

# Introduction to Nuclear Fusion as An Energy Source

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**Po-Yu Chang**

**Institute of Space and Plasma Sciences, National Cheng Kung University**

**Lecture 7**

**2024 spring semester**

**Wednesday 9:10-12:00**

**Materials:**

**<https://capst.ncku.edu.tw/PGS/index.php/teaching/>**

**Online courses:**

**<https://nckucc.webex.com/nckucc/j.php?MTID=ma76b50f97b1c6d72db61de9eaa9f0b27>**

# Note

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- **課程公告:Tokamak Physics**
- **講者:陳騷 院士**
- **課程日期:2023/4/23 至 6 月底**
- **課程時間:每周二與周四,上午 10 點至 12 點**
- **課程地點:科學四館 203 教室**
- **報名方式:無須報名,於上課時間前往 203 教室即可**
- **相關資訊:此次課程延續 2023 年 10 月至 12 月的內容,上課影片可以至下列**
- **Youtube 連結觀看 [https://youtube.com/playlist?list=PLB7ffAhbDy-lcd4oWWcWsPpEmