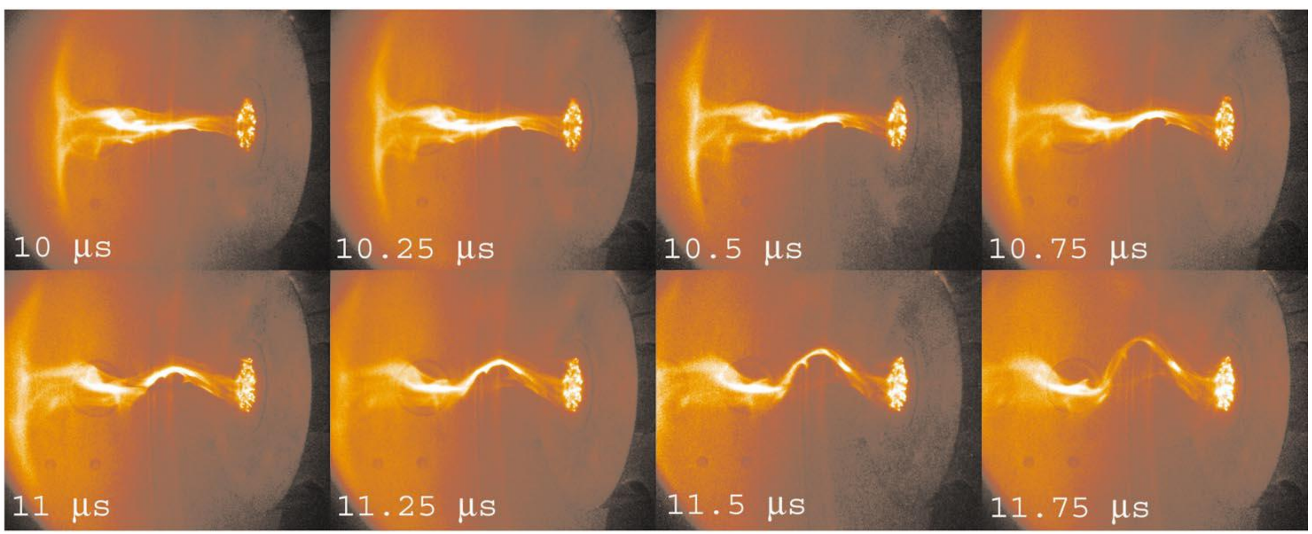


F. N. Beg, et. Al., Plasma Phys. Control. Fusion **46**, 1 (2004)

Kink instabilities in the lab

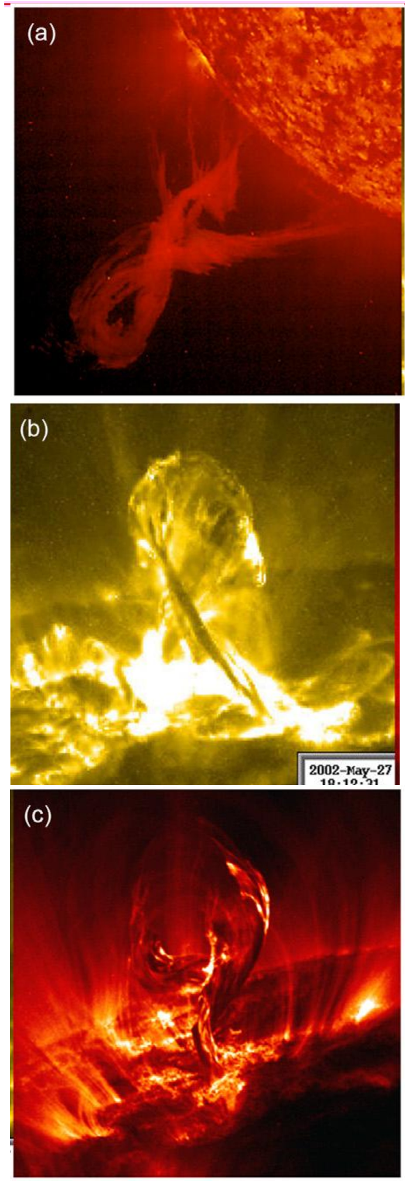


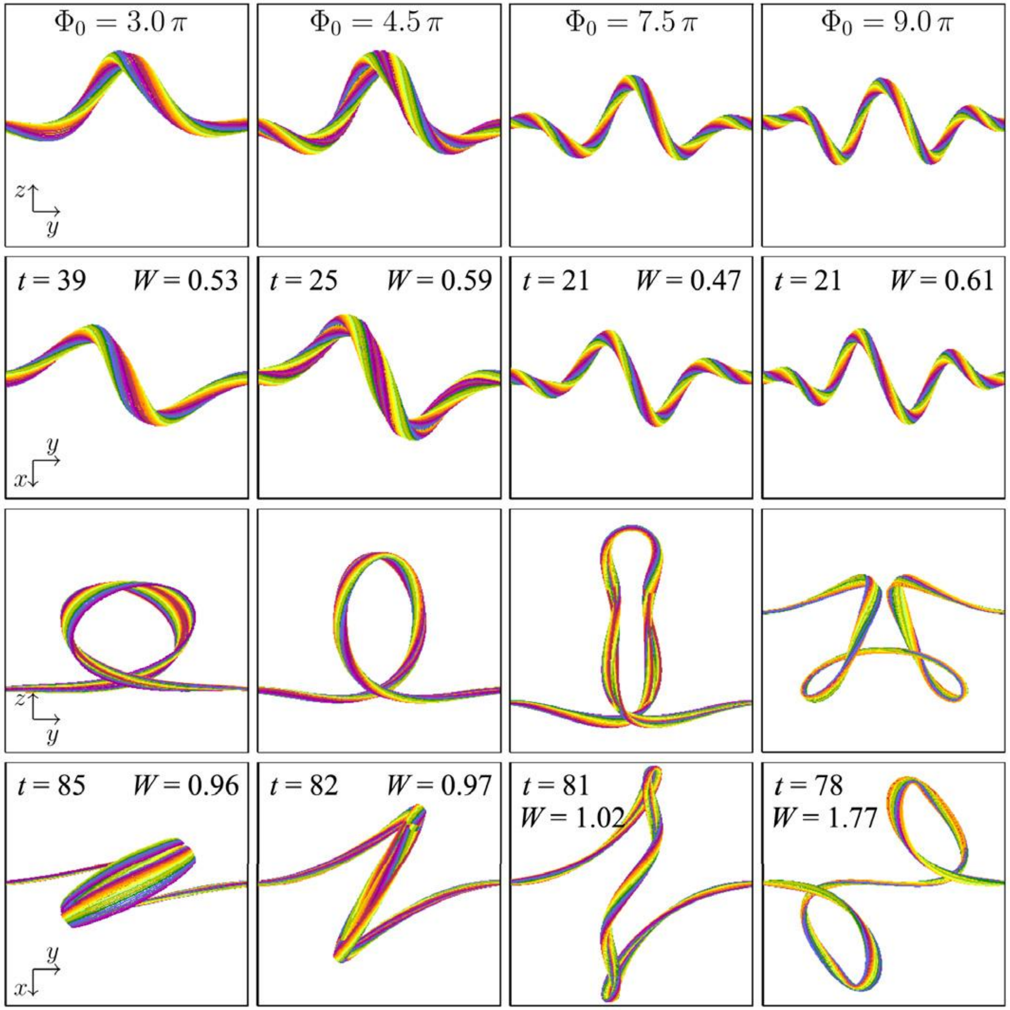




S. C. Hsu, P. M. Bellan, Phys. Plasma **12**, 032103 (2005)

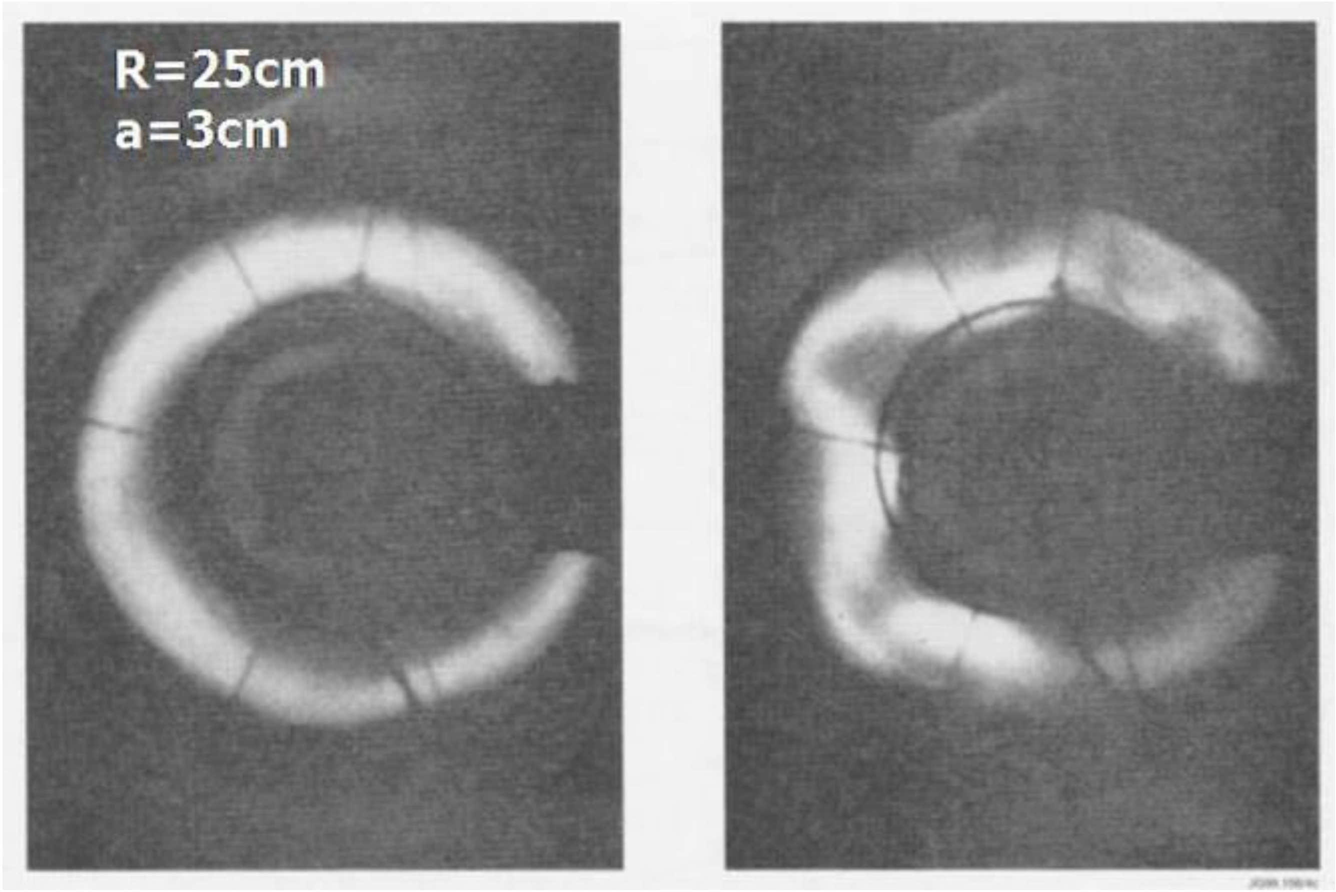
Kink instabilities in space





T Török *et al*, Plasma Phys. Control. Fusion **56**, 064012 (2014)

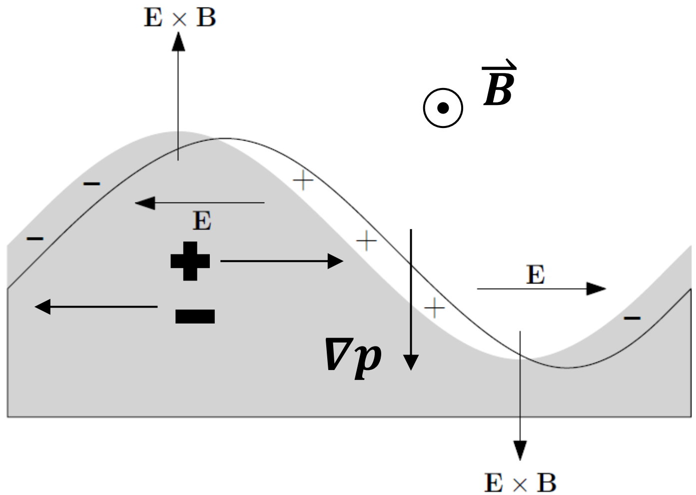
Kink instability in Tokamak

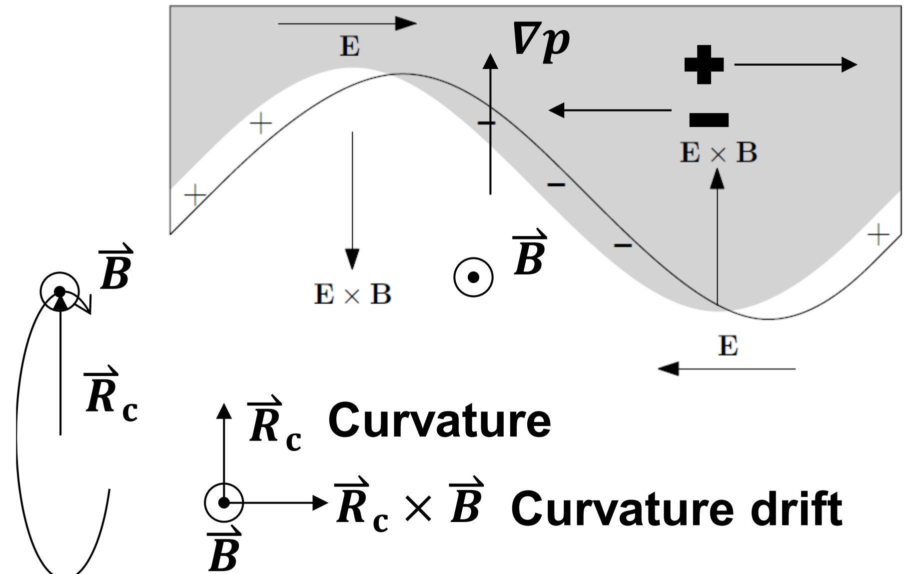


https://en.wikipedia.org/wiki/Kink_instability

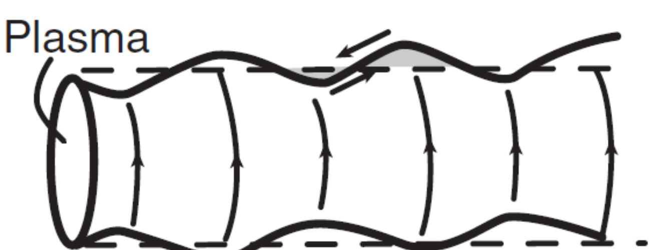
Pressure driven instability – interchange perturbations

- **Unstable: bad curvature** $\vec{R}_c \cdot \nabla p < 0$
- **stable: good curvature** $\vec{R}_c \cdot \nabla p > 0$





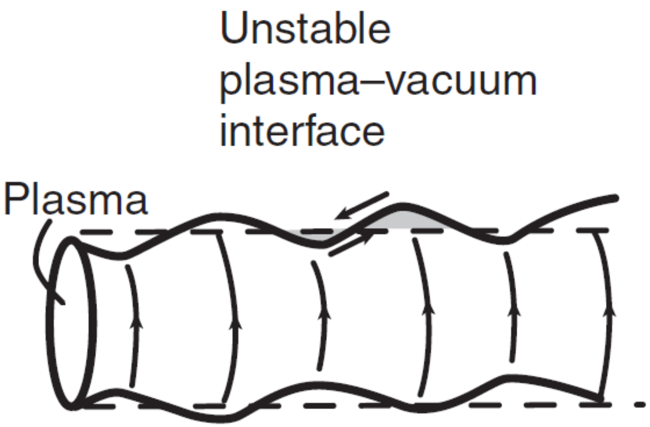
Unstable
plasma–vacuum
interface



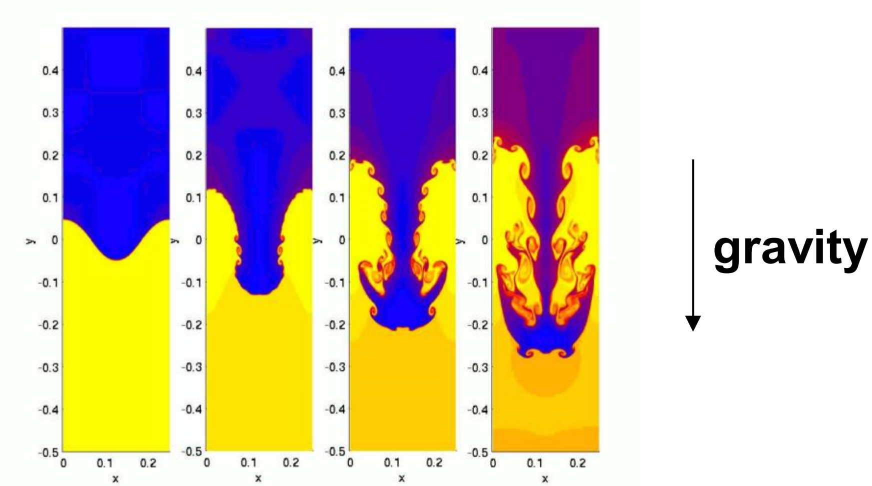
Stable
plasma–vacuum
interface

Ben Dudson, Lecture note of Magnetic Confinement Fusion 2014

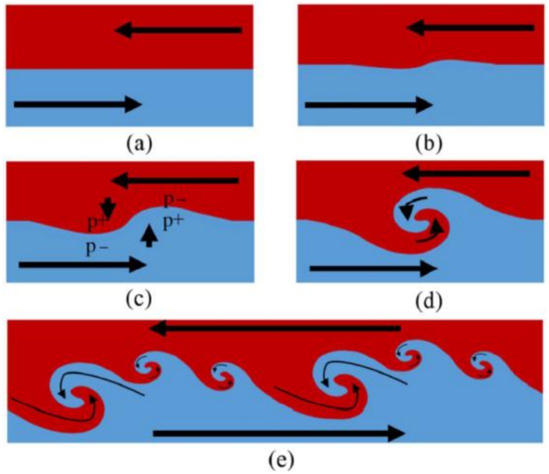
Rayleigh-Taylor instability



- **Rayleigh-Taylor instability**

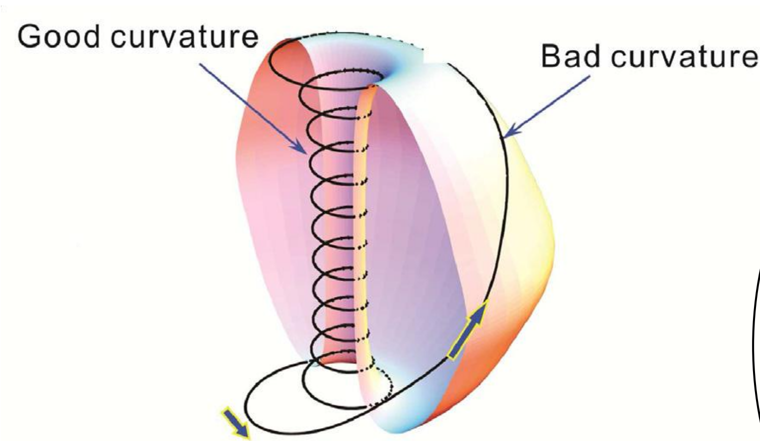


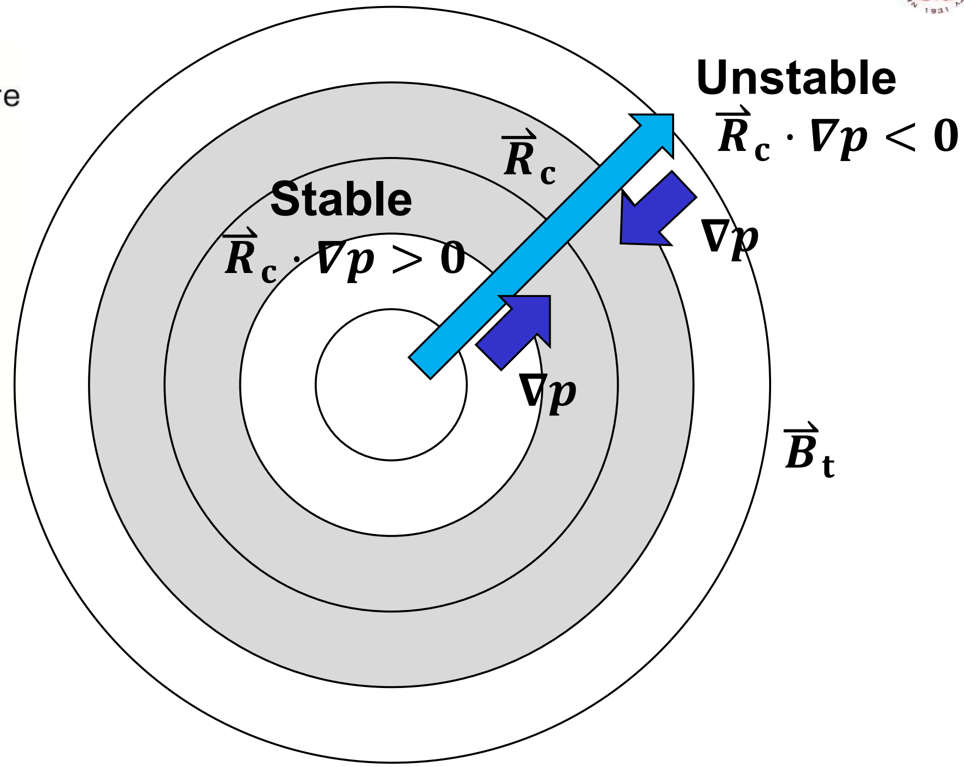
- **Kelvin-Helmholtz instability**



https://en.wikipedia.org/wiki/Rayleigh%E2%80%93Taylor_instability
https://en.wikipedia.org/wiki/Kelvin%E2%80%93Helmholtz_instability
Xie Lei et al, Energy Report **7**, 2262 (2021)

Pressure driven instability – interchange perturbations

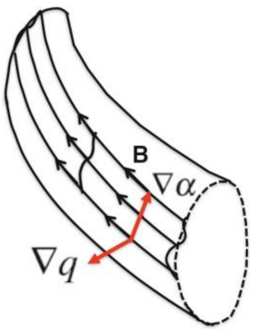


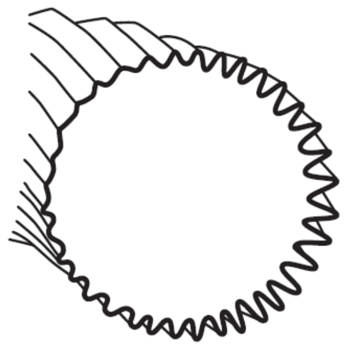


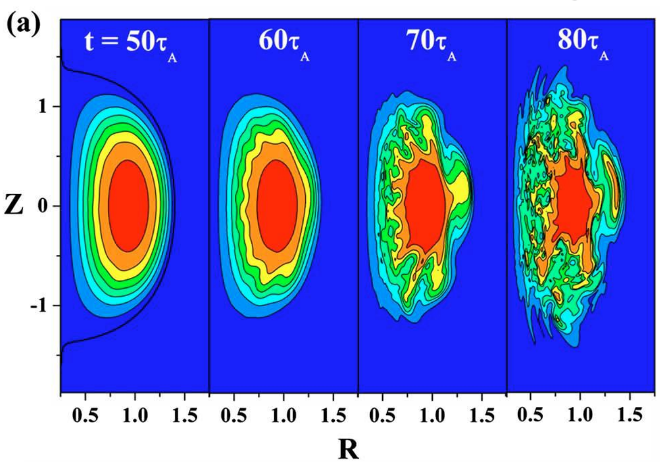
- **Suydam criterion for cylindrical plasmas:** $$\mu_o \frac{2r^2}{{B_\theta}^2}\frac{1}{s^2}\vec{R}_c \cdot \nabla p > -\frac{1}{4}$$ **Shear:** $$s = \frac{r}{q}\frac{dq}{dr}$$
- **Mercier criterion for tokamak:** $$D = -\mu_o \frac{2r}{B^2}\frac{1}{s^2}\frac{dp}{dr}\left(1 - q^2\right) < \frac{1}{4}$$

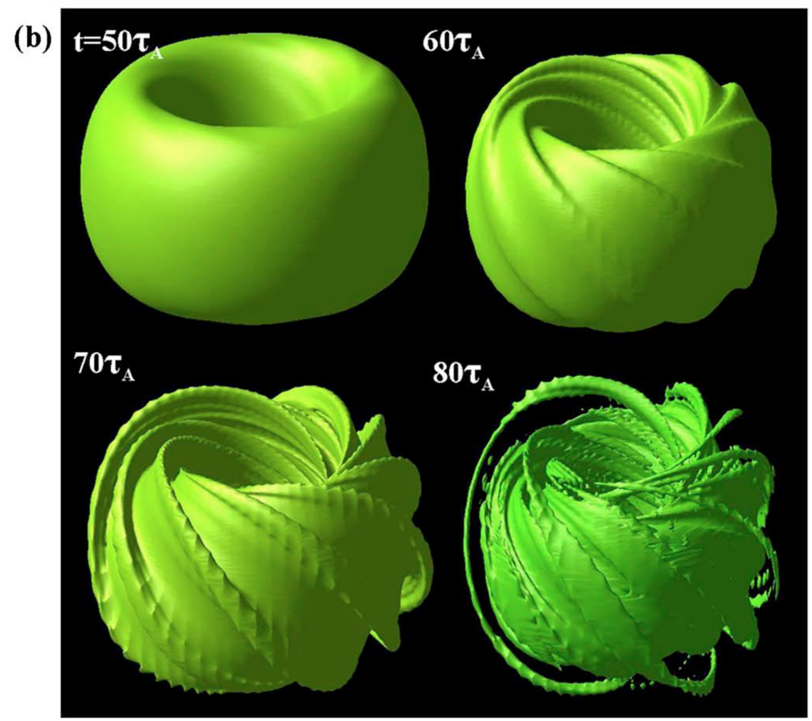
Ballooning mode – show wavelength mode

$$\delta W_F = \frac{1}{2}\int d\,\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_o} + \frac{B^2}{\mu_o}\left|\nabla\cdot\vec{\xi}_\perp + 2\,\vec{\xi}_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\vec{\xi}\right|^2 \right.$$
$$\left. -2\left(\vec{\xi}_\perp\cdot\nabla p\right)\left(\vec{\kappa}\cdot\vec{\xi}_\perp^{\,*}\right) - J_{||}\left(\vec{\xi}_\perp^{\,*}\times\vec{b}\right)\cdot\vec{Q}_\perp\right]$$



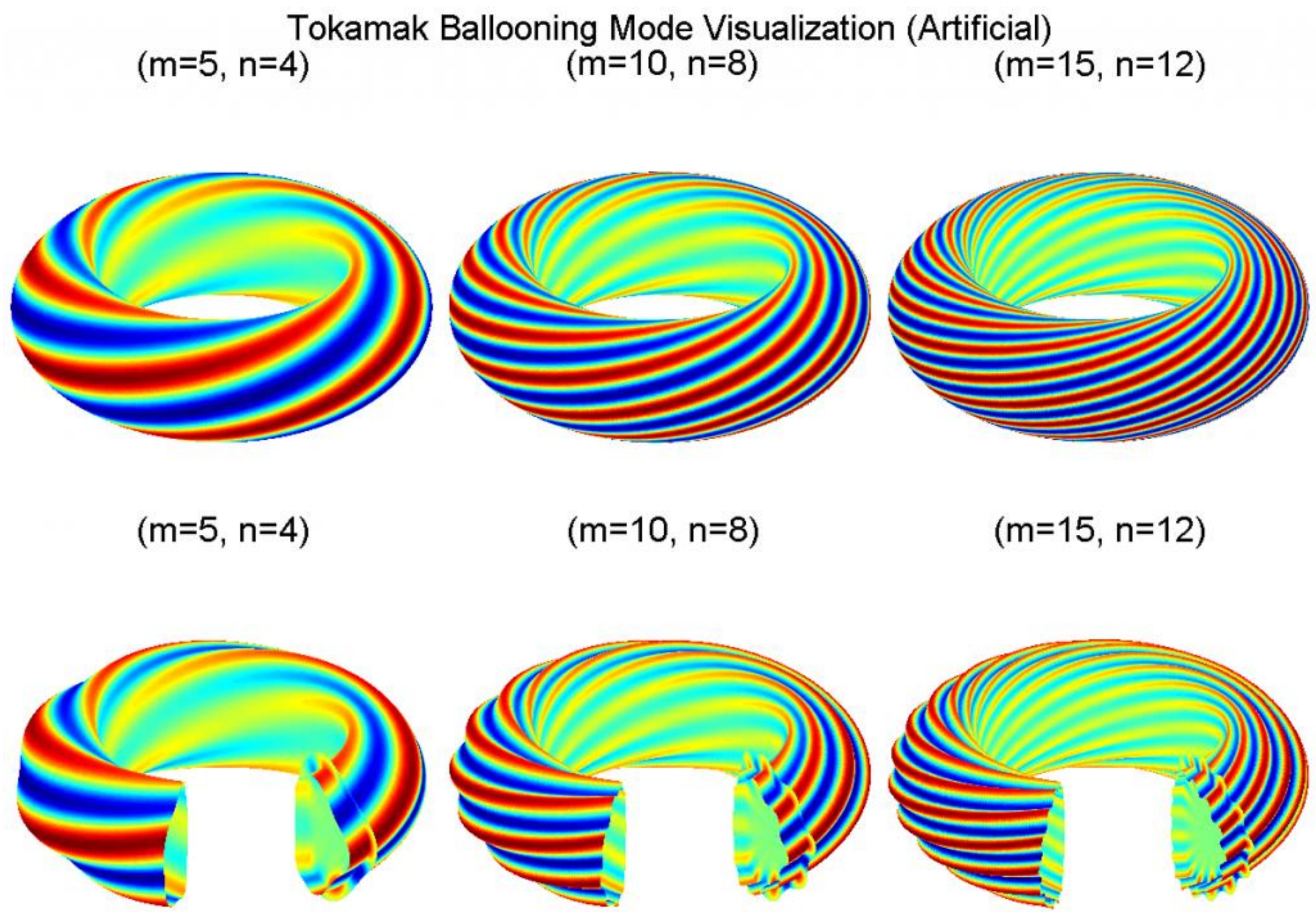






R. Khan et al, Phys. Plasmas **14**, 062302 (2007)

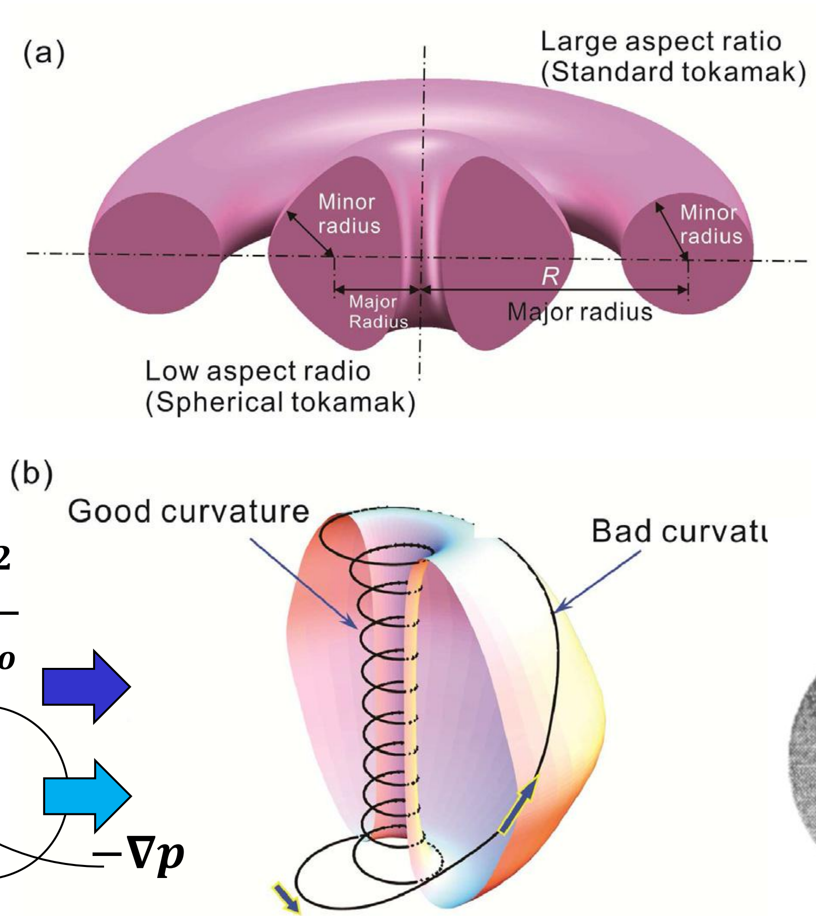
Ballooning mode – show wavelength mode

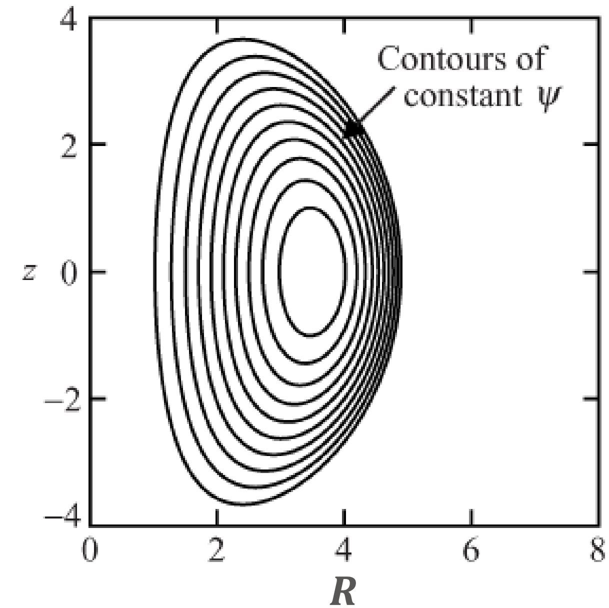


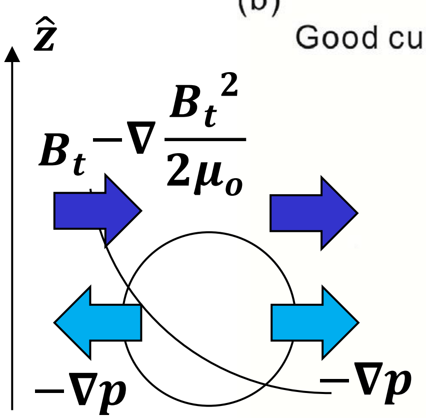
https://blog.sciencenet.cn/home.php?mod=space&uid=222053&do=album&picid=308476#pic_block

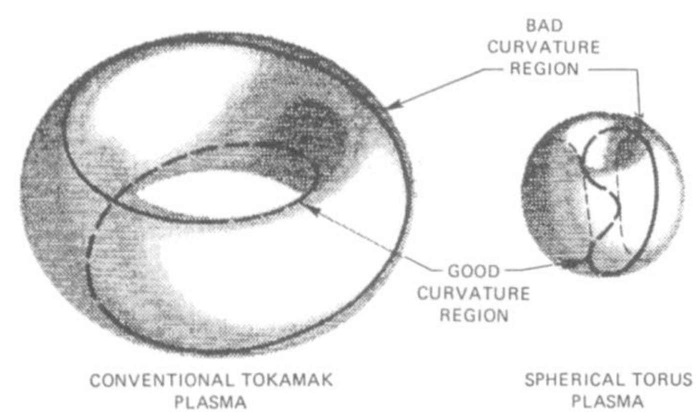
The Spherical tokamak

- **Aspect ratio $R_o/a \sim 1.6$**







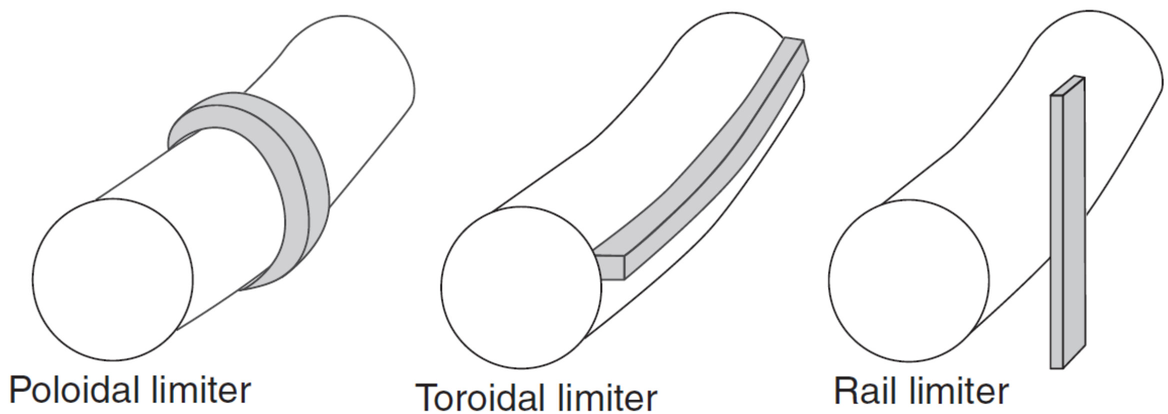


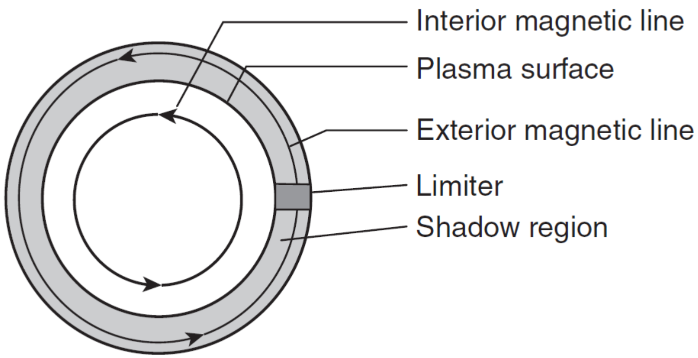
Zhe Gao, Matter and Radia. Extrem., **1**, 153 (2016)
Y-k. M.Peng and D. J. Strickler, Nucl. Fusion **26**, 769 (1986)

The Spherical tokamak

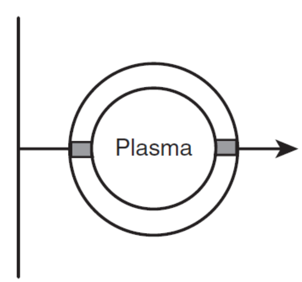
- **Aspect ratio $R_o/a \sim 1.6$**
- **Advantages:**
  - **Higher β_t limit.**
  - **A compact design almost spherical in appearance.**
- **Challenges:**
  - **Minimum space is given in the center of the torus to accommodate the toroidal field coils.**
  - **With a very compact design the technology associated with the construction and maintenance of the device may be more difficult than for a "normal" tokamak.**
  - **Large currents will have to be driven noninductively, a costly and physically difficult requirement.**

Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma

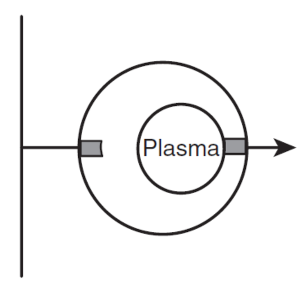




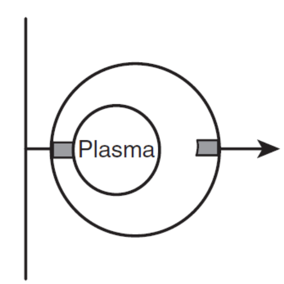
- **Vertical field is correct.**



- **Vertical field is too small.**

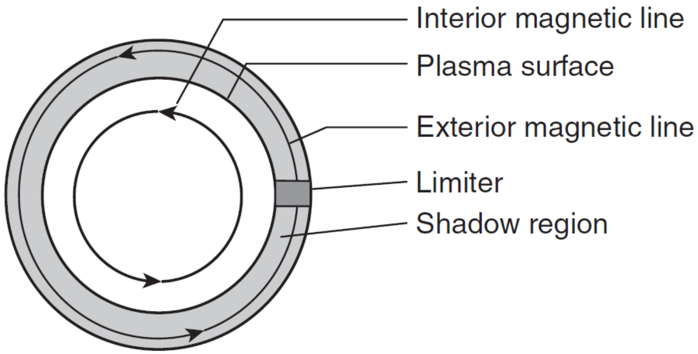


- **Vertical field is too large.**

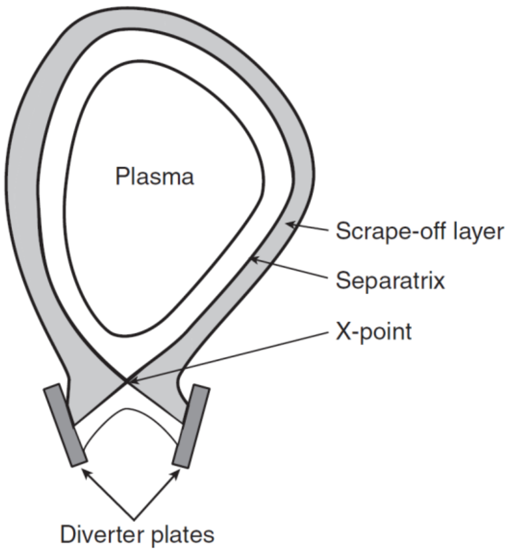


Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma

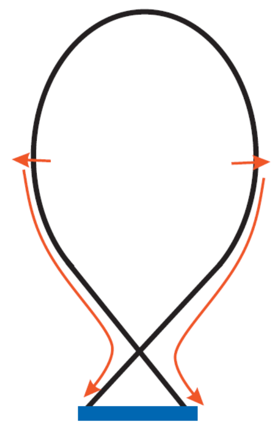
- **A mechanical limiter is a robust piece of material, often made of tungsten, molybdenum, or graphite placed inside the vacuum chamber.**
- **Some of the particles of the limiter surface may escape. Neutral particles can penetrate some distance into the plasma before being ionized.**
- **The high-z impurities can lead to significant additional energy loss in the plasma through radiation.**
- **In ignition experiments and fusion reactors, the bombardment is more intense and extends over longer periods of time. In addition, if the impurity level is too high, it may not be possible to achieve a high enough temperature to ignite.**



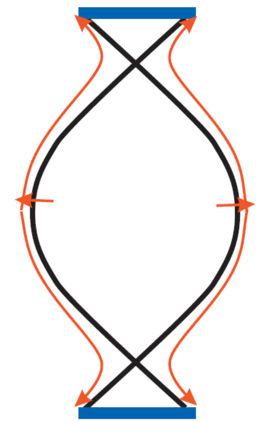
The magnetic divertor – guide a narrower layer of magnetic lines away from the edge of the plasma



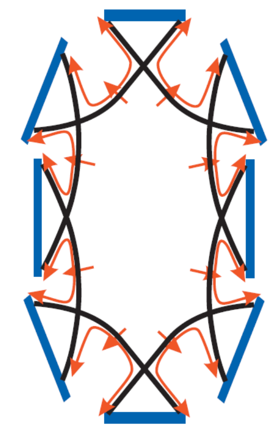
- **Single-null poloidal-field divertors for tokamak**



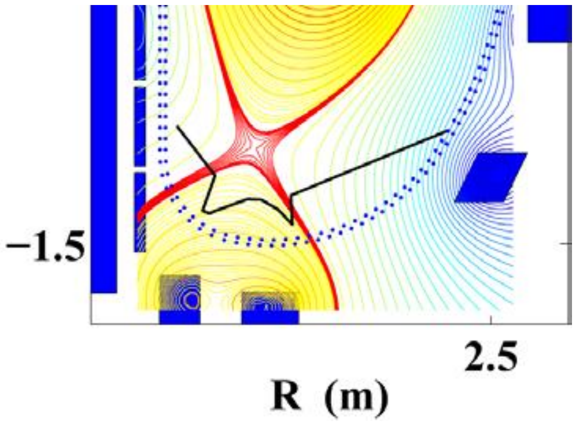
- **Double-null poloidal-field divertors for tokamak**



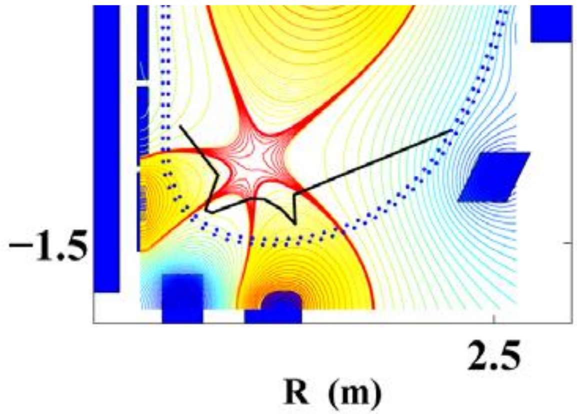
- **Island divertor for stellerators**



- **Standard**



- **Snowflake**



Y. Feng, Nucl. Fusion, **46**, 807 (2006)
L Xue *et al*, Plasma Phys. Control. Fusion **58**, 055005 (2016)

Pros and cons of a divertor

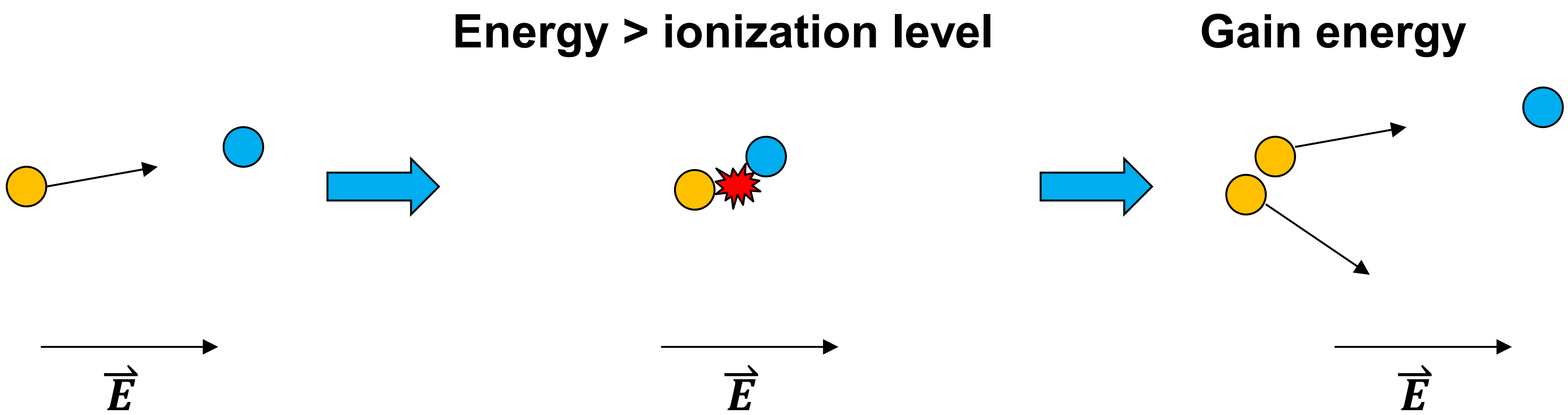
- **Advantages:**
  - The collector plate is remote from the plasma. There is space available to spread out the magnetic lines.
  - A lower intensity of particles and energy bombard the collector plate leading to a longer replacement time.
  - It is more difficult for impurities to migrate into the plasma.
  - There are longer distance distances to travel and if a neutral particle becomes ionized before or during the time it crosses the divertor layer on its way toward the plasma, its parallel motion then carries it back to the collector plate.
  - The larger divertor chamber provides more access to pump out impurities.
  - The plasma edge is not in direct contact with a solid material such as a limiter.
- **Disadvantages: larger and more complex system and more expensive.**

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Collisions play an important role in ionization process

- **At the microscopic level, breakdown requires the presence of sufficiently energy charge particles that have acquired enough energy from the applied electric field between two energy-dissipating collisions to ionize the material and to create more charge particles.**



In most cases, <u>electrons</u> dominate the breakdown process since its mobility is much larger than that of ions

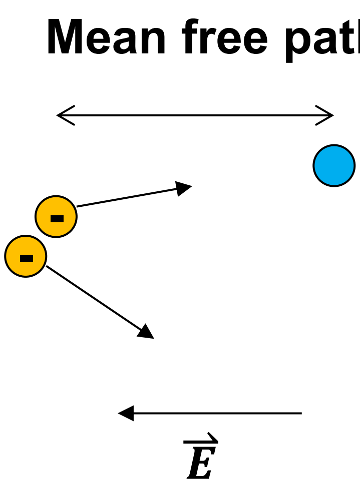
$$E_{\mathrm{k}} = \frac{1}{2}mv^2 \qquad v = \sqrt{\frac{2E_{\mathrm{k}}}{m}} \qquad E_{\mathrm{k}} \sim kT$$

Collision time: $t = \dfrac{s}{\sqrt{\dfrac{2E_{\mathrm{k}}}{m}}} \sim \dfrac{n^{-1/3}}{\sqrt{T}}\sqrt{m}$ $\qquad n = \dfrac{\#/}{V} \sim \dfrac{\#//}{S^3}$ $\qquad s \sim n^{-1/3}$

$$\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}} \sim 2000 \times \text{Atomic mass} \qquad \frac{t_{\mathrm{i}}}{t_{\mathrm{e}}} \sim 45 \times \sqrt{A}$$

Mean free path is important in ionization process

- **For an electron to acquire enough energy between collisions, its <u>mean free path</u> in the material must be sufficiently long.**



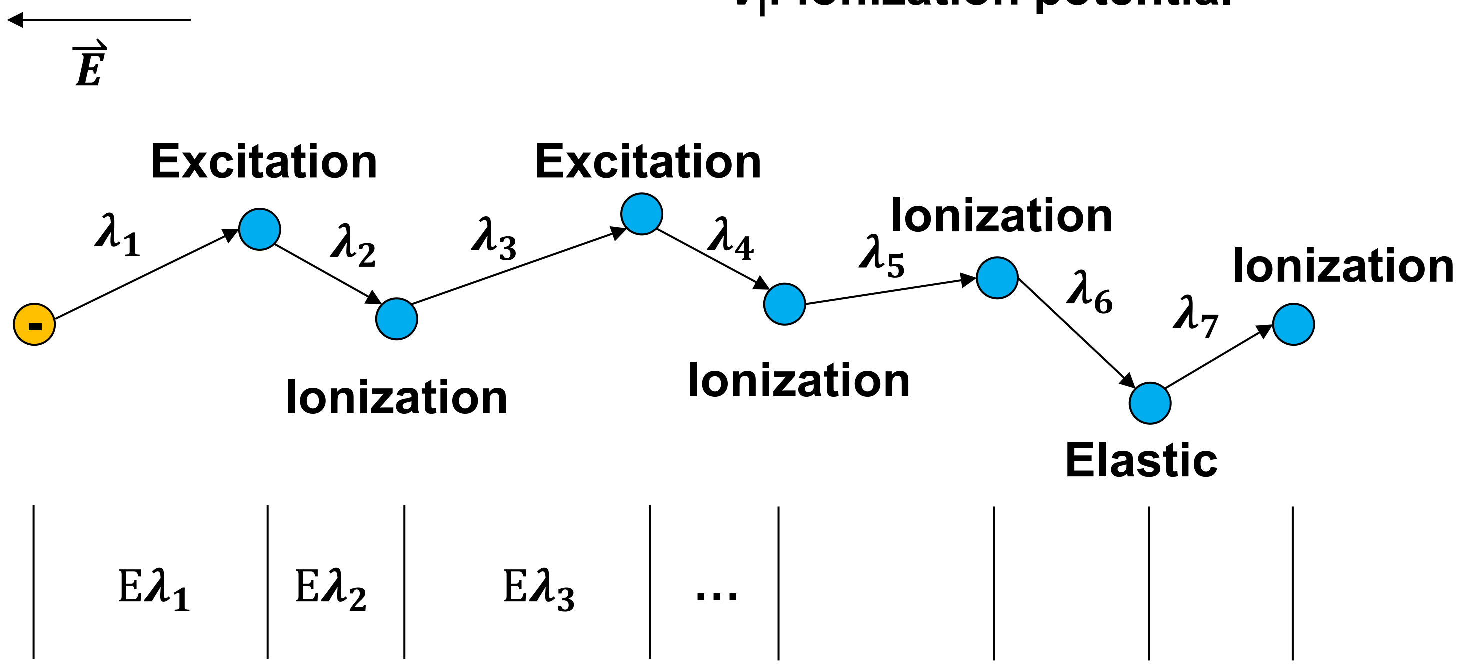
$$E_{\mathbf{k}} = e \times E \times \lambda = \mathrm{eV}$$

Electron impact ionization is the most important process in a breakdown of gases

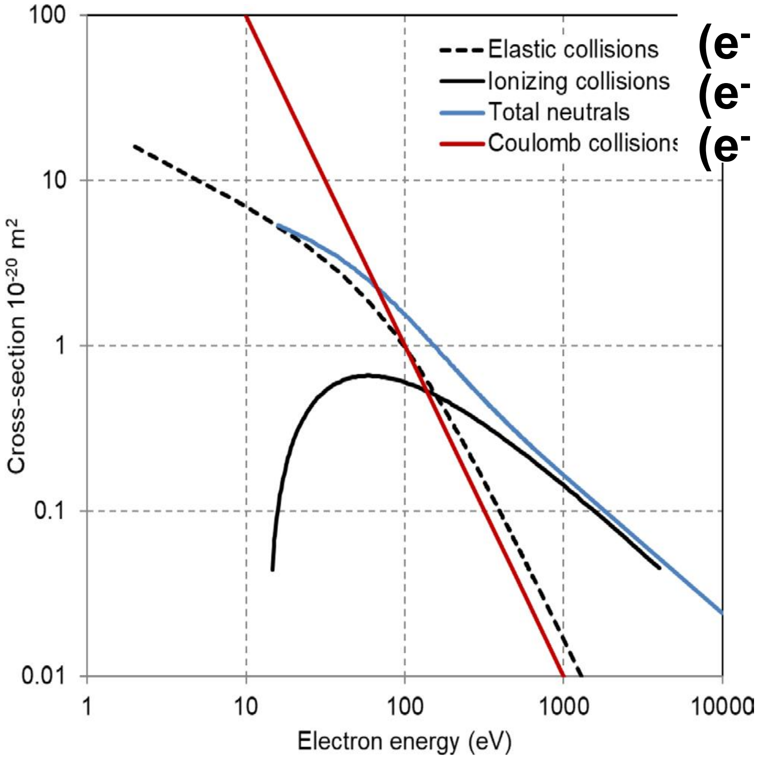
- **Electron impact ionization:** $A + e^- \rightarrow A^+ + e^- + e^-$
  - The most important process in the breakdown of gases but is not sufficient alone to result in the breakdown.

$$eE\lambda_{e,i} \geq eV_i$$

V_i: ionization potential



Collision cross-sections of elastic, ionizing collisions between e$^-$ and H_2 and coulomb collisions



(e$^-$ vs H_2)

(e$^-$ + $H_2 \rightarrow 2e^- + H^+ + H$)

(e$^-$ vs H^+)

$$\sigma_{\mathrm{elastic,m^2}} = \frac{1.75 \times 10^{-16}}{\left(W_{\mathrm{e,eV}}{}^{1.5} + 750\right)\sqrt{W_{\mathrm{e,eV}}}}$$

$$\sigma_{\mathrm{ionizing,m^2}} \sim 3 \times 10^{-20}\left(ln\epsilon - 0.69 + \frac{0.66}{\epsilon}\right)\epsilon^{-1}$$

$$\epsilon = \frac{W_{\mathrm{e}}}{E_{\mathrm{ion}}} \quad E_{\mathrm{ion}} \sim 15 \text{ eV for } \mathrm{H_2}$$

$$\sigma_{\mathrm{coulomb}} = \frac{e^4}{4\pi\epsilon_{\mathrm{o}}{}^2 m_{\mathrm{e}}{}^2 v_{\mathrm{e}}{}^2}\ln\Lambda$$

$$= \frac{e^4}{16\pi\epsilon_{\mathrm{o}}{}^2 W_{\mathrm{e}}{}^2}\ln\Lambda \sim 10^{-16} W_{\mathrm{e}}{}^{-2}$$

$$\text{for } \ln\Lambda \sim 13 - 15$$

$$\nu = n v_E \sigma$$

$$\lambda = \frac{1}{n\sigma}$$

P. C. de Vries and Y. Gribov, Nucl. Fusion, **59**, 096043 (2019)

Townsend avalanche process for Tokamak breakdown

- **The first Townsend coefficient α: the number of ionizing collisions made on the average by an electron as it travels 1 m along the electric field:**

$$\alpha \sim \frac{1}{\lambda_\mathrm{i}} = \frac{\nu_\mathrm{ei}}{\bar{v}_\mathrm{e}} = \frac{n_0\langle\sigma v_\mathrm{e}\rangle_\mathrm{ne}}{\bar{v}_\mathrm{e}} = \frac{p}{T}\frac{\langle\sigma \mathrm{v}\rangle_\mathrm{ne}}{\bar{v}_\mathrm{e}} \equiv Ap \qquad A \equiv \frac{1}{T}\frac{\langle\sigma \mathrm{v}\rangle_\mathrm{ne}}{\bar{v}_\mathrm{e}}$$

- **Number of primary electrons with energy higher than the ionization potential:**

$$dn_\mathrm{e} = -n_\mathrm{e}\frac{\mathrm{d}x_\mathrm{i}}{\lambda_\mathrm{i}} \quad \Rightarrow \quad \frac{n_\mathrm{e}(x_\mathrm{i})}{n_\mathrm{e0}} = \exp\left(-\frac{x_\mathrm{i}}{\lambda_\mathrm{i}}\right)$$

$$\alpha \equiv \frac{\#/\text{ ionization collisions}}{\text{per electron}} \times (\#/\text{electron with E} > \text{ionization potential})$$

$$= \frac{1}{\lambda_\mathrm{i}}\frac{n_\mathrm{e}(x_\mathrm{i})}{n_\mathrm{e0}} = \frac{1}{\lambda_\mathrm{i}}\exp\left(-\frac{x_\mathrm{i}}{\lambda_\mathrm{i}}\right)$$

$$\alpha = Ap\exp(-Apx_\mathrm{i})$$

A = 3.83 m^{-1}Pa^{-1} = 1060 m^{-1}Torr^{-1}

B = 96.6 Vm^{-1}Pa^{-1} = 35000 m^{-1}Torr^{-1}

$$\alpha = Ap\exp\left(-\frac{AV^*}{E/p}\right) \equiv Ap\exp\left(-\frac{B}{E/p}\right) \qquad x_\mathrm{i} \approx \frac{V^*}{E} \text{ where } V^* > V_\mathrm{i}$$

- **The parameters *A* and *B* must be experimentally determined.**

Paschen's curve for minimum breakdown voltage

$$\alpha \sim \frac{1}{\lambda_\mathrm{i}} = Ap \qquad \alpha = Ap\exp\left(-\frac{B}{E/p}\right)$$

***A* = 3.83 m^{-1}Pa^{-1} = 1060 m^{-1}Torr^{-1}**

***B* = 96.6 Vm^{-1}Pa^{-1} = 35000 m^{-1}Torr^{-1}**

- **For *p*=1 mPa,** $\lambda_\mathrm{i} \sim 262\ \mathrm{m}$ **, for ITER,** $2\pi r_\mathrm{o} \sim 38\ \mathrm{m}$ $\quad \dfrac{\lambda_\mathrm{i}}{2\pi r_\mathrm{o}} \sim 7$

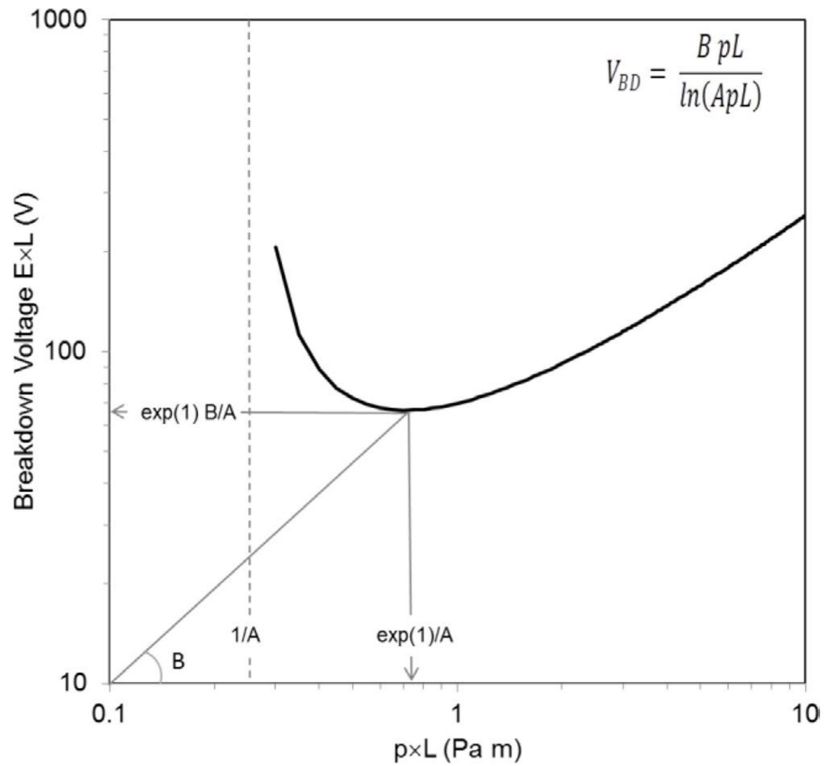
- **For breakdown to happen:**

$$\alpha L > 1 \qquad \alpha L = ApL\exp\left(-\frac{BpL}{V_\mathrm{BD}}\right) > 1$$

$$\exp\left(-\frac{BpL}{V_{BD}}\right) > \frac{1}{ApL}$$

$$-\frac{BpL}{V_\mathrm{BD}} > -\ln(ApL)$$

$$V_\mathrm{BD} > \frac{BpL}{\ln(ApL)} \qquad E_\mathrm{BD} > \frac{Bp}{\ln(ApL)}$$



P. C. de Vries and Y. Gribov, Nucl. Fusion, **59**, 096043 (2019)

Perpendicular stray-field (B_z) needs to be as small as possible

- For p=1 mPa, $\lambda_{\mathrm{i}} \sim 262\ \mathrm{m}$, for ITER, $2\pi r_{\mathrm{o}} \sim 38\ \mathrm{m}$ $\quad \dfrac{\lambda_{\mathrm{i}}}{2\pi r_{\mathrm{o}}} \sim 7$

$$\frac{B_z}{B_T} \sim 10^{-3} \qquad \lambda_i \times \frac{B_z}{B_T} = 0.26\ \mathrm{m}$$

- For ITER,

$$E \sim E_{\mathrm{loop}} = 0.3\ \mathrm{V/m}$$

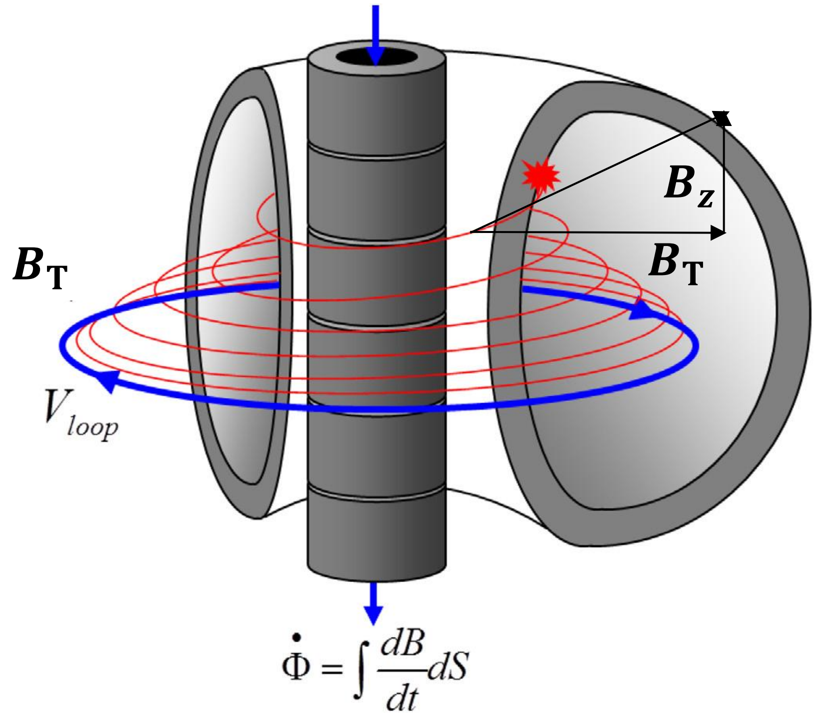
$$p = 1\ \mathrm{mPa} \qquad L_{\mathrm{BD}} = 357\ \mathrm{m}$$

- Required loop field:

$$E_{\mathrm{BD}} > \frac{Bp}{\ln(ApL)}$$

$$E_{\mathrm{BD}} > \frac{1.25 \times 10^4 P_{\mathrm{Torr}}}{\ln(510 P L_{\mathrm{c}})}$$

$$L_{\mathrm{c}} = 0.25 a_{\mathrm{eff}} \left(\frac{B_z}{B_T}\right)$$



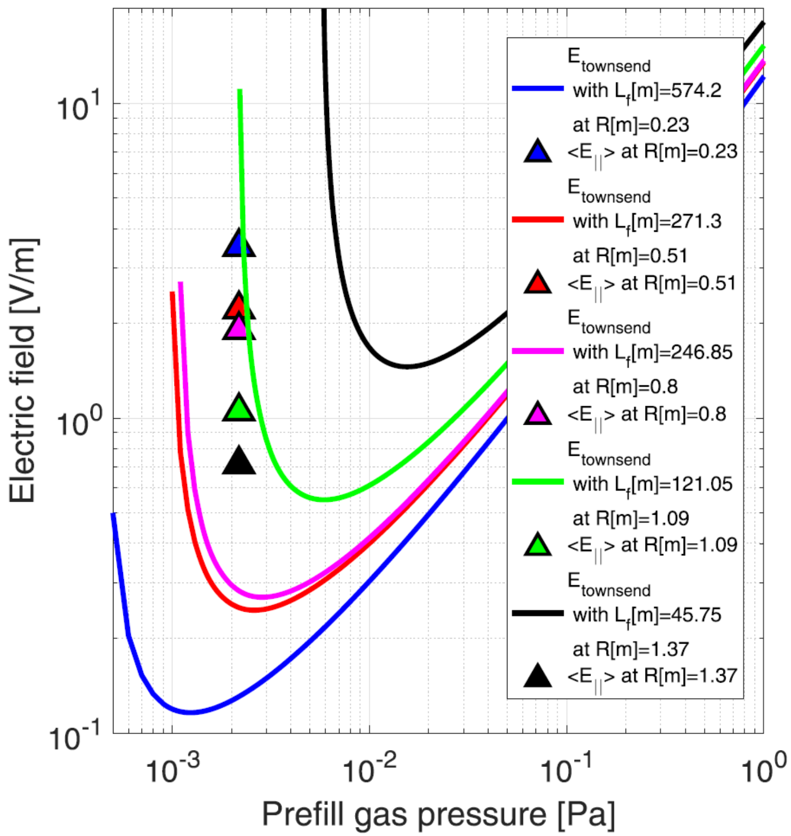
- **W/ preionization**: $E_{\mathrm{T}} \dfrac{B_{\mathrm{T}}}{B_z} \geq 100\ \mathrm{V/m}$
- **Purely Ohmic discharges**: $E_{\mathrm{T}} \dfrac{B_{\mathrm{T}}}{B_z} \geq 100\ \mathrm{V/m}$

P. C. de Vries and Y. Gribov, Nucl. Fusion, **59**, 096043 (2019)
S. J. Doyle et al, Fusion Eng. Des. **171**, 112706 (2021)

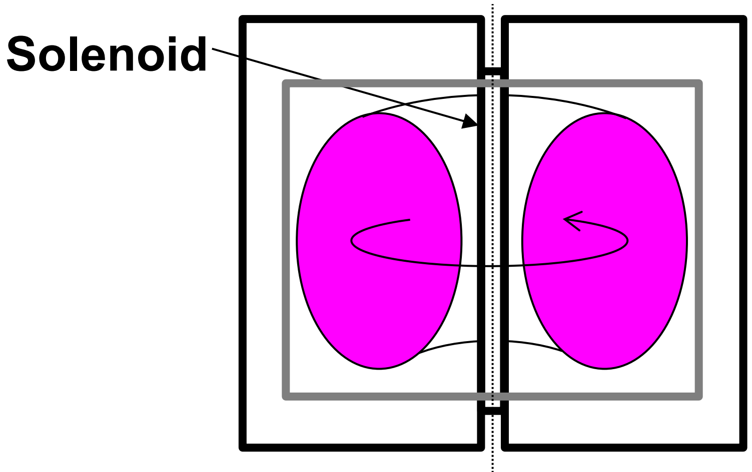
Examples or required loop electric fields

$$E_{\mathrm{BD}} > \frac{1.25 \times 10^4 P_{\mathrm{Torr}}}{\ln(510 P L_c)}$$

$$L_c = 0.25 a_{\mathrm{eff}} \left(\frac{B_z}{B_T}\right)$$



- W/ preionization: $E_{\mathrm{T}} \frac{B_{\mathrm{T}}}{B_{\mathrm{z}}} \geq 100\,\mathrm{V/m}$
- Purely Ohmic discharges: $E_{\mathrm{T}} \frac{B_{\mathrm{T}}}{B_{\mathrm{z}}} \geq 100\,\mathrm{V/m}$



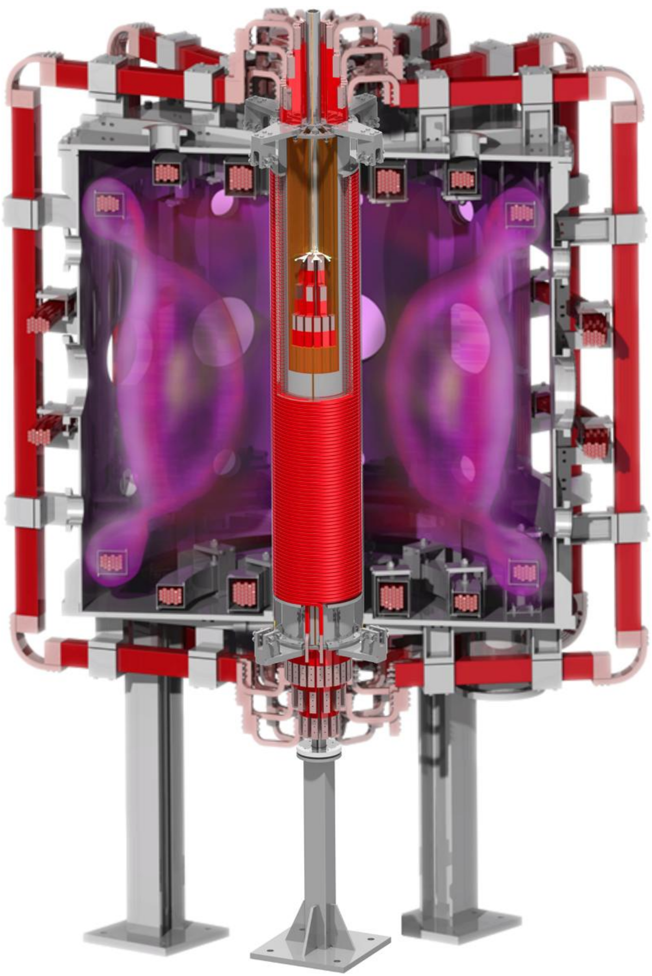
$$V_{\phi} = -\frac{\partial \phi}{\partial t} \equiv M \frac{\partial I_{CS}}{\partial t}$$

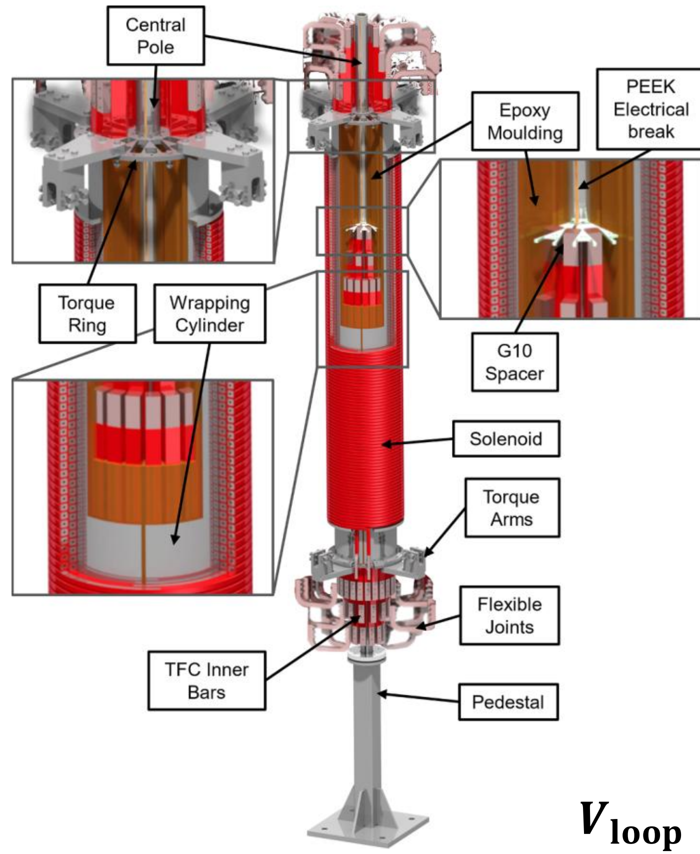
$$E_{\phi} = -\frac{1}{2\pi r}\frac{\partial \phi}{\partial t}$$

H.-T. Kim, etc., Nucl. Fusion **62**, 126012 (2022)
S. J. Doyle et al, Fusion Eng. Des. **171**, 112706 (2021)

Central solenoid can be used to provide the required loop voltage for breakdown

- **SMART:**





$$V_{\text{loop}} = \frac{A_{\text{sol}} \mu N_{\text{sol}}}{L_{\text{sol}}} \frac{dI_{\text{sol}}}{dt}$$

J. Segado-Fernandez et al, Fusion Eng. Des. **193**, 113832 (2023)

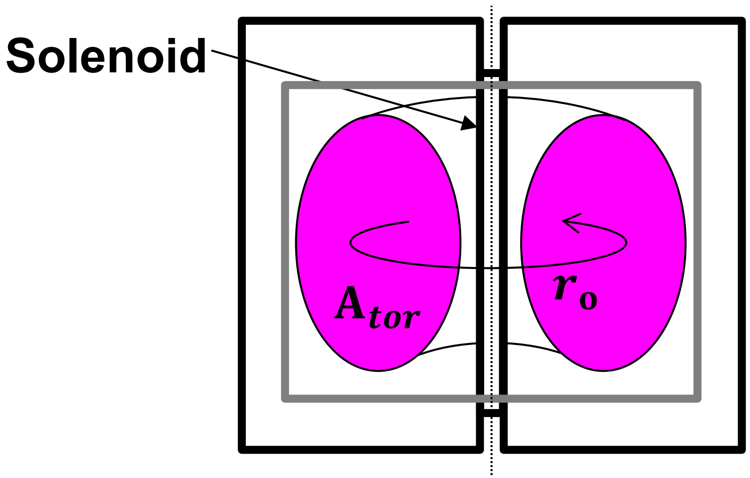
Solenoid can be used to drive the plasma current

$$L_{\mathrm{tor}}\frac{dI}{dt}+IR=V_{\mathrm{loop}}=M\frac{dI_{\mathrm{sol}}}{dt}$$

$$L_{\mathrm{tor}}=\mu_{\mathrm{o}}r_{\mathrm{o}}\left(\ln\left(\frac{8r_{\mathrm{o}}}{a}\right)-1.5\right)$$

$$R_{\mathrm{spitzer}}=\eta_{\mathrm{spiter}}\frac{2\pi r_{\mathrm{o}}}{A_{\mathrm{tor}}}$$

$$\eta_{\mathrm{spiter}}=5.2\times10^{-3}Z\ln\Lambda\,{T_{e,(\mathrm{eV})}}^{-3/2}$$



Current is initially driven at the surface and then diffuses into the plasma

- **Simplified Ohm's law:** $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$
- **Assuming a stationary plasma:** $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\eta \vec{j}\right)$
- **Assuming a constant η:**

$$-\frac{\partial}{\partial t} \nabla \times \vec{B} = \eta \nabla \times \nabla \times \vec{j} = \eta\left(\nabla\left(\nabla \cdot \vec{j}\right) - \nabla^2 \vec{j}\right)$$

$$\frac{\partial \vec{j}}{\partial t} = \frac{\eta}{\mu_o} \nabla^2 \vec{j}$$

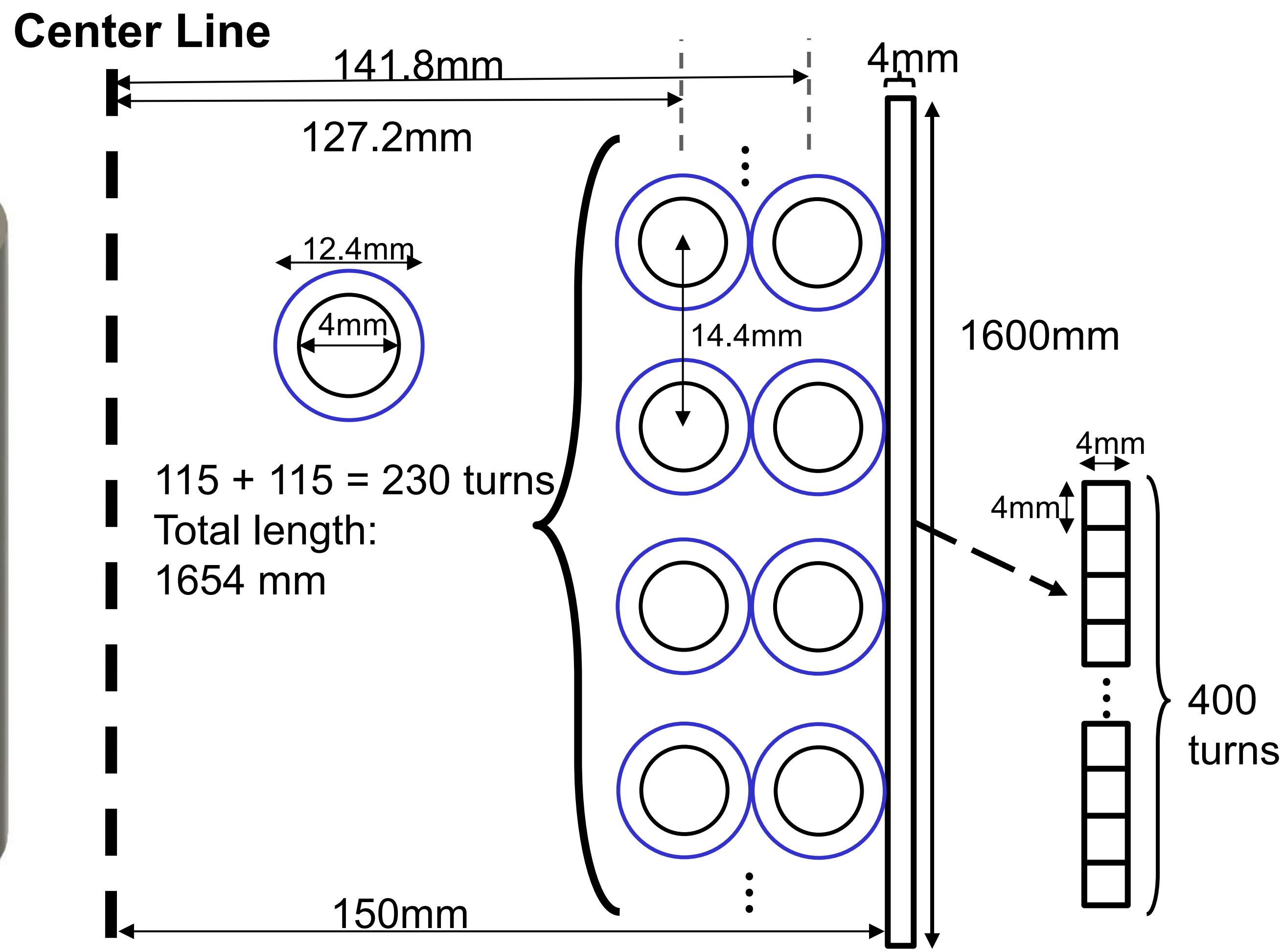
- **Assuming non-constant η:**

$$\frac{\partial \vec{j}}{\partial t} = \frac{1}{\mu_o} \nabla^2\left(\eta \vec{j}\right) - \nabla\left(\nabla \cdot \left(\mu \vec{j}\right)\right) \qquad \frac{\partial j_T}{\partial t} = \frac{1}{\mu_o} \nabla^2 (\eta j_T)$$

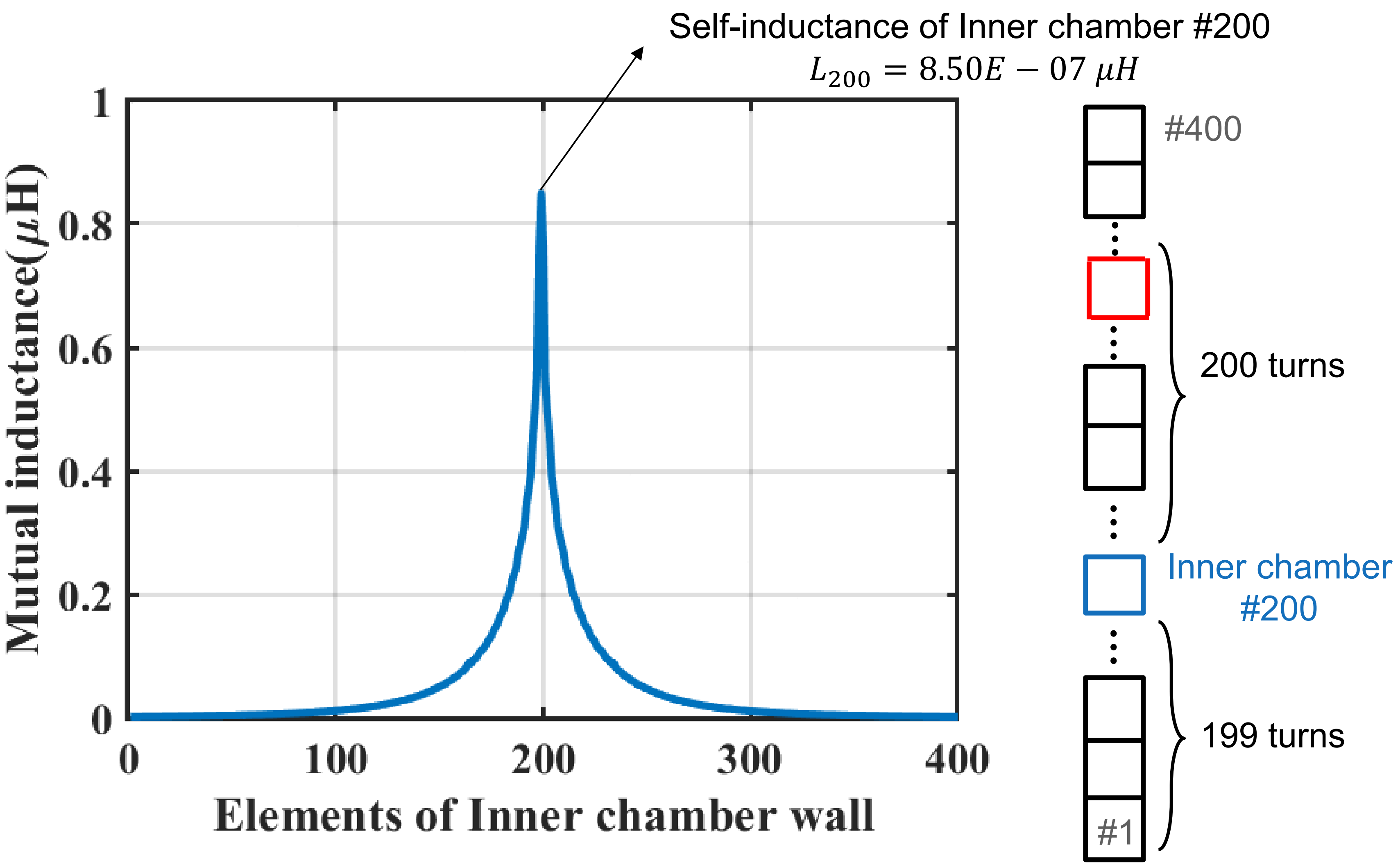
- **Since $\eta \propto T^{-3/2}$, resistance drops with higher temperature. The typical limited temperature is ~3 keV.**

Eddy current needed to be considered

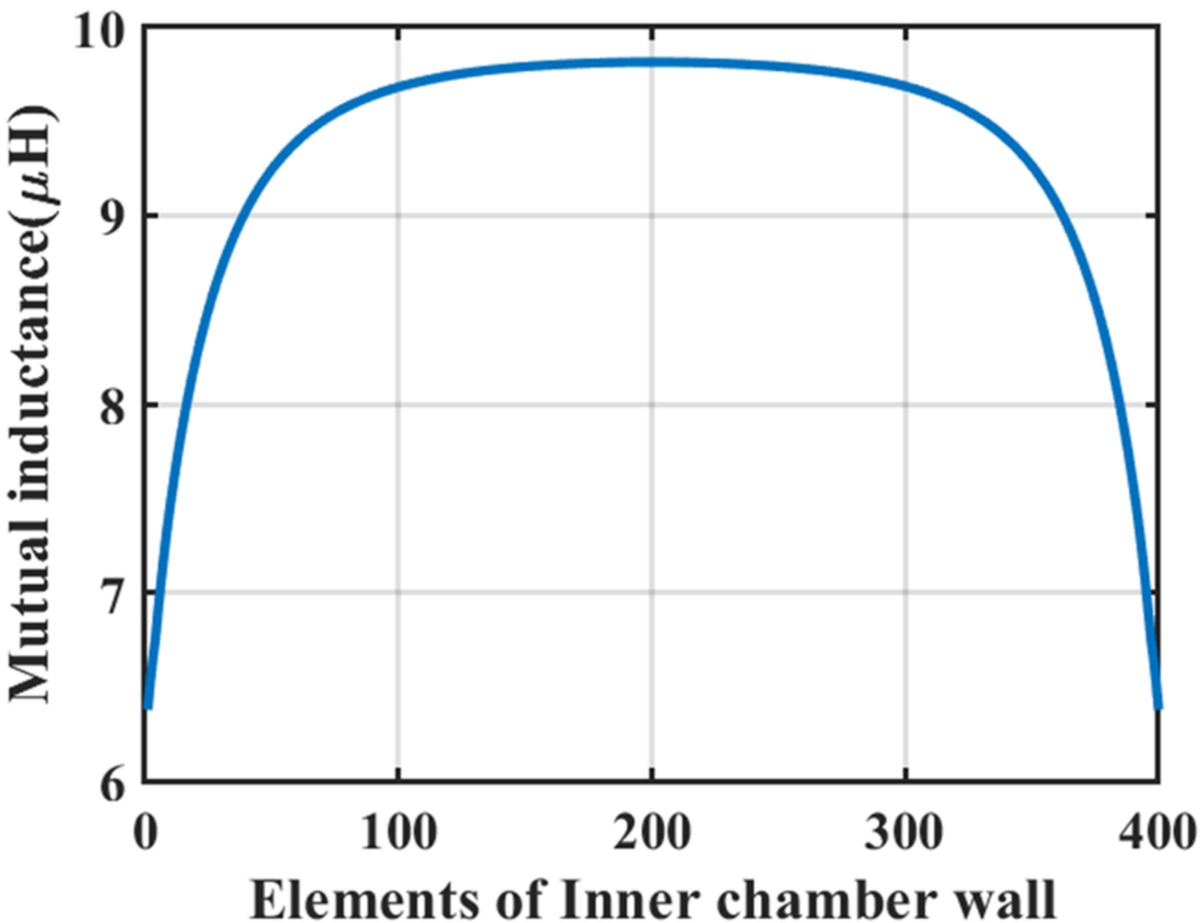
$$\overleftrightarrow{M}\frac{d\vec{I}}{\mathrm{dt}} + \overleftrightarrow{R}_{\Omega}\vec{I} = \vec{V} \qquad \overleftrightarrow{M}\frac{\vec{I'} - \vec{I}}{\Delta\mathrm{t}} + \overleftrightarrow{R}_{\Omega}\vec{I} = \vec{V} \qquad \vec{I'} = \left(\overleftrightarrow{1} - \Delta\mathrm{t}\,\overleftrightarrow{M}^{-1}\overleftrightarrow{R}_{\Omega}\right)\vec{I} + \Delta\mathrm{t}\,\overleftrightarrow{M}^{-1}\vec{V}$$

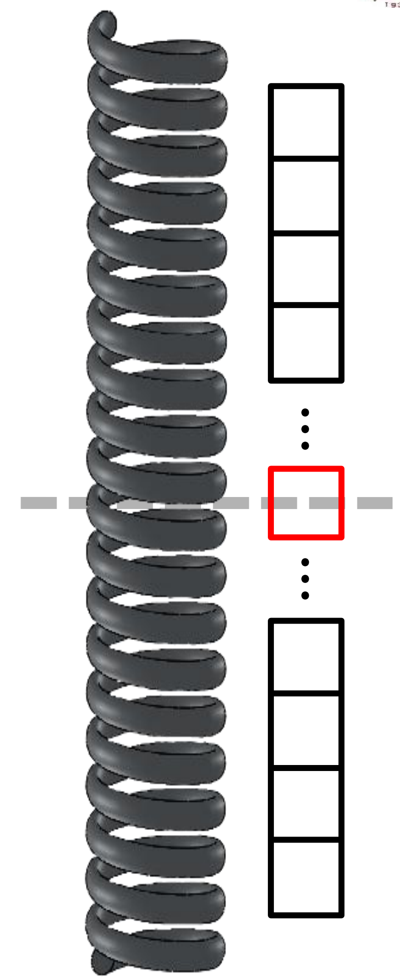


The mutual inductance between inner chamber element #200 and other elements

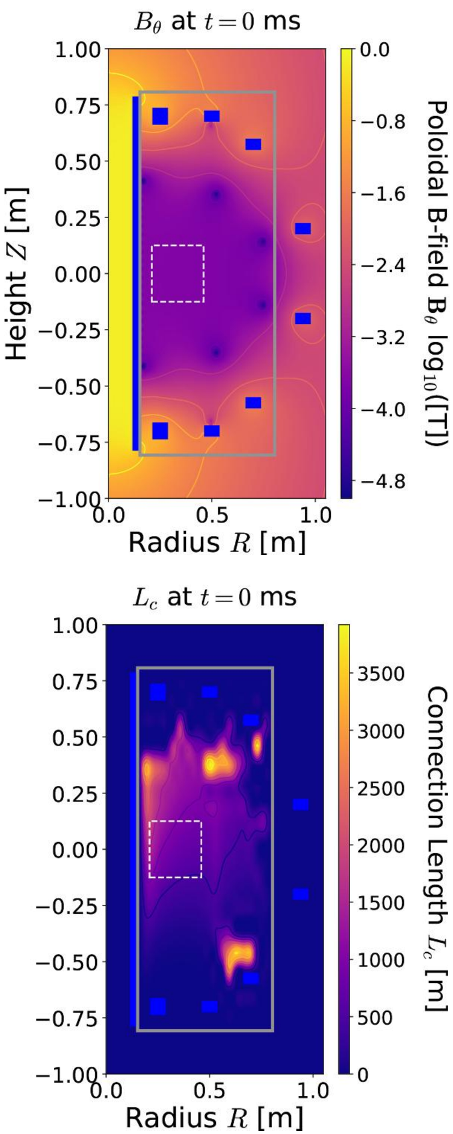


The mutual inductance between the central solenoid and other chamber wall elements

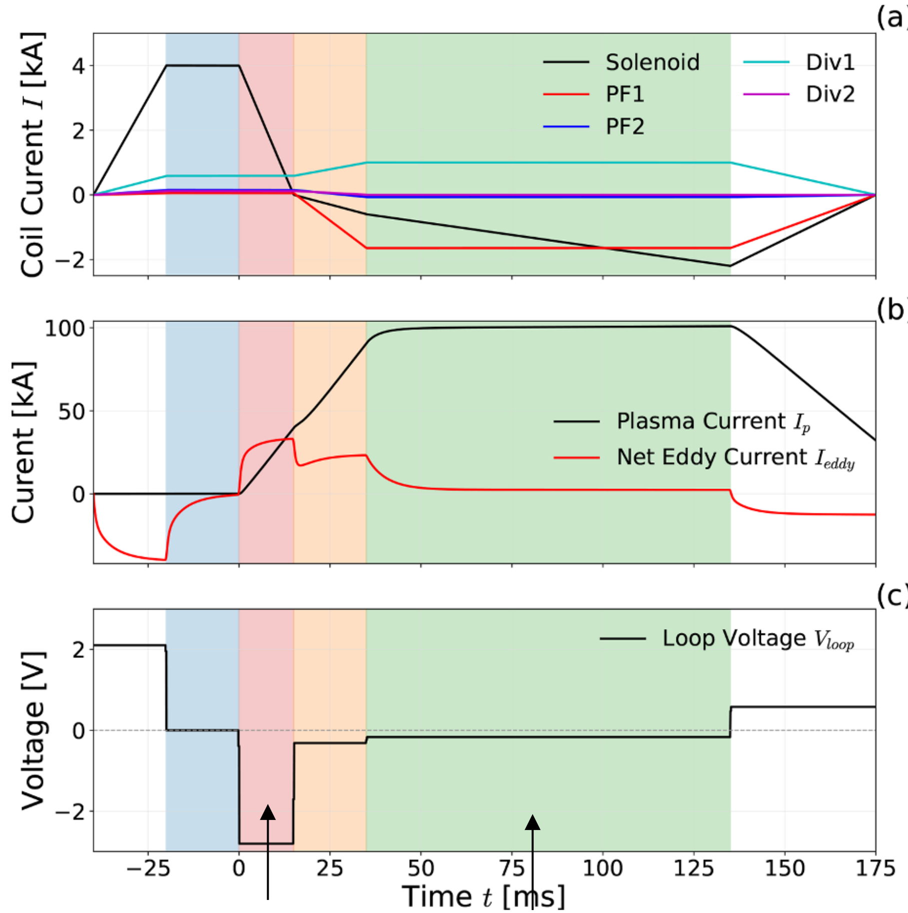




Poloidal coils are used to reduce the stray field during breakdown



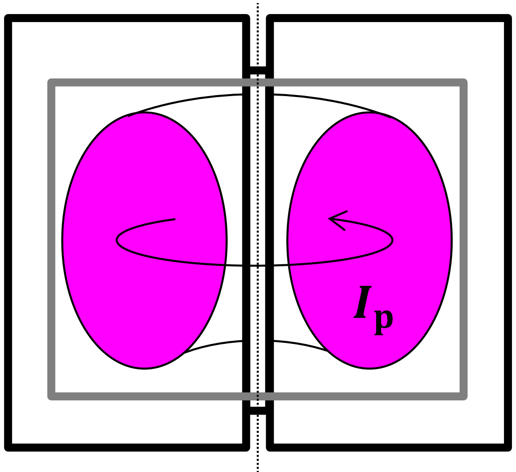
- **Currents of SMART**



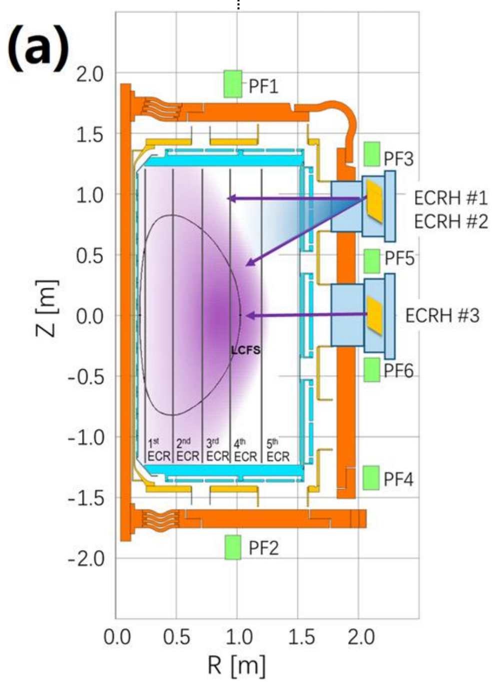
Breakdown **Plasma current is driven**

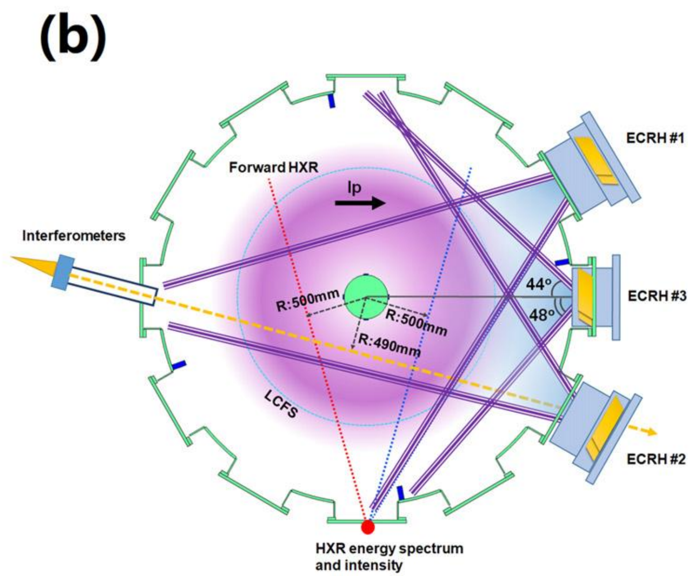
S. J. Doyle, etc., Fusion Eng. Des. **171**, 112706 (2021)

Momentum exchange may be needed to drive plasma current



$$\vec{j}_{\mathrm{p}} = \Sigma q n \vec{v} = -e n_{\mathrm{e}} \vec{v}_{\mathrm{e}} + e n_{\mathrm{i}} \vec{v}_{\mathrm{i}}$$

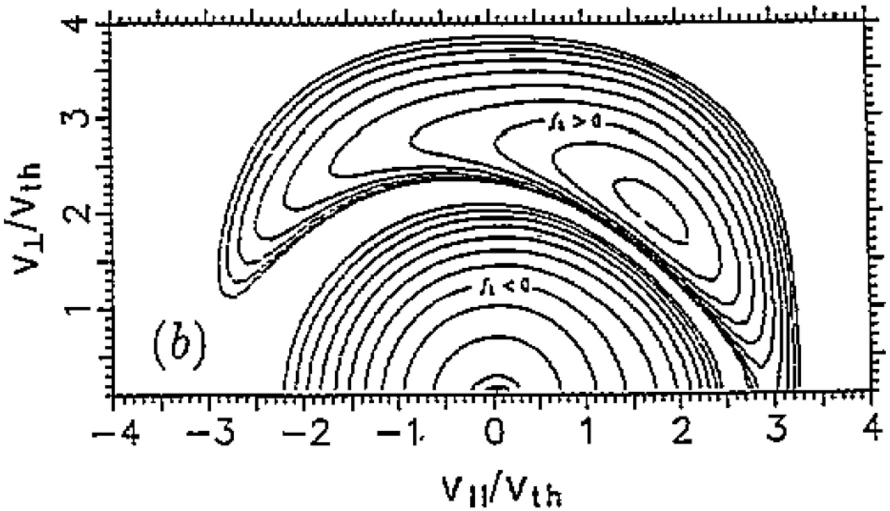




Yuejiang Shi et al, Nucl. Fusion **62**, 086047 (2022)

The collisional re-distribution of the ECRH-driven anisotropy in $E_\perp$ causes some parallel momentum to flow from e^- to ions

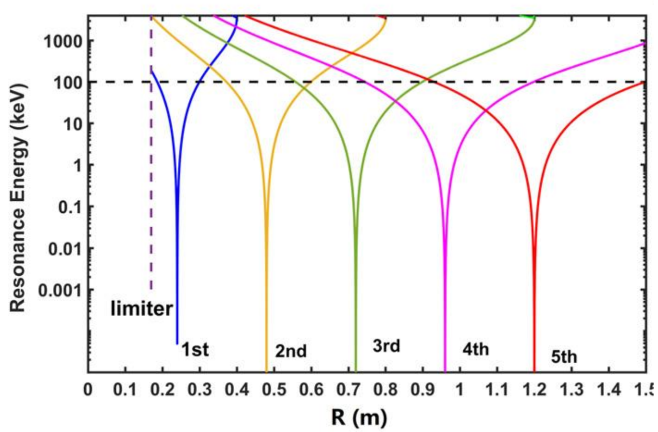
- **Coulomb collisions are more efficient at lower energies.**

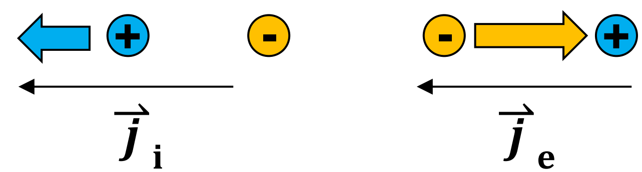


Velocity: $v_2 > v_1$

Collisions: $\nu_2 < \nu_1$

- **Electron cyclotron current drive:**



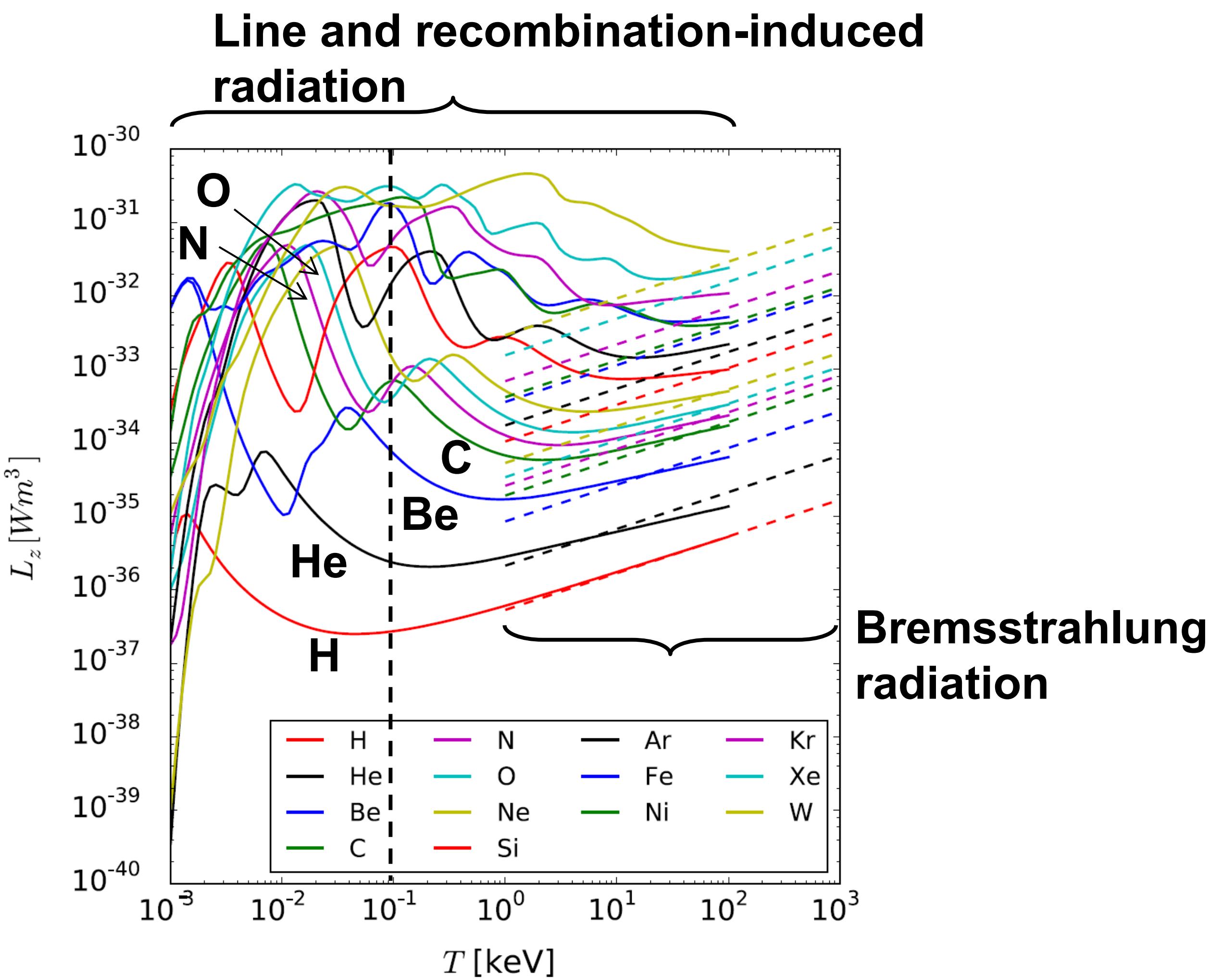


$$\vec{j}_p = -en_e\vec{v}_e + en_i\vec{v}_i$$

$$\vec{P} = n_e m_e\vec{v}_e + n_i m_i\vec{v}_i \approx 0$$

V. Erckmann and U. Gasparion, Plasma Phys. Control. Fusion **36**, 1869 (1994)
Yuejiang Shi et al, Nucl. Fusion **62**, 086047 (2022)

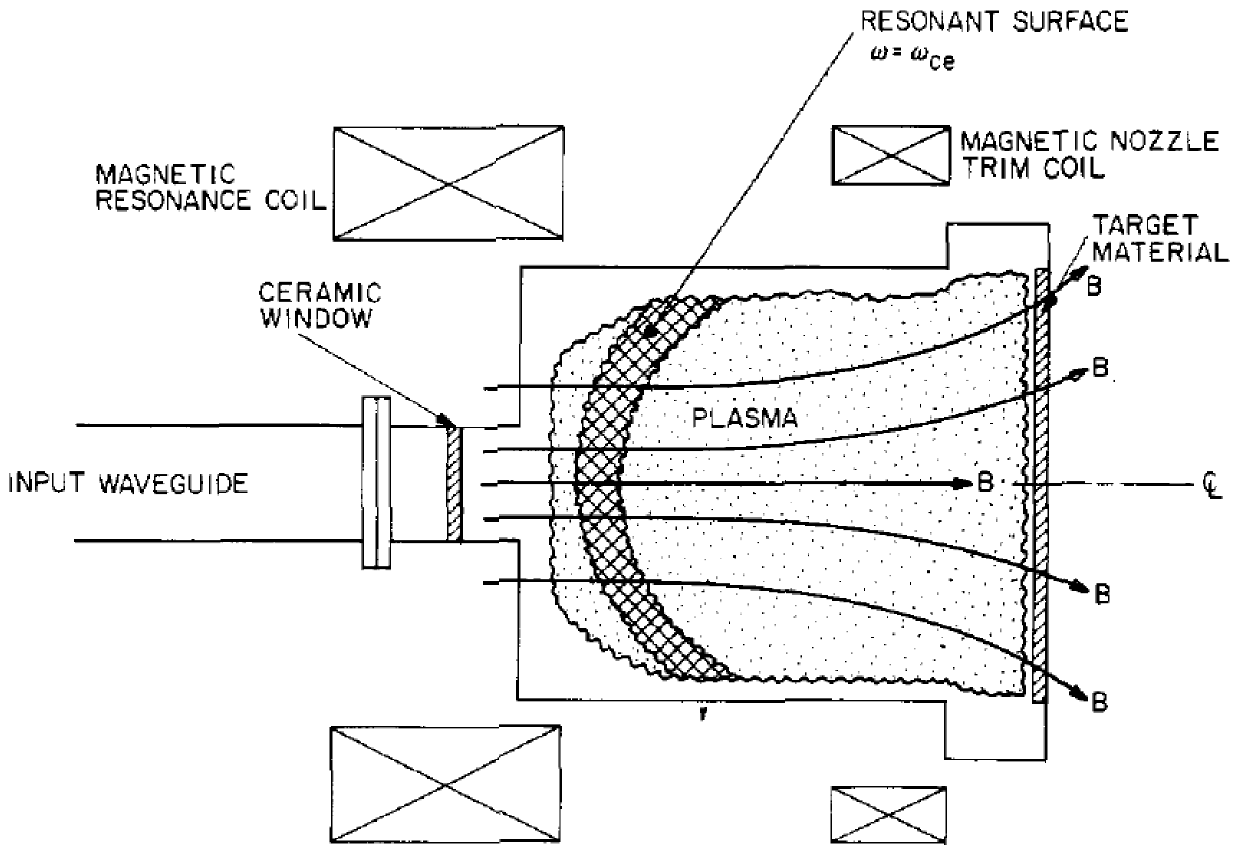
Temperature of 100 eV is the threshold of radiation barrier by impurities



H. Lux, etc., Fusion Eng. Des. **42**, 101 (2015)

Strong absorption occurs when the frequency matches the electron cyclotron frequency

- **Electron cyclotron resonance (ECR) plasma reactor**



Electron cyclotron frequency depends on magnetic field only

$$m_e \frac{d\vec{v}}{\mathrm{dt}} = -\frac{e}{c}\,\vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

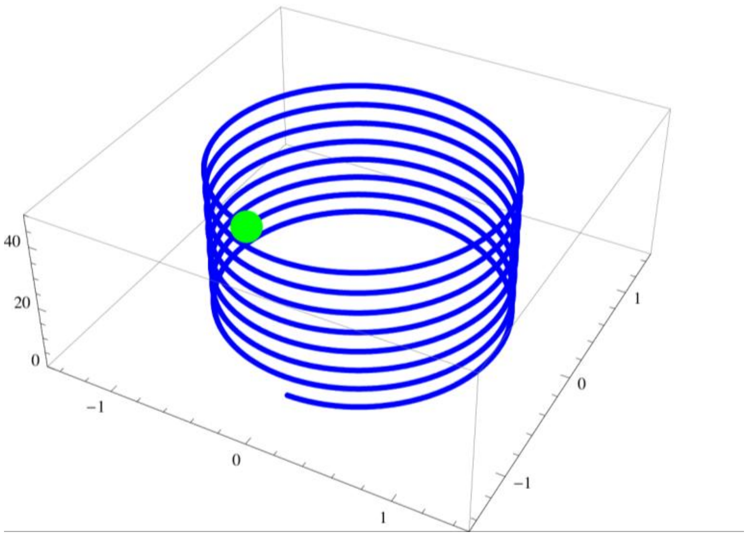
$$m_e \dot{v}_x = -\frac{e}{c} B v_y$$

$$m_e \dot{v}_y = \frac{e}{c} B v_x$$

$$m_e \dot{v}_z = 0$$

$$\ddot{v}_x = -\frac{eB}{m_e c}\dot{v}_y = -\left(\frac{eB}{m_e c}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{eB}{m_e c}\dot{v}_x = -\left(\frac{eB}{m_e c}\right)^2 v_y$$



- Therefore

$$\omega_{\mathrm{ce}} = \frac{eB}{m_e c}$$

Electrons keep getting accelerated when a electric field rotates in electron's gyrofrequency

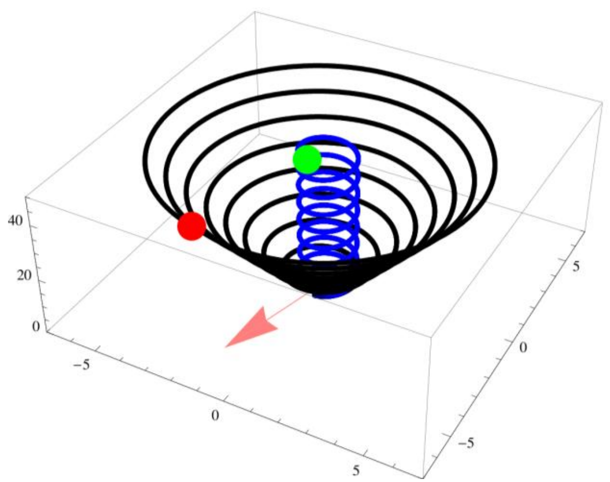
$$m_e \frac{d\vec{v}}{dt} = -\frac{e}{c}\,\vec{v}\times\vec{B} - e\,\vec{E} \qquad \vec{B} = B_0\hat{z} \qquad \vec{E} = E_0[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)]$$

$$m_e\dot{v}_x = -\frac{e}{c}Bv_y + E_0\cos(\omega t) \qquad m_e\dot{v}_y = \frac{e}{c}Bv_x + E_0\sin(\omega t) \qquad m_e\dot{v}_z = 0$$

$$\ddot{v}_x = -\frac{eB}{m_e c}\dot{v}_y - \frac{E_0}{m_e}\omega\sin(\omega t) = -\omega_{ce}{}^2 v_x - \frac{E_0}{m_e}(\omega_{ce}+\omega)\sin(\omega t)$$

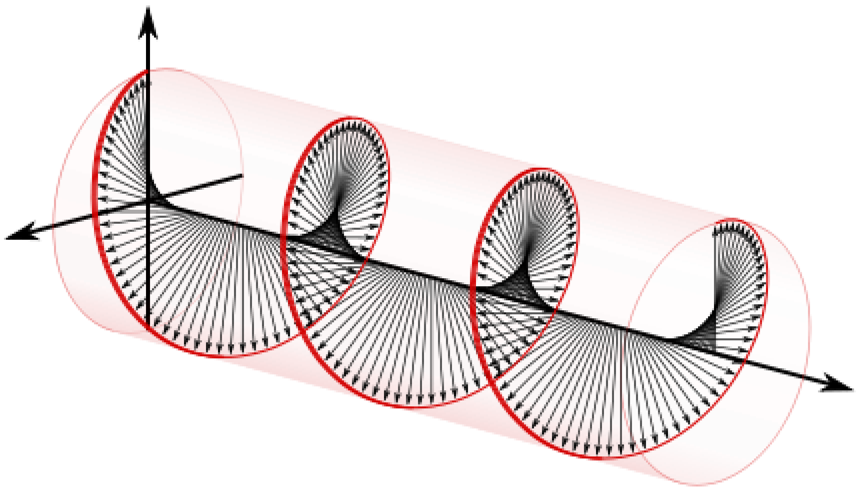
$$\ddot{v}_y = -\frac{eB}{m_e c}\dot{v}_x + \frac{E_0}{m_e}\omega\cos(\omega t) = -\omega_{ce}{}^2 v_y + \frac{E_0}{m_e}(\omega_{ce}+\omega)\cos(\omega t)$$

$$\omega_{ce} = \frac{eB}{m_e c}$$

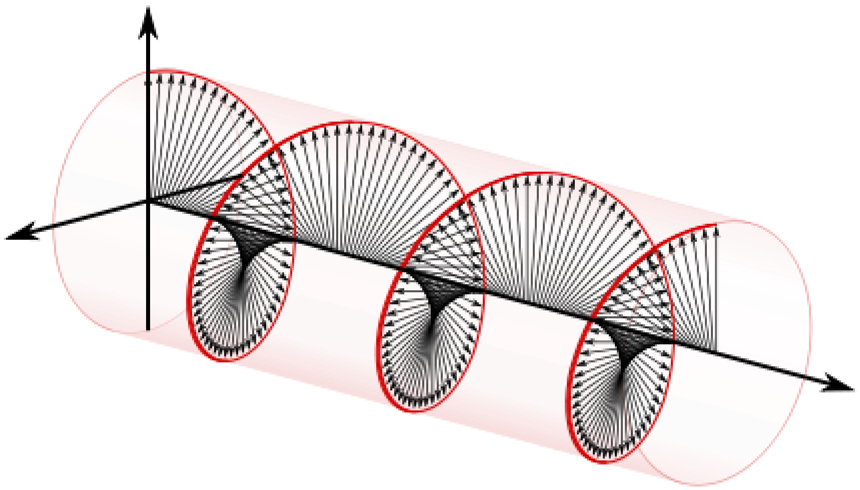


Electric field in a circular polarized electromagnetic wave keeps rotating as the wave propagates

- **Right-handed polarization**



- **Left-handed polarization**



https://en.wikipedia.org/wiki/Circular_polarization

Only right-handed polarization can resonance with electron's gyromotion

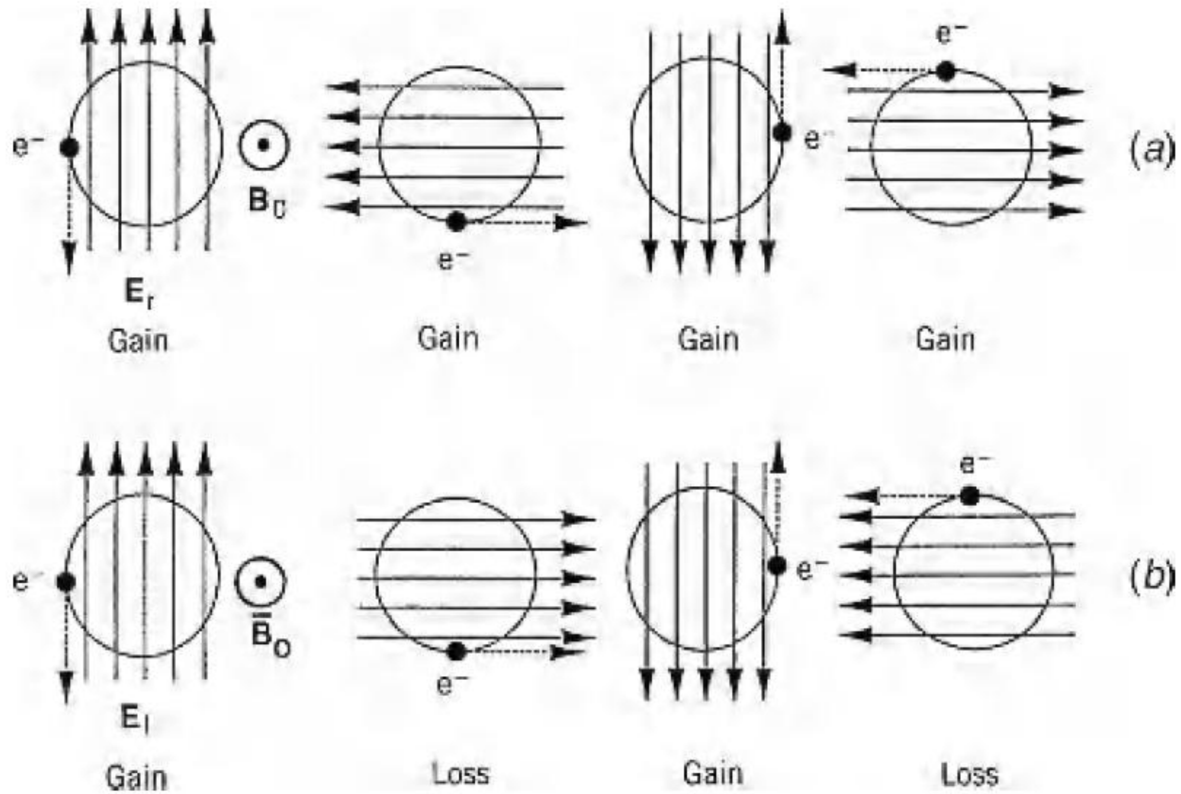
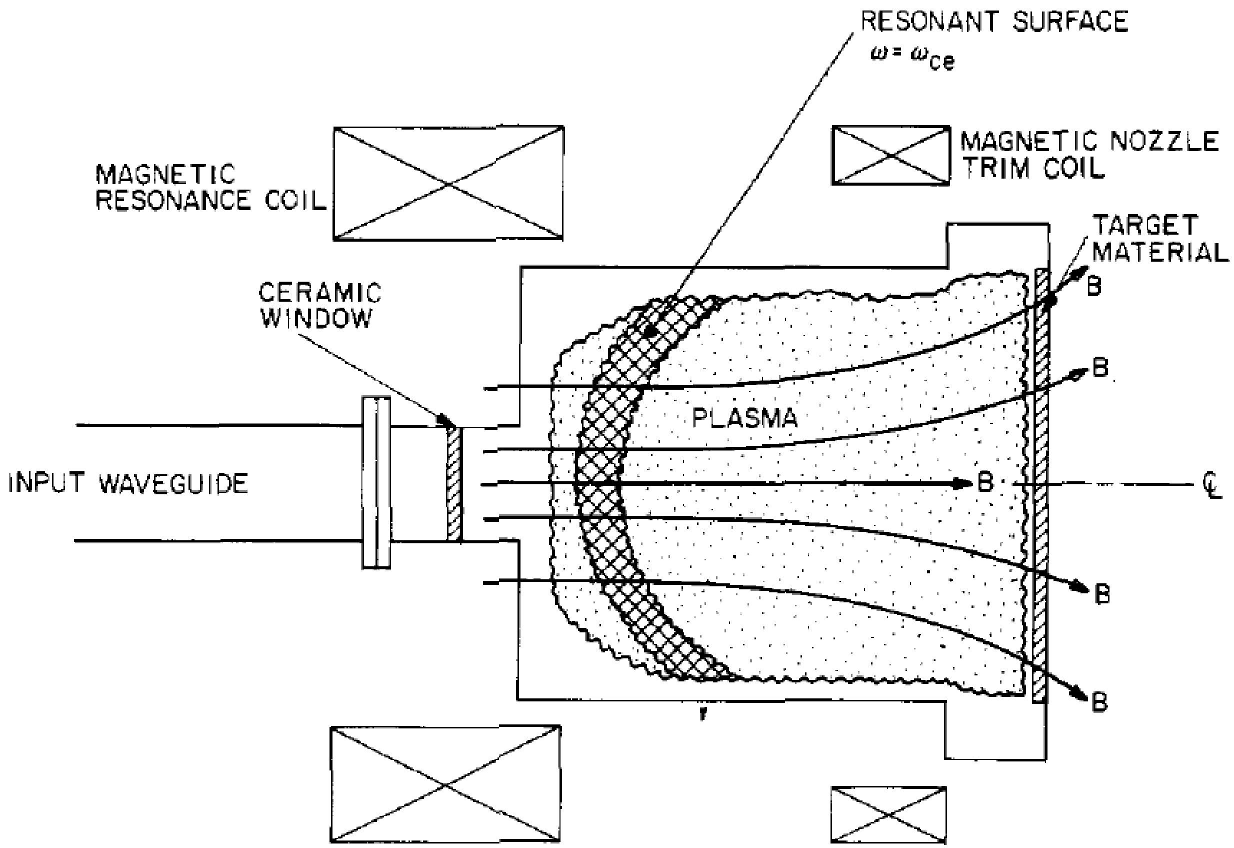


FIGURE 13.5. Basic principle of ECR heating: (*a*) continuous energy gain for right-hand polarization; (*b*) oscillating energy for left-hand polarization (after Lieberman and Gottscho, 1994).

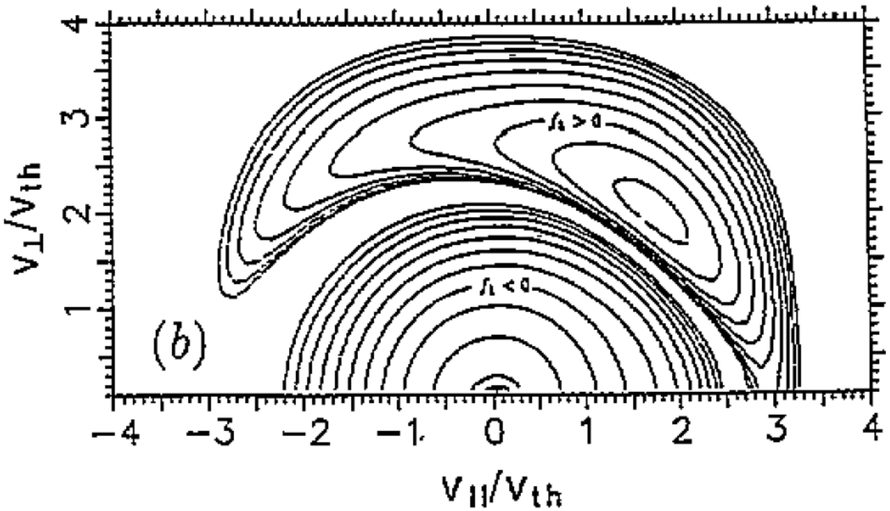
Strong absorption occurs when the frequency matches the electron cyclotron frequency

- **Electron cyclotron resonance (ECR) plasma reactor**



The collisional re-distribution of the ECRH-driven anisotropy in $E_\perp$ causes some parallel momentum to flow from e^- to ions

- **Coulomb collisions are more efficient at lower energies.**



Velocity: $v_2 > v_1$

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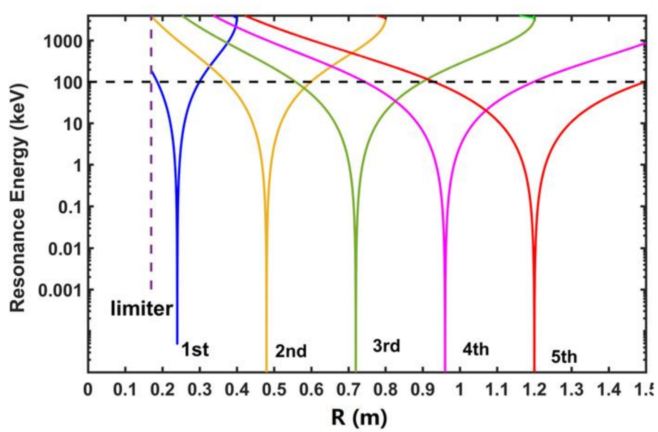
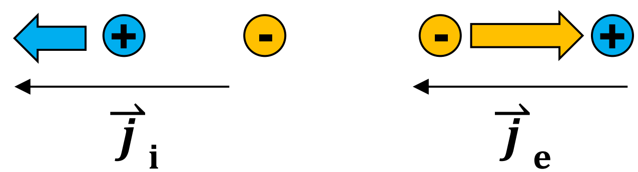
- **Electron cyclotron current drive:**





$$\vec{j}_{\mathrm{p}} = -en_{\mathrm{e}}\,\vec{v}_{\mathrm{e}} + en_{\mathrm{i}}\,\vec{v}_{\mathrm{i}}$$

$$\vec{P} = n_{\mathrm{e}}m_{\mathrm{e}}\,\vec{v}_{\mathrm{e}} + n_{\mathrm{i}}m_{\mathrm{i}}\,\vec{v}_{\mathrm{i}} \approx 0$$

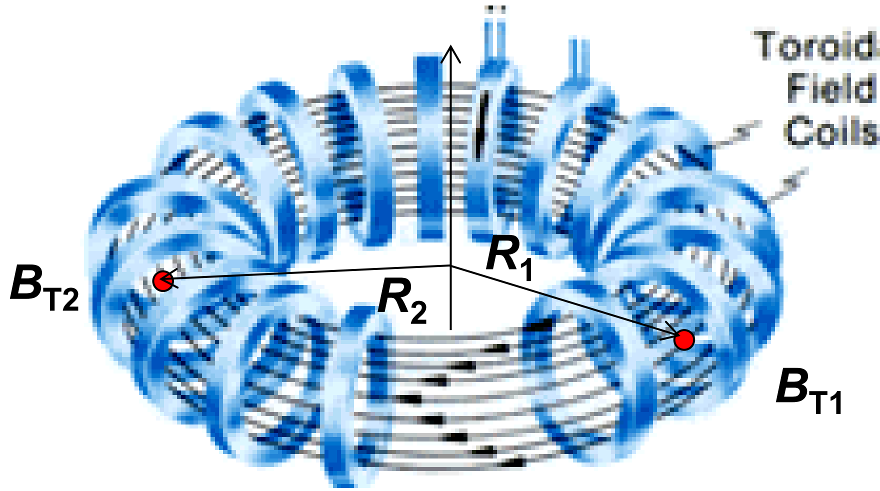
V. Erckmann and U. Gasparion, Plasma Phys. Control. Fusion **36**, 1869 (1994)
Yuejiang Shi et al, Nucl. Fusion **62**, 086047 (2022)

With poloidal fields, charged particles see nonuniform toroidal magnetic field

- **W/o poloidal field**

$$R_1 = R_2$$

$$B_{T1} = B_{T2}$$



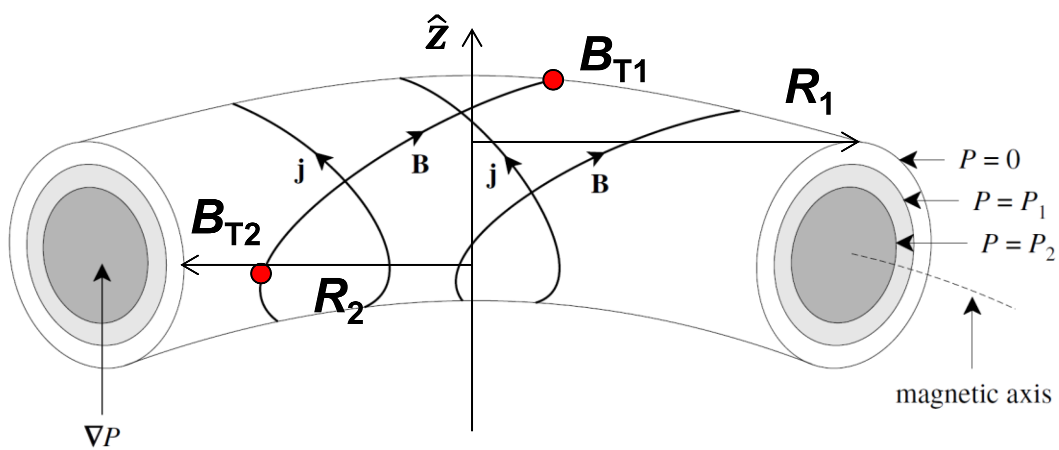
$$B_T \propto \frac{1}{R}$$

$$B_T \gg B_p$$

- **W/ poloidal field**

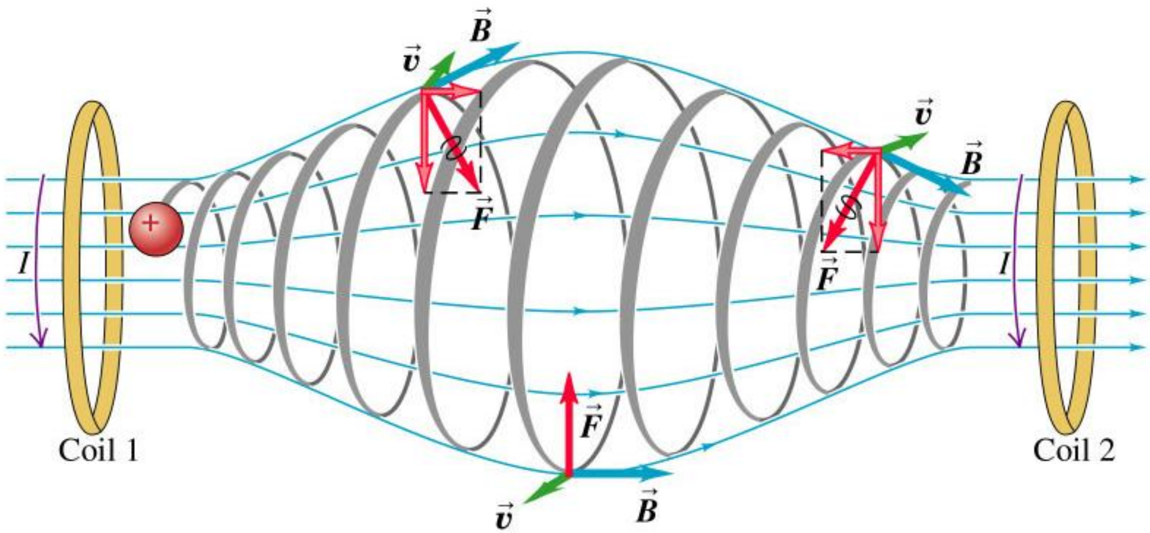
$$R_1 > R_2$$

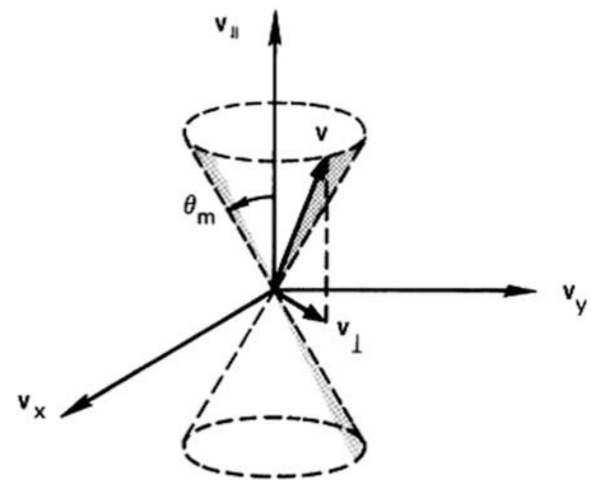
$$B_{T1} < B_{T2}$$



Charged particles can be partially confined by a magnetic mirror machine

- Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.





$$v'^{\,2}_{\perp} = v_{\perp o}^{\;2} + v_{||o}^{\;2} \equiv v_o^{\,2}$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}^{\;2}}{v'^{\,2}_{\perp}} = \frac{v_{\perp o}^{\;2}}{v_o^{\,2}} \equiv sin^2\theta$$

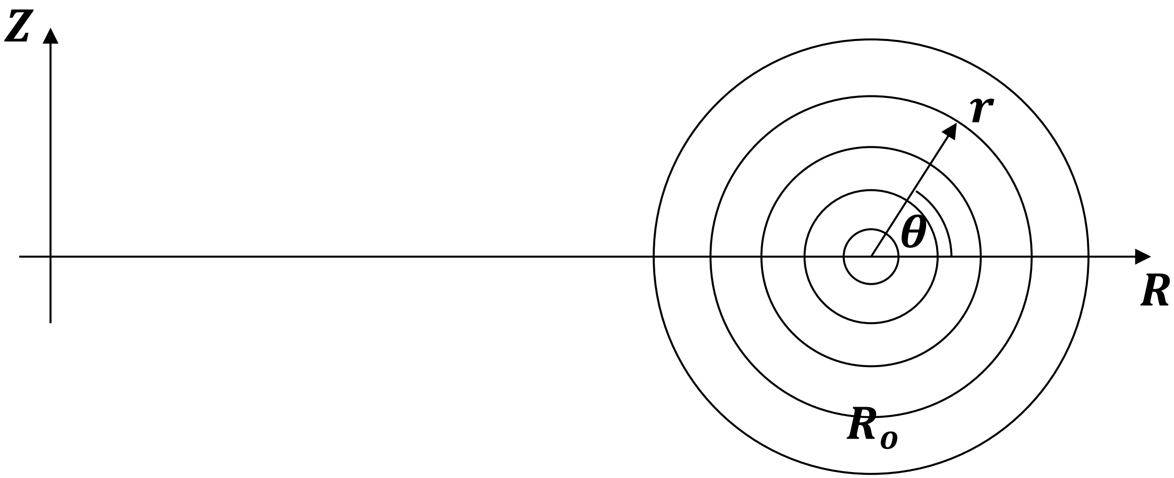
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}^{\;2} + \frac{1}{2}mv_{\perp}^{\;2} \qquad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_{\perp}^{\;2}}{B}$$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = sin^2\theta_m$$

- Large $v_{||}$ may occur from collisions between particles.
- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Parallel velocity changes when particles follow field the field line



$R \gg r$

$B_T \gg B_p$

$R = R_o + r\cos\theta = R_o(1 + \epsilon\cos\theta)$ Inverse aspect ratio: $\epsilon \equiv \frac{r}{R_o}$

$$B \simeq \frac{B_o}{1+\epsilon\cos\theta} \simeq B_o(1-\epsilon\cos\theta)$$

Invarient: $\mu \equiv \frac{1}{2}\frac{m{v_\perp}^2}{B}$ $\quad \frac{{v_\perp}^2}{B_o(1-\epsilon cos\theta)} = \frac{{v_{\perp 0}}^2}{B_o(1-\epsilon)}$ $\quad {v_\perp}^2 = \frac{{v_{\perp 0}}^2(1-\epsilon cos\theta)}{1-\epsilon}$

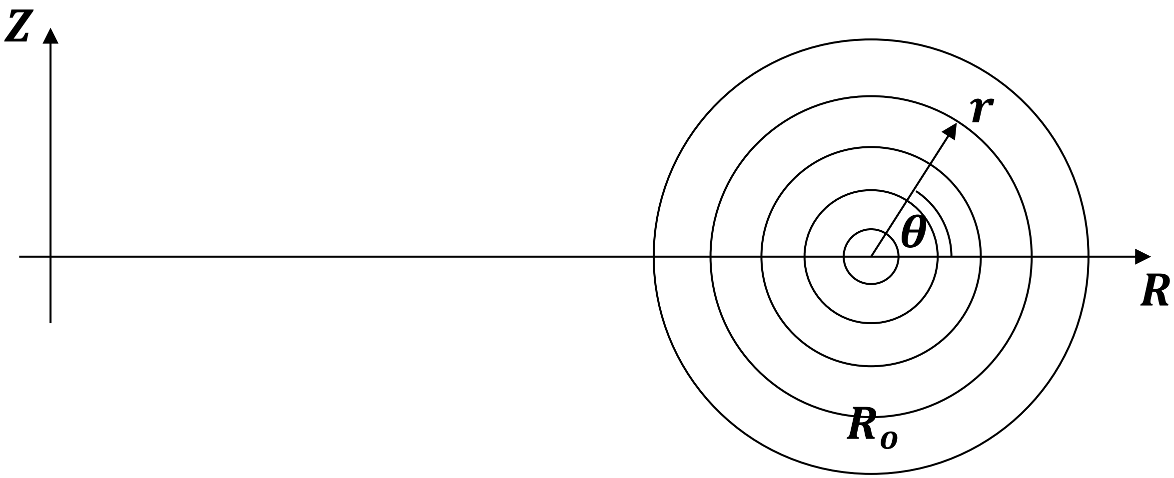
$$v^2 = {v_\perp}^2 + {v_{||}}^2 = {v_{\perp o}}^2 + {v_{||o}}^2$$

$${v_{||}}^2 = v^2\left(1 - \frac{{v_\perp}^2}{v^2}\right)$$

$${v_{||}}^2 = v^2\left[1 - \frac{{v_{\perp 0}}^2}{v^2}\frac{(1-\epsilon\cos\theta)}{1-\epsilon}\right]$$

$$\approx v^2\left\{1 - \frac{{v_{\perp 0}}^2}{v^2}\left[1 + 2\epsilon\sin^2\left(\frac{\theta}{2}\right)\right]\right\}$$

Particles may be trapped by nonuniform magnetic field



$R \gg r$

$B_\mathrm{T} \gg B_\mathrm{p}$

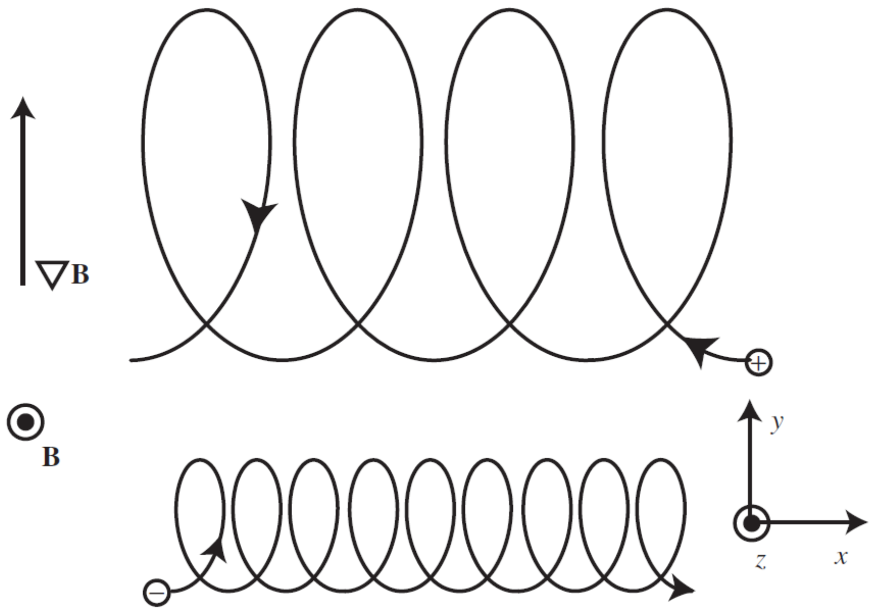
$\epsilon \equiv \dfrac{r}{R_\mathrm{o}}$

$$v_{||}^2 \approx v^2\left\{1 - \frac{{v_{\perp 0}}^2}{v^2}\left[1 + 2\epsilon\sin^2\left(\frac{\theta}{2}\right)\right]\right\}$$

- For ${v_{||}}^2 \geq 0$, particles are passing.
- For ${v_{||}}^2 \leq 0$, particles are trapped.

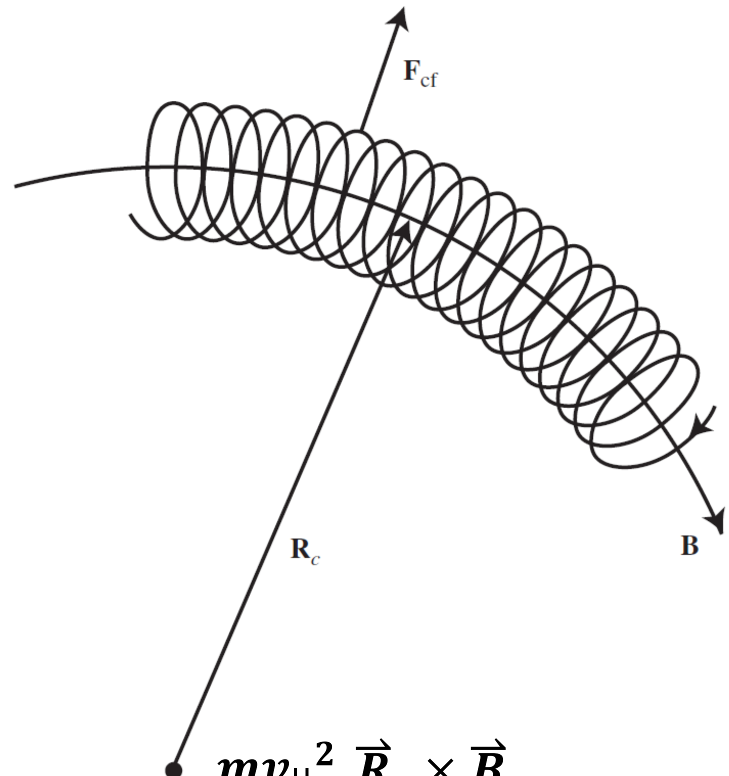
Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**



$$\vec{v}_{\nabla} = \frac{m v_{\perp}{}^{2}}{2q} \frac{\vec{B} \times \nabla B}{B^{3}}$$

- **Curvature drift**

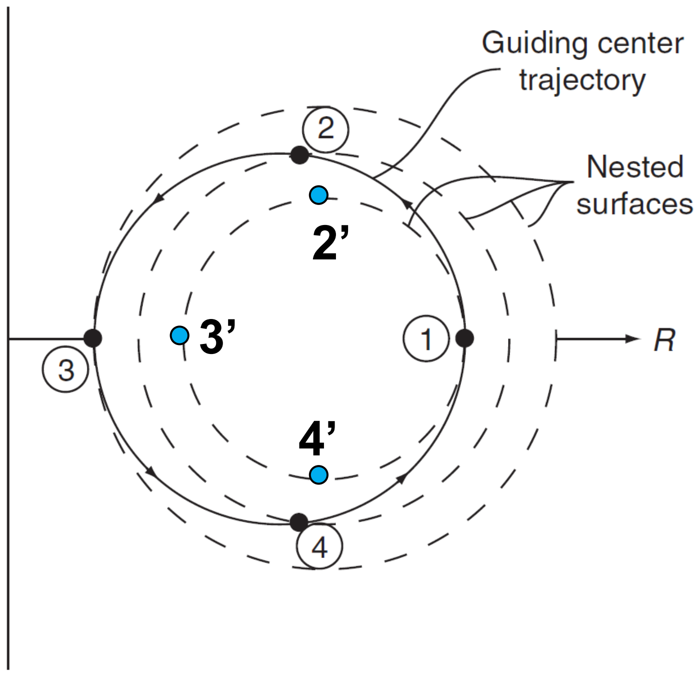


$$\vec{v}_{R} = \frac{m v_{||}{}^{2}}{2q} \frac{\vec{R}_{c} \times \vec{B}}{R_{c} B^{2}}$$

$$\vec{v}_{\text{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{c} B^{2}} \left(v_{||}{}^{2} + \frac{1}{2} v_{\perp}{}^{2} \right) = \frac{m}{q} \frac{\vec{R}_{c} \times \vec{B}}{R_{c}{}^{2} B^{2}} \left(v_{||}{}^{2} + \frac{1}{2} v_{\perp}{}^{2} \right)$$

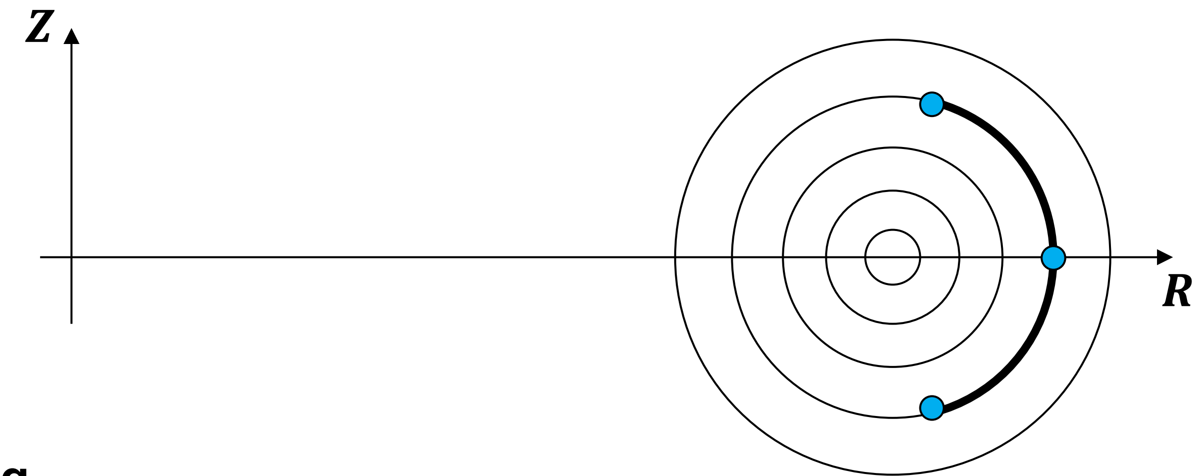
For passing particles, they drift back to the original position with a “semicircle” orbit

$$\vec{v}_{\text{total}} = \vec{v}_{\text{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_c B^2}\left(v_{||}^2 + \frac{1}{2} v_{\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_c \times \vec{B}}{{R_c}^2 B^2}\left(v_{||}^2 + \frac{1}{2} v_{\perp}^2\right)$$

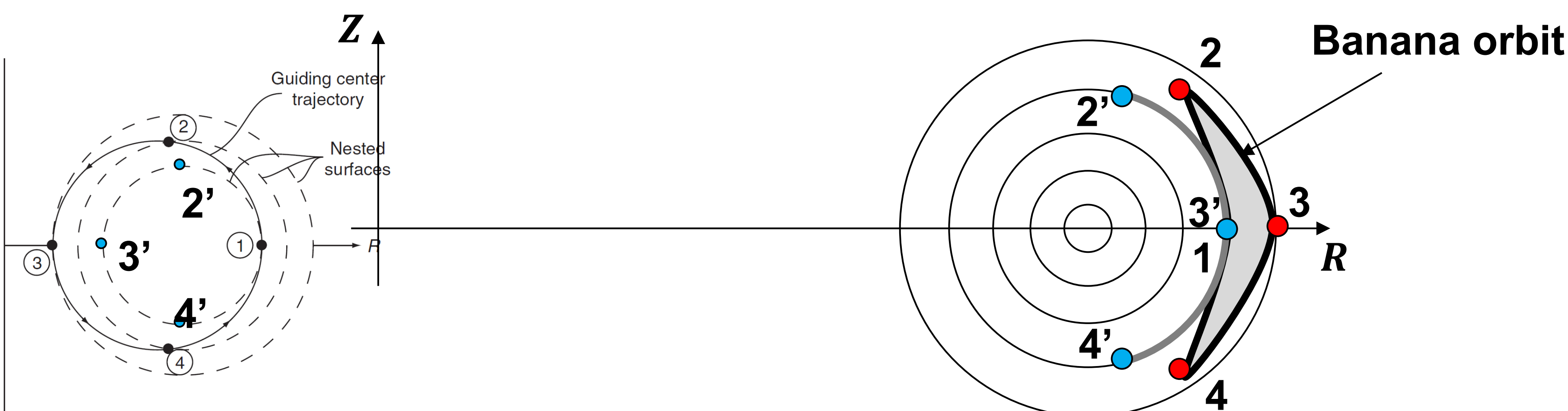


For trapped particles, they drift back to the original position with a banana orbit

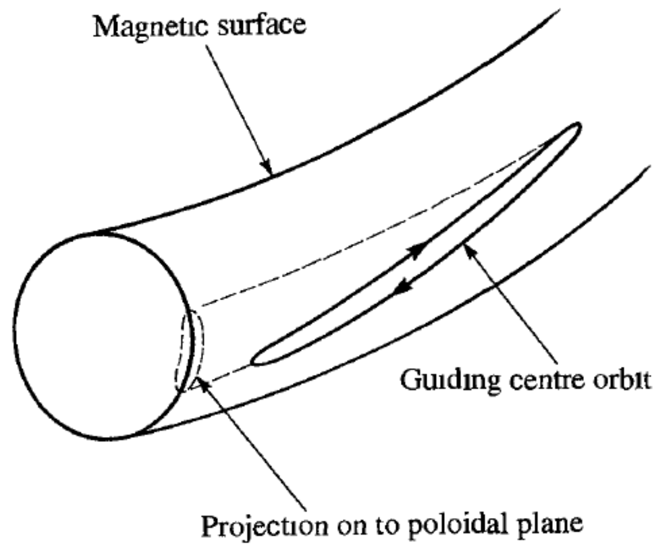
- W/o drifting

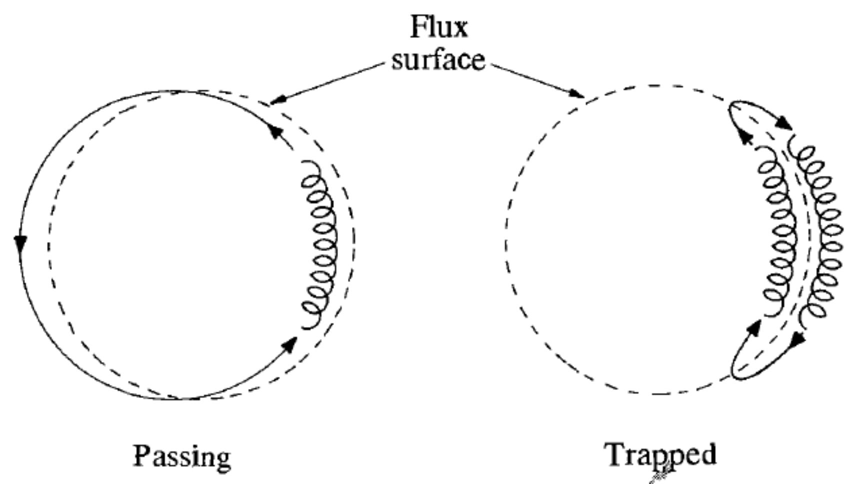


- W/ drifting



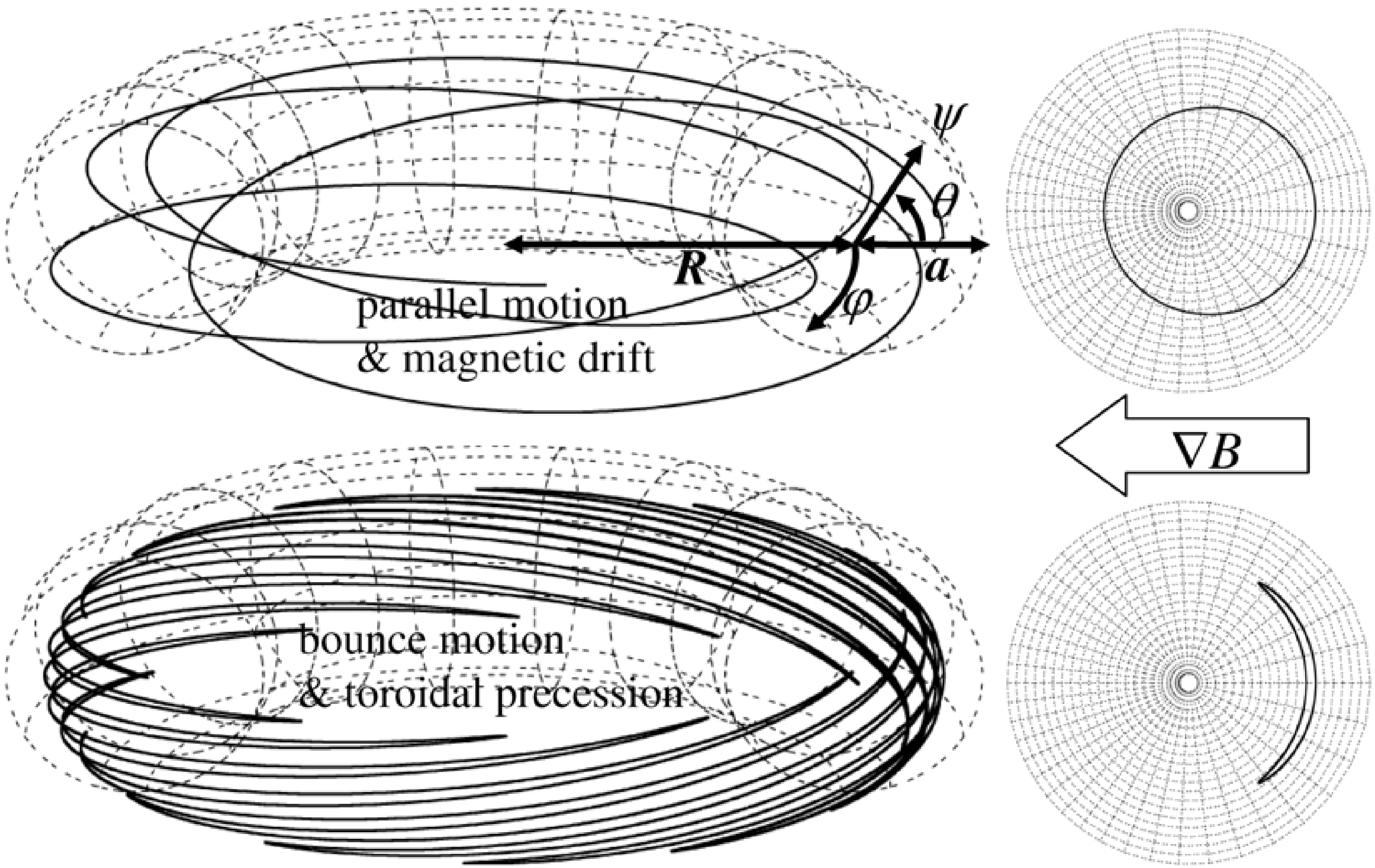
Trajectories of charged particles





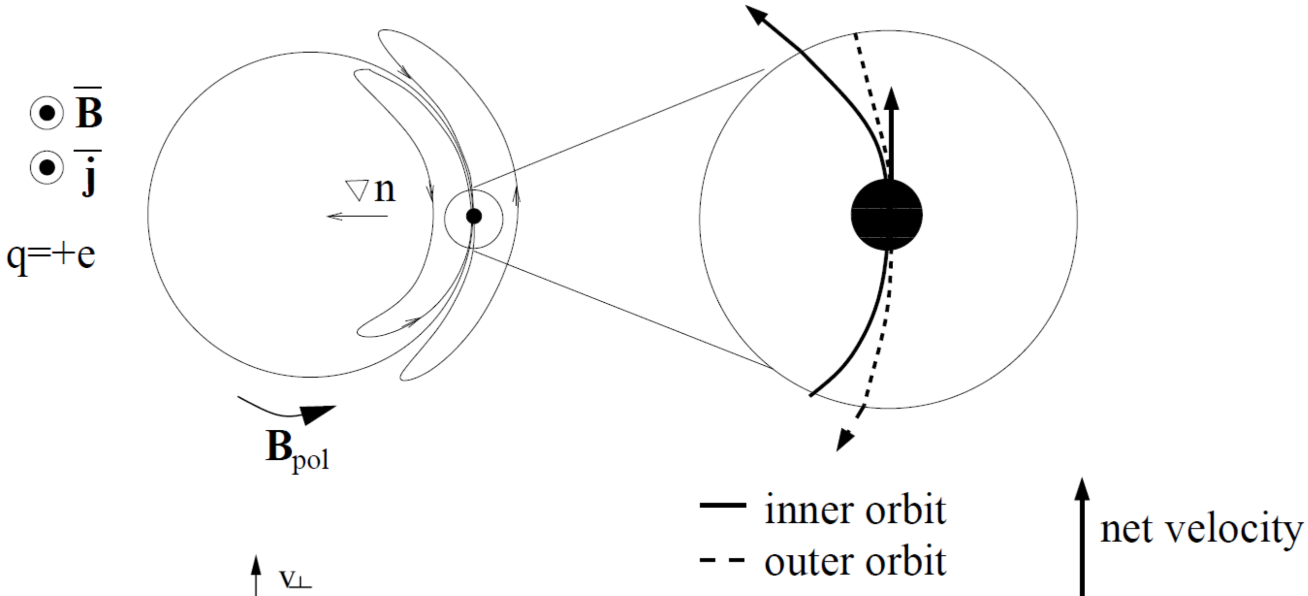
John Wesson, Tokamaks 3rd edition

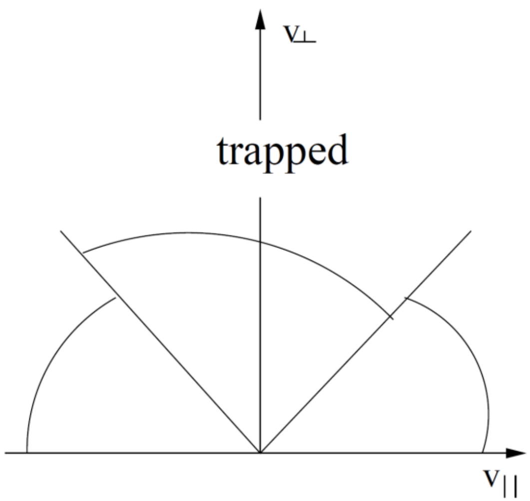
The trajectories of charged particles follow the toroidal field lines



X. Garbet et al, Nucl. Fusion **50**, 043002 (2010)

A banana current is generated when there is a pressure gradient in the plasma

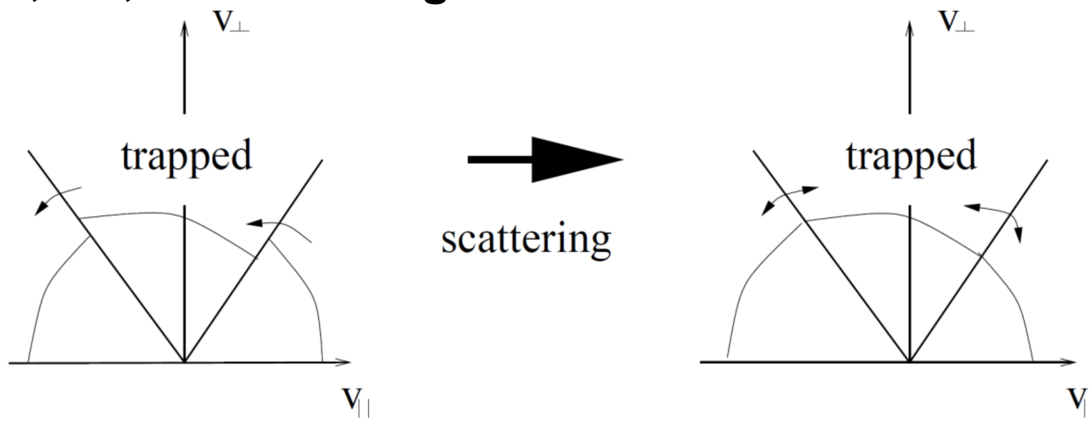


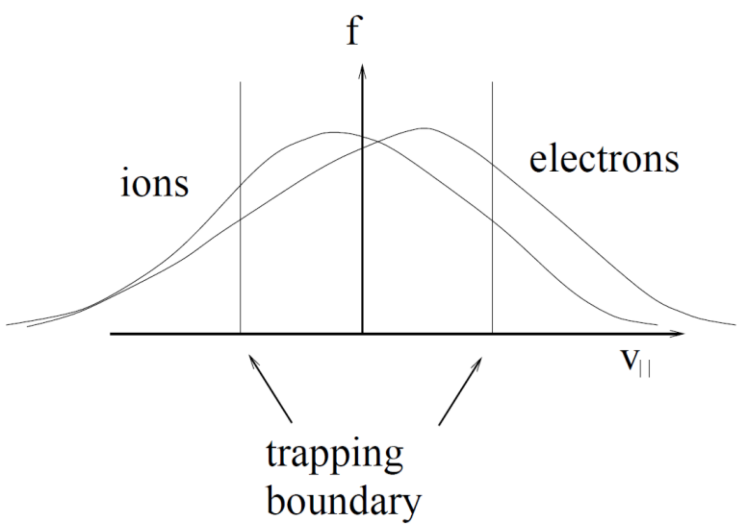


Samuli Saarelma, Helsinki University of Technology, PhD Thesis 2015

Bootstrap current is generated when passing particles are scattered by the trapped particles

- **Scattering smooths the velocity distribution and shifts it in the parallel direction, i.e., a current is generated. It is called the bootstrap current.**



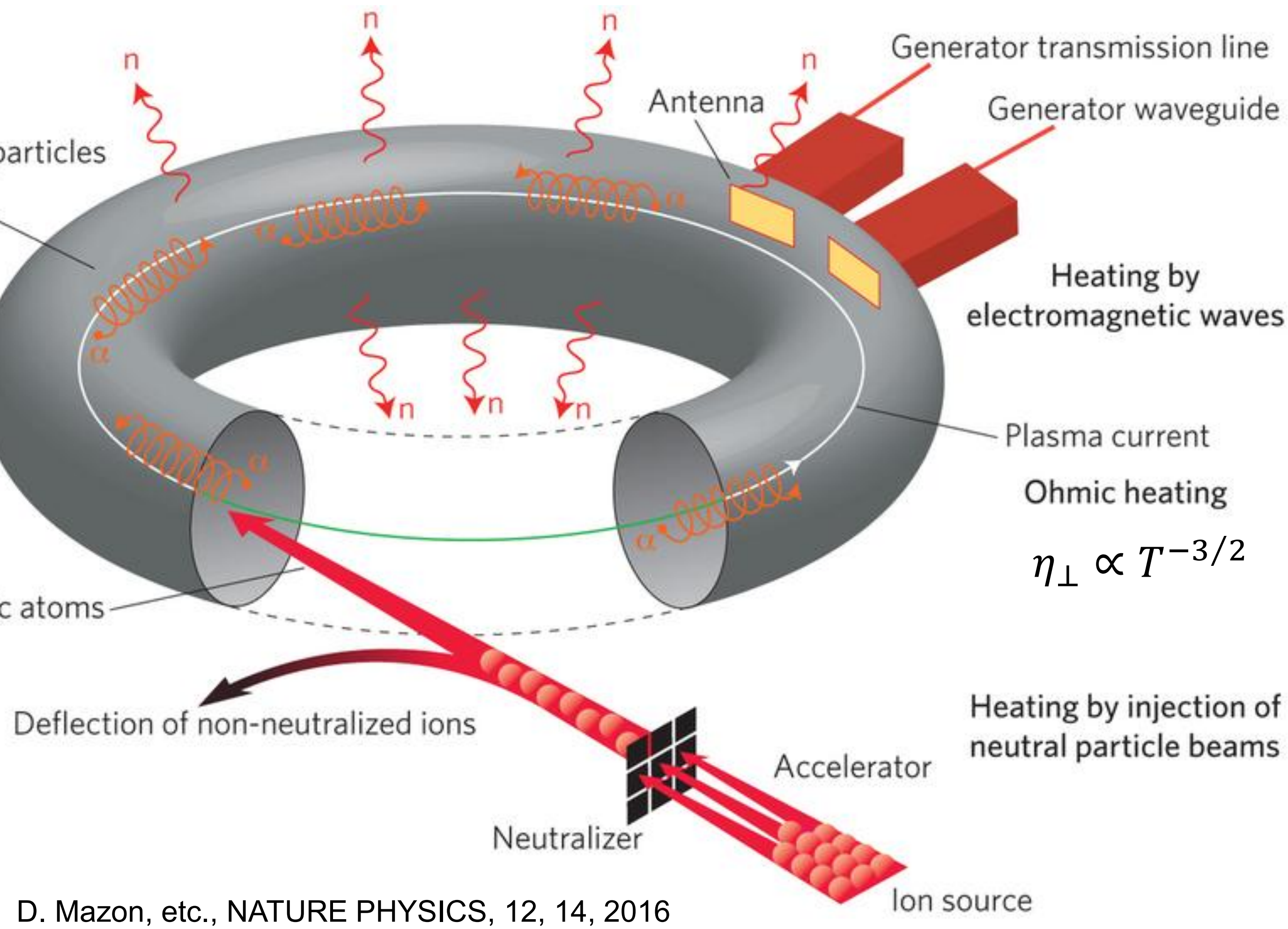


$$\boldsymbol{j = -enu_{||\mathrm{e}} + enu_{||\mathrm{i}} = 4\epsilon^{3/2}\frac{1}{B_\mathrm{p}}T\frac{dn}{dr}}$$

- **The bootstrap current is vital for steady-state operation.**

Samuli Saarelma, Helsinki University of Technology, PhD Thesis 2015

Neutral beam injector is one of the main heat mechanisms in MCF



$\eta_{\perp} \propto T^{-3/2}$

D. Mazon, etc., NATURE PHYSICS, 12, 14, 2016
https://zh.wikihow.com/%E5%9C%A8%E6%89%8B%E6%8C%87%E4%B8%8A%E8%BD%AC%E7%AF%AE%E7%90%83

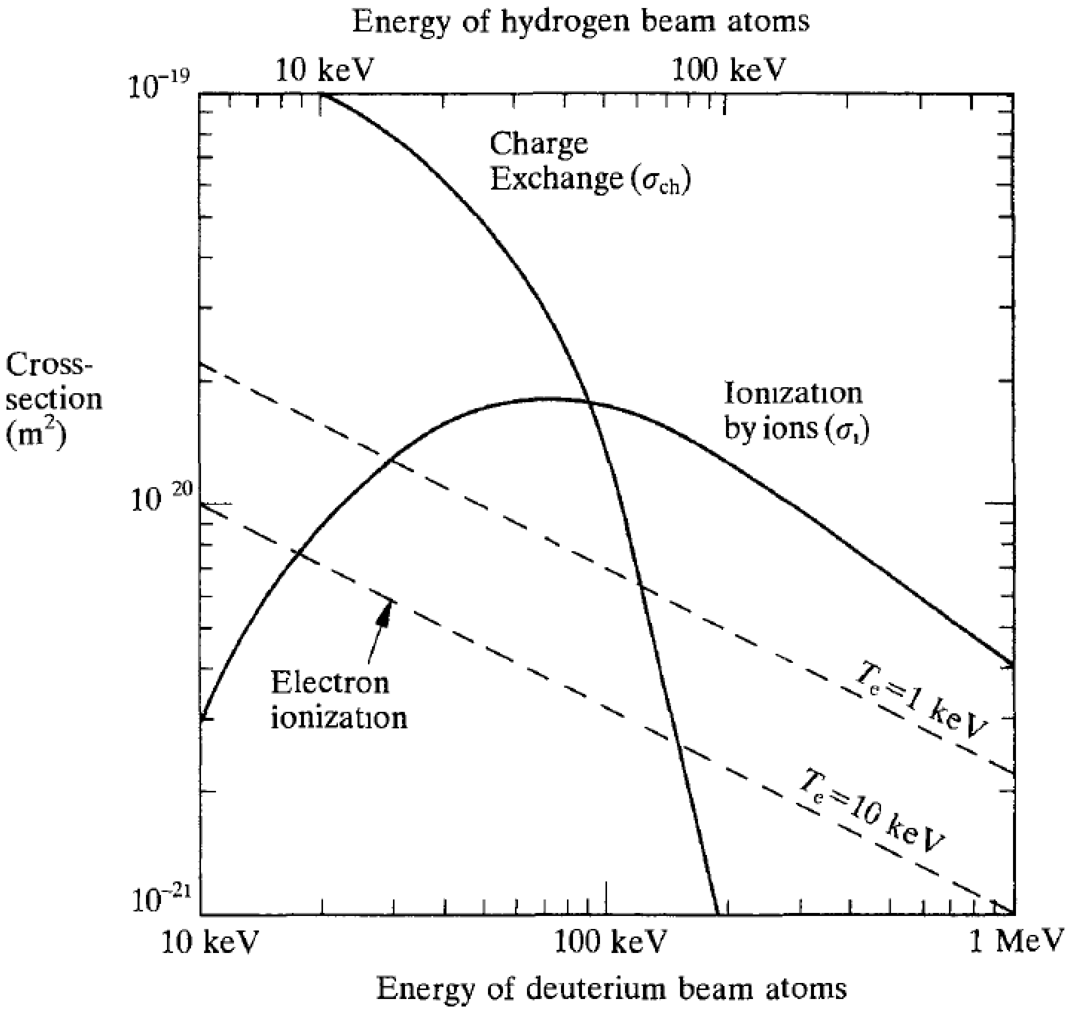
Varies way of heating a MCF device

System		Frequency/ energy	Maximum power coupled to plasma	Overall system efficiency	Development/ demonstration required	Remarks
ECRF	Demonstrated in tokamaks	28–157 GHz	2.8 MW, 0.2 s	30–40%	Power sources and windows, off-axis CD	Provides off-axis CD
	ITER needs	150–170 GHz	50 MW, S S			
ICRF	Demonstrated in tokamaks	25–120 MHz	22 MW, 3 s (L-mode); 16.5 MW, 3 s (H-mode)	50–60%	ELM tolerant system	Provides ion heating and smaller ELMs
	ITER needs	40–75 MHz	50 MW, S S			
LHRF	Demonstrated in tokamaks	1.3–8 GHz	2.5 MW, 120 s; 10 MW, 0.5 s	45–55%	Launcher, coupling to H-mode	Provides off-axis CD
	ITER needs	5 GHz	50 MW, S S			
NBI +ve ion	Demonstrated in tokamaks	80–140 keV	40 MW, 2 s; 20 MW, 8 s	35–45%	None	Not applicable
	ITER needs	None	None			
NBI −ve ion	Demonstrated in tokamaks	0.35 MeV	5.2 MW, D^-, 0.8 s (from 2 sources)			
	ITER needs	1 MeV	50 MW, S S	∼37%	System, tests on tokamak, plasma CD	provides rotation

‘S S’ indicates steady state

Nuclear Fusion, 39, 2495, 1999

Neutral atoms are ionized by collisions in the plasma



- **Charge exchange:**

  $H_b + H_p^+ \rightarrow H_b^+ + H_p$

- **Ionization by ions**

  $H_b + H_p^+ \rightarrow H_b^+ + H_p^+ + e^-$

- **Ionization by electrons**

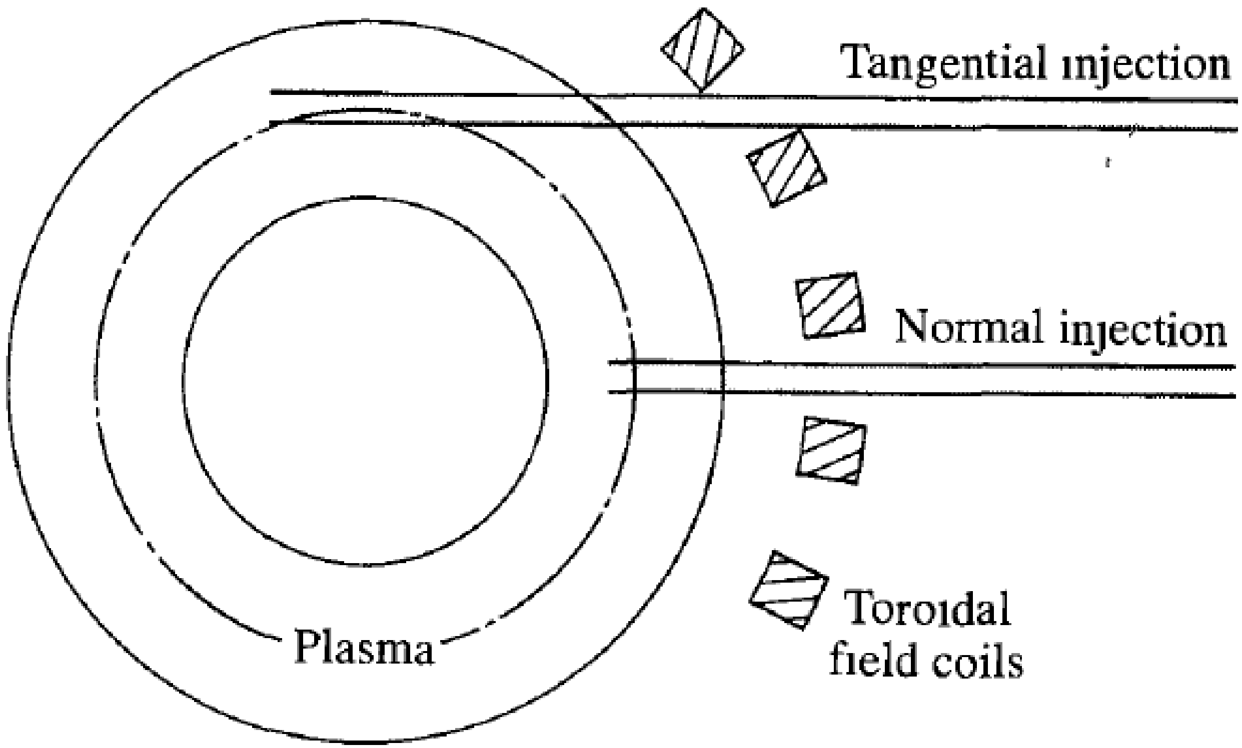
  $H_b + e^- \rightarrow H_b^+ + 2e^-$

b: beam
p: plasma

John Wesson, Tokamaks 3rd edition

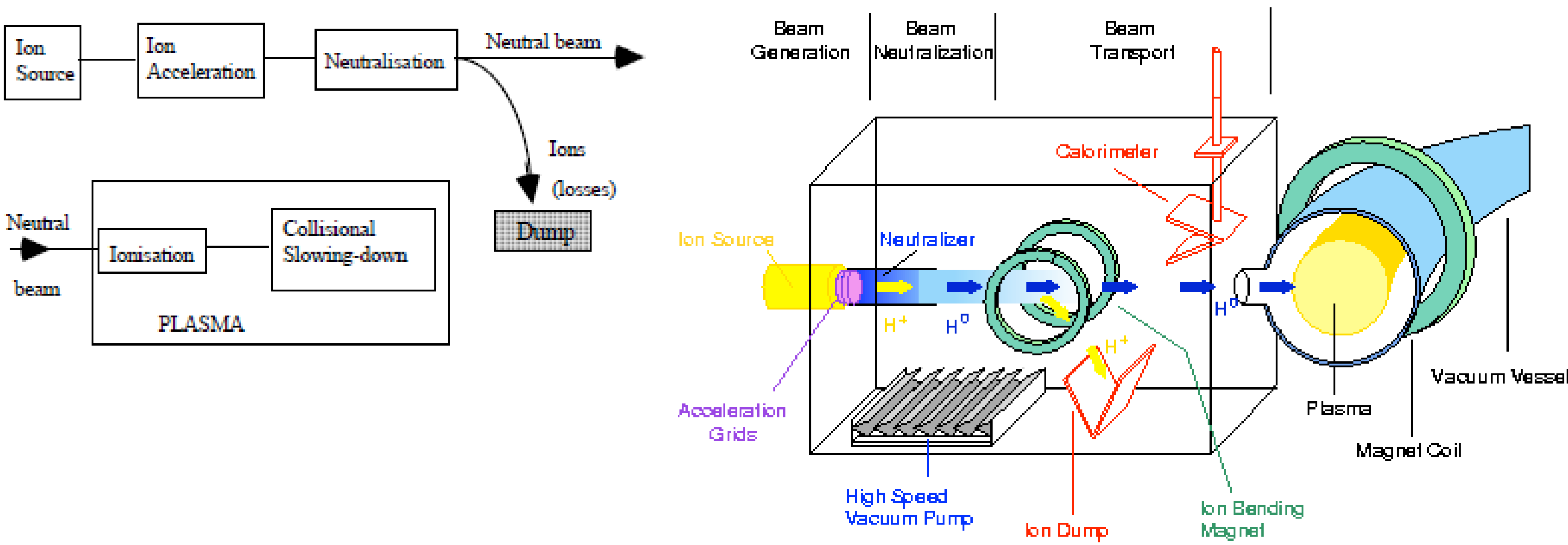
Neutral beam absorption length increases with tangential injection

- **It is more difficult to access through the toroidal field coils with tangential injection.**



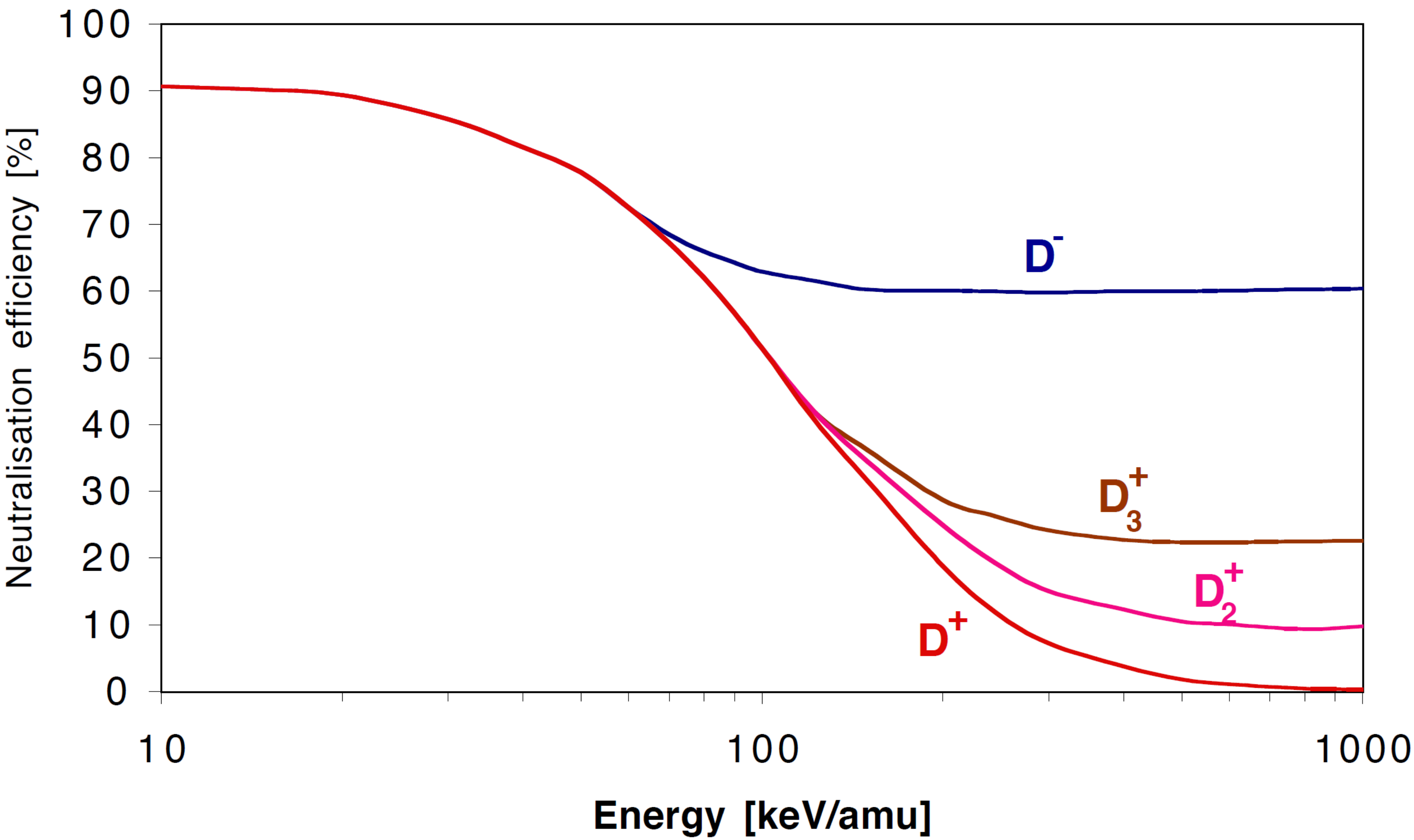
John Wesson, Tokamaks 3rd edition

Neutral particles heat the plasma via coulomb collisions



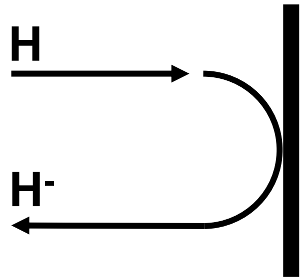
1. create energetic (fast) neutral ions
2. ionize the neutral particles
3. heat the plasma (electrons and ions) via Coulomb collisions

Negative ion source is preferred due to higher neutralization efficiency



There are two ways to make negative ions – surface and volume production

- **Surface production, depends on :**
  - **Work function Φ**
  - **Electron affinity level, 0.75 eV for H^-**
  - **Perpendicular velocity**
  - **Work function can be reduced by covering the metal surface with cesium**



$$H + e^- \longrightarrow H^-$$

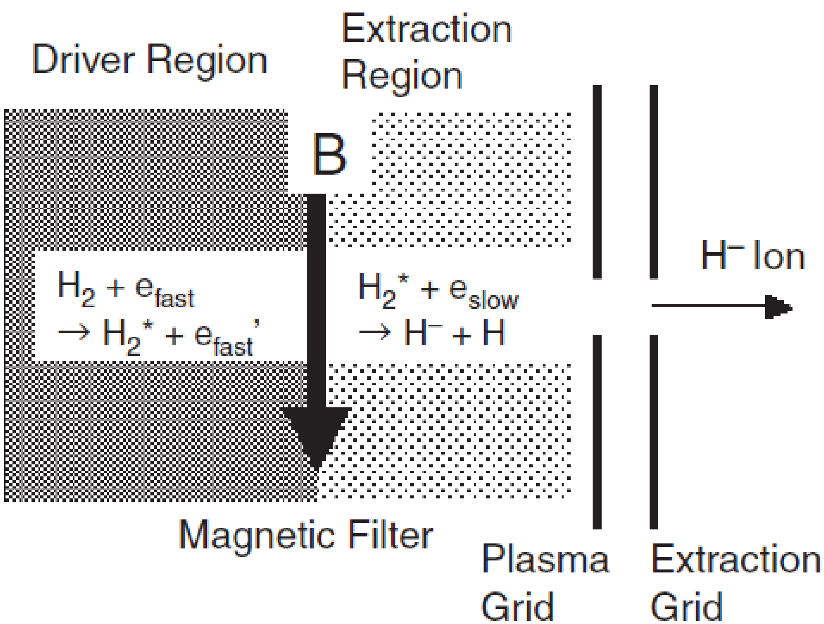
$$H^+ + 2e^- \longrightarrow H^-$$

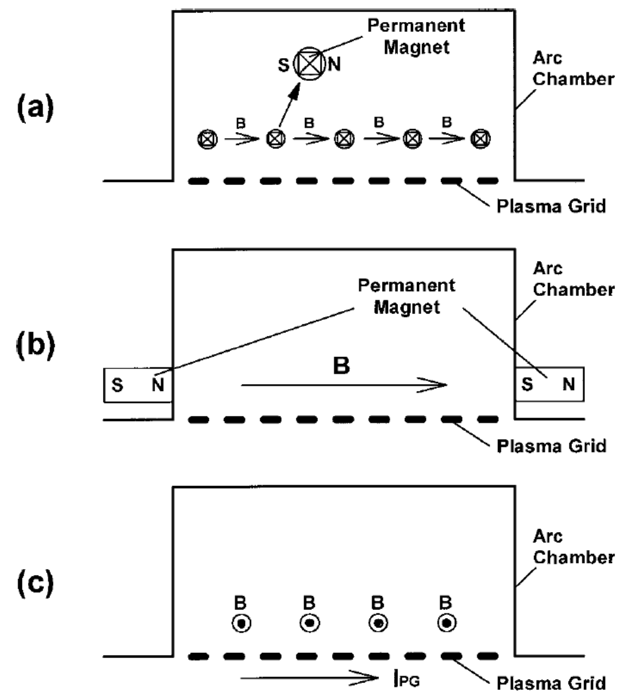
- **Volume production:**

$$H_2 + e_{fast}(>20\ \mathrm{eV}) \rightarrow H_2^*(\text{excited state}) + e_{fast},$$

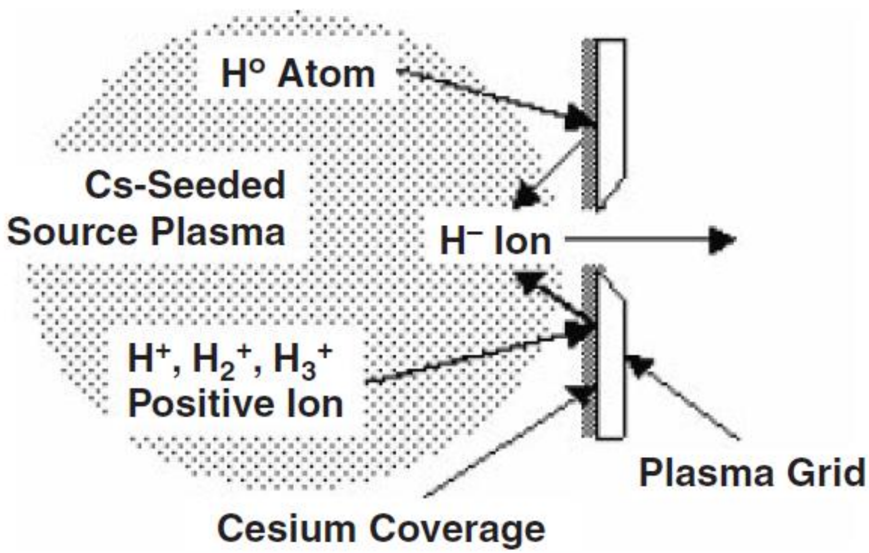
$$H_2^*(\text{excited state}) + e_{slow}(\approx 1\ \mathrm{eV}) \rightarrow H^- + H.$$

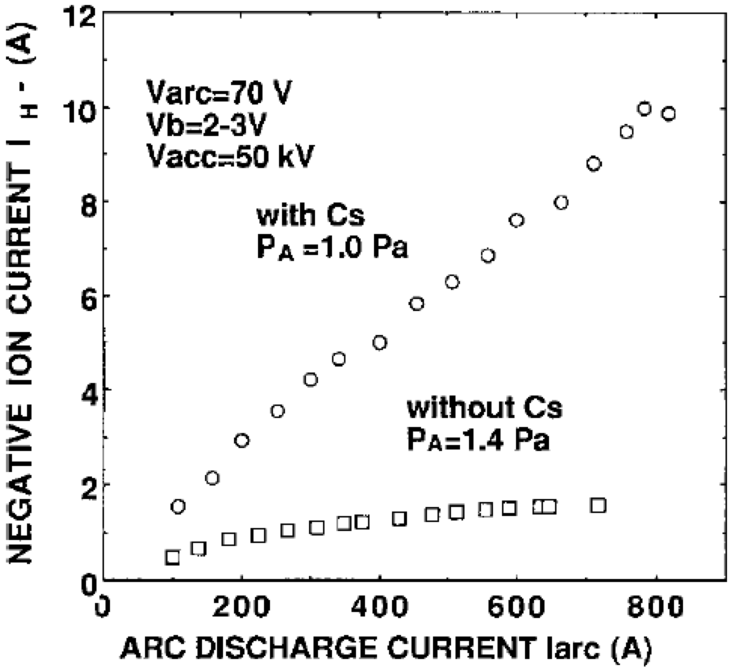
Two-chamber method of negative ions in volume production with a magnetic filter



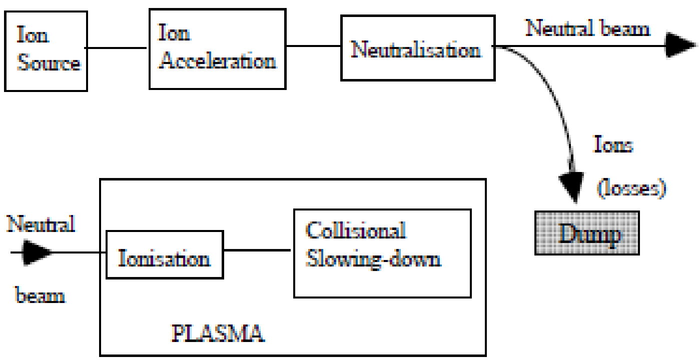


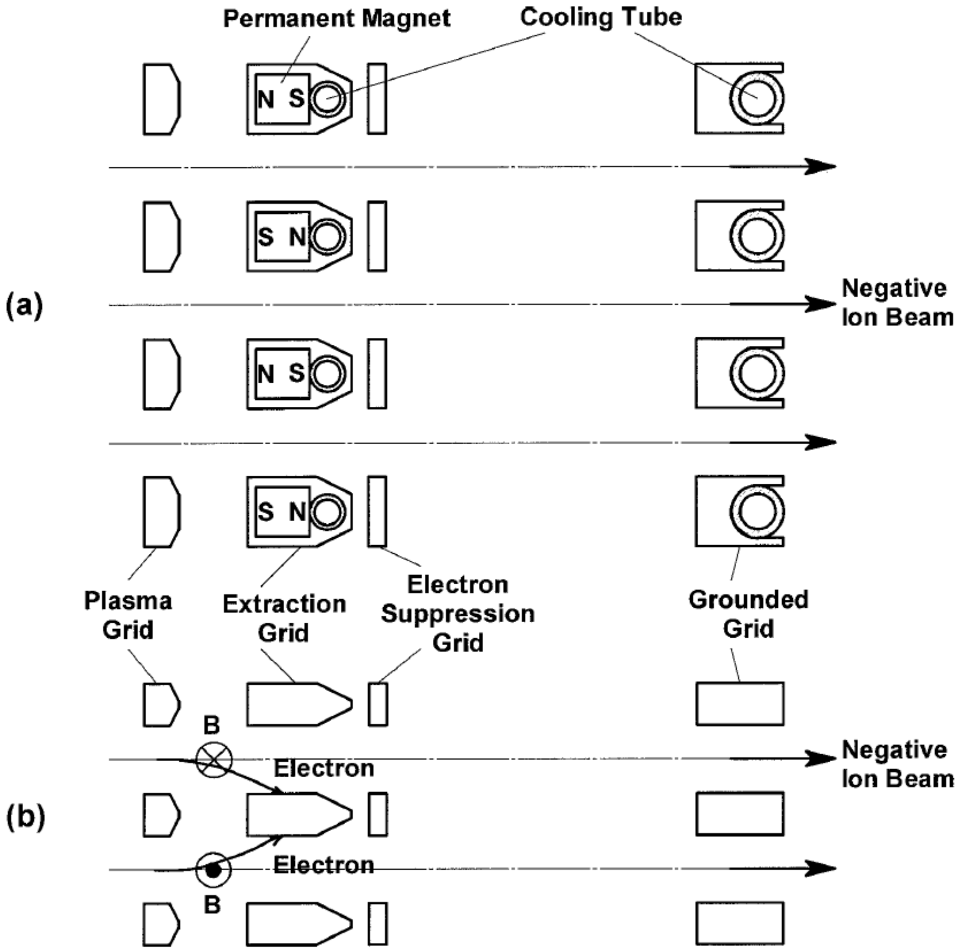
Adding cesium increases negative ion current





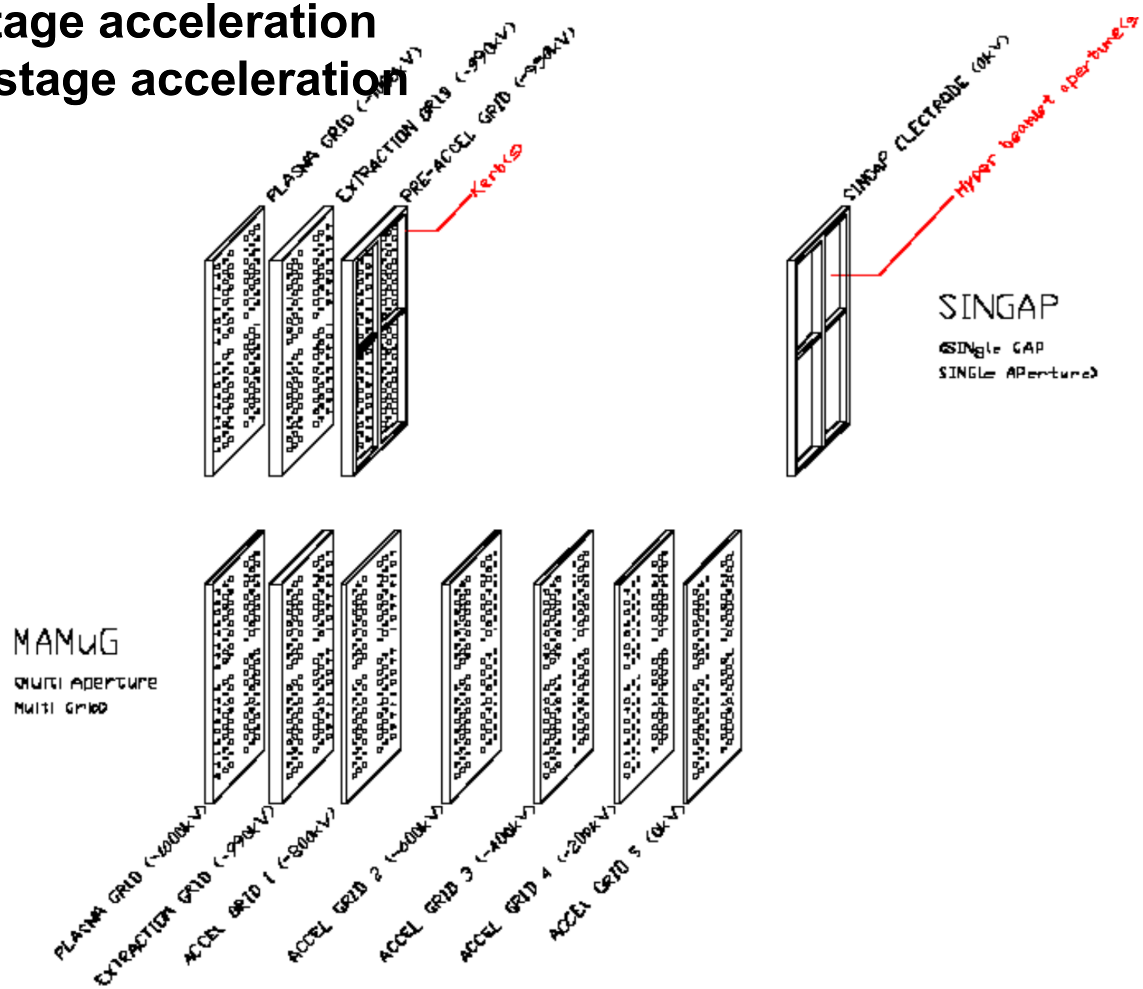
Electrons need to be filtered out since they are extracted together with negative ions





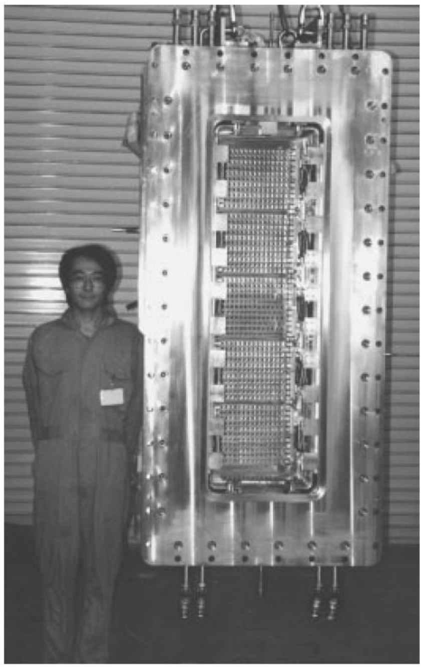
Acceleration

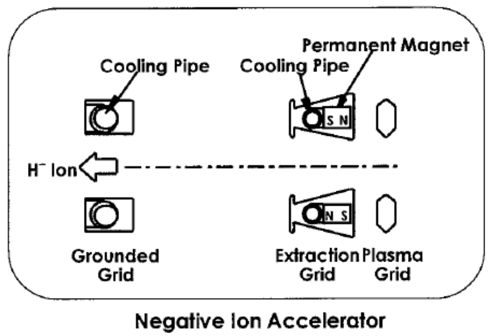
- **Multi-stage acceleration**
- **Single-stage acceleration**

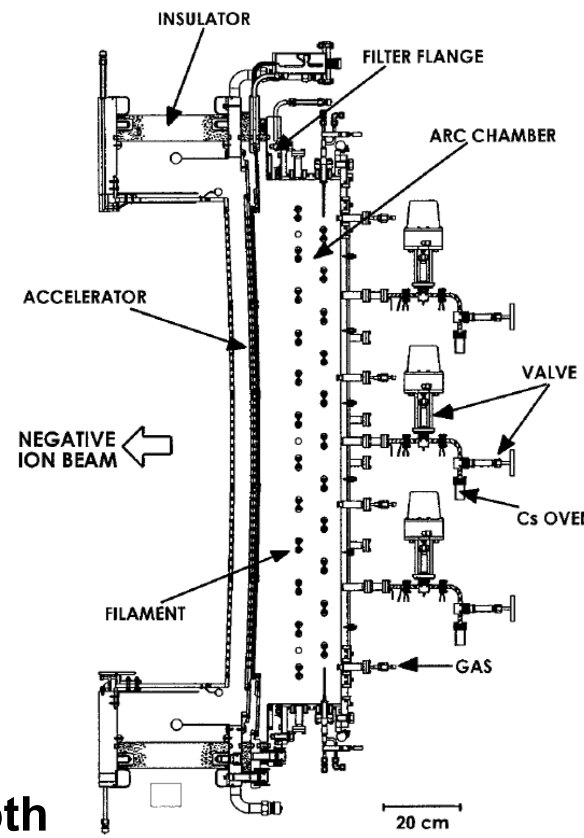


The ITER neutral beam system: status of the project and review of the main technological issues, presented by V. Antoni

NBI system of the LHD fusion machine





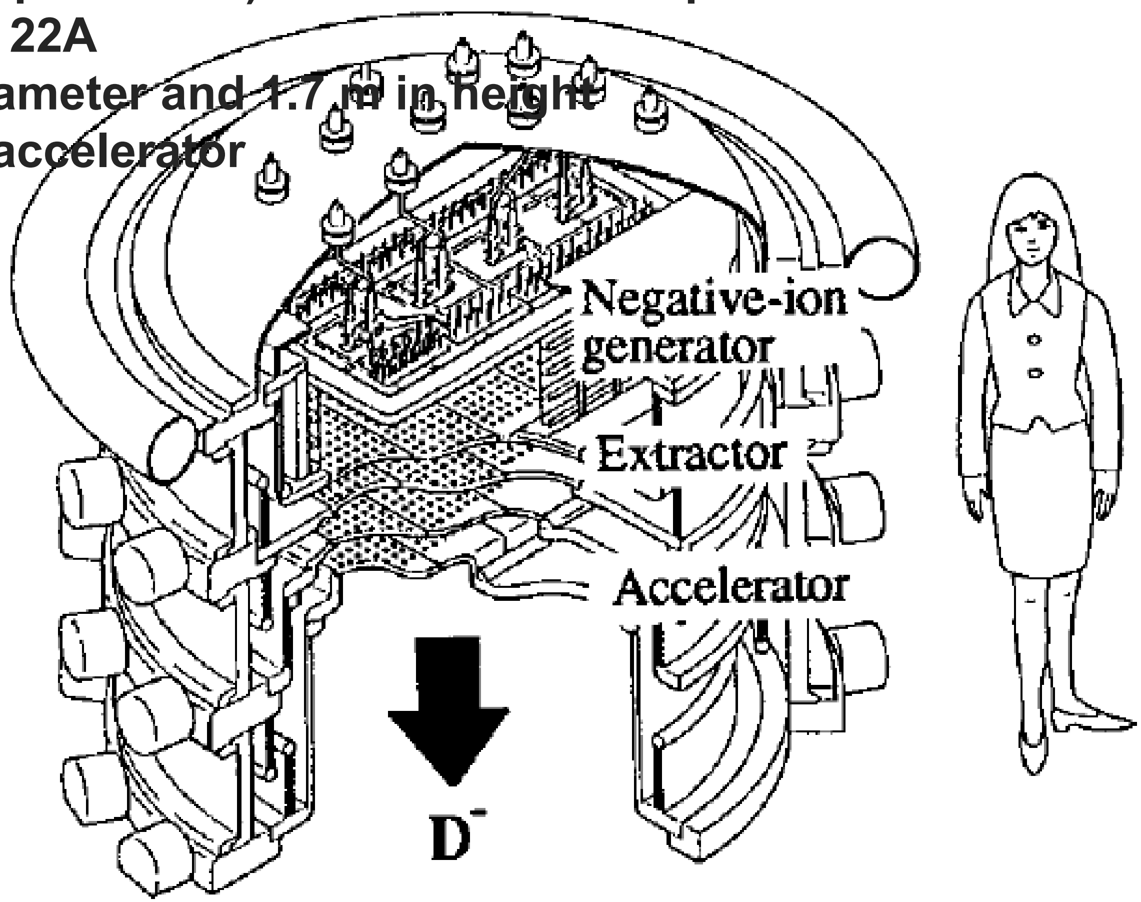


- 180 keV and 30 A
- Arc chamber: 35 cm x 145 cm, 21cm in depth
- Single stage accelerator

Y. Takeiri, etc., Nucl. Fusion, 46, S199, 2006

JT60U NBI system

- JT-60 (Japan-Torus) is a tokamak in Japan.
- 550 keV, 22A
- 2m in diameter and 1.7 m in height
- 3-stage accelerator



Neutralization

- **Gas neutralization**
  - **Collisions between fast negative ions and atoms**

$$H^- + H_2 \longrightarrow H + H_2 + e^-$$

  - **Fast ions can lose another electron after neutralized**

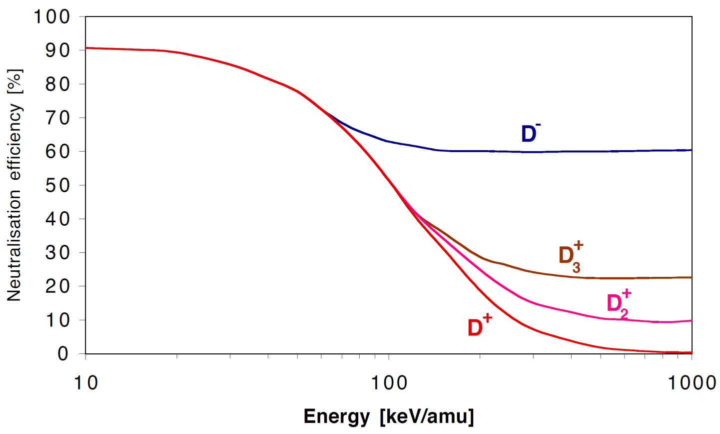
$$H + H_2 \longrightarrow H^+ + H_2 + e^-$$

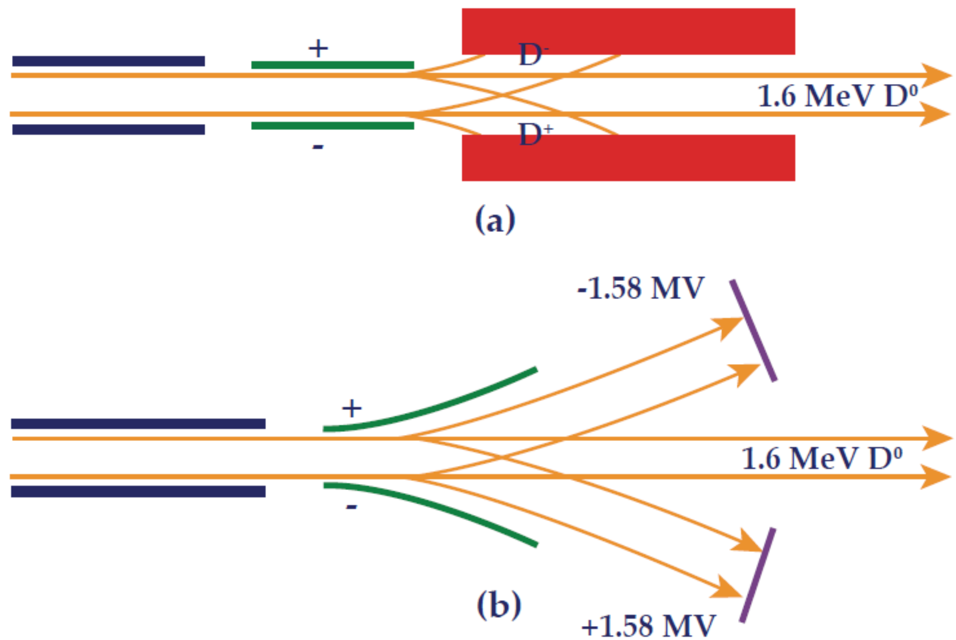
- **Plasma neutralization**
  - **Collisions with charged particles in plasma**

$$H^- + X\left(e, \mathrm{Ar}, H^+, {H_2}^+\right) \longrightarrow H + X + e^-$$

  - **The efficiencies reach up to 85% for fully ionized hydrogen plasma**

Beam dump

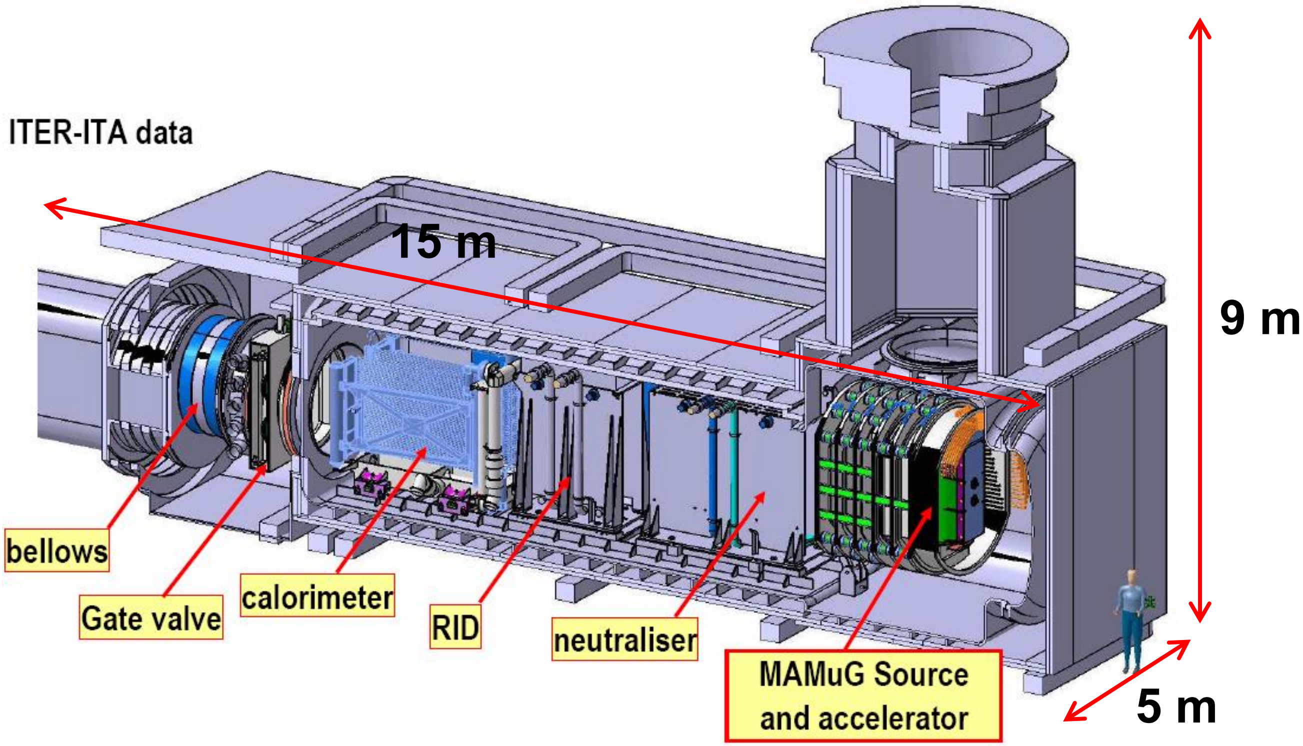




R. W. Moir, Fusion Technology, 25, 129, 1994

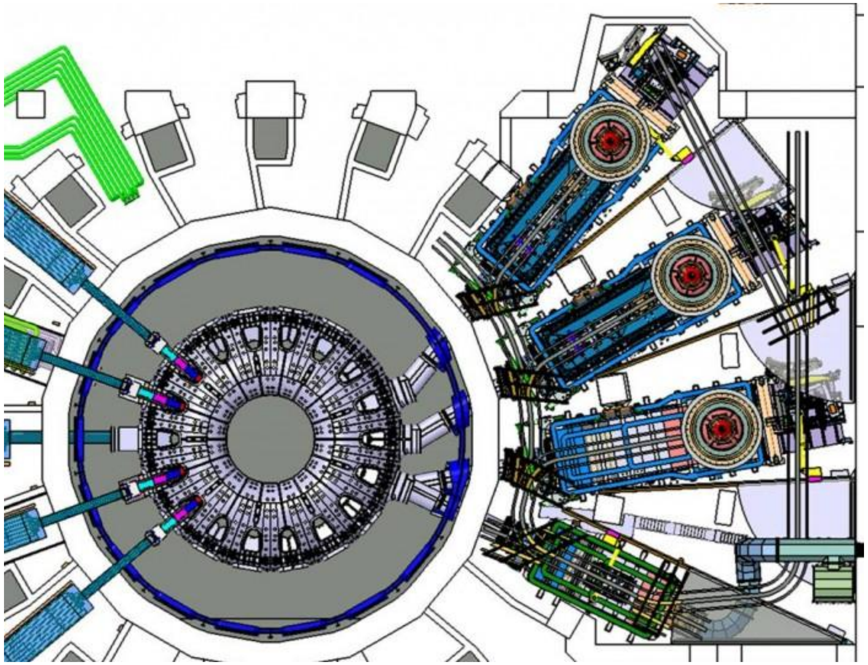
NBI for ITER

- **beam components (Ion Source, Accelerator, Neutralizer, Residual Ion Dump and Calorimeter)**
- **other components (cryo-pump, vessels, fast shutter, duct, magnetic shielding, and residual magnetic field compensating coils)**



The ITER neutral beam system: status of the project and review of the main technological issues, presented by V. Antoni

Neutral beam penetration



- Parallel direction
  - Longest path through the densest part of the plasma
  - Harder to be built
- Perpendicular direction
  - Path is short
  - Larger perpendicular energies leads to larger losses
  - Easier to be built

https://www.iter.org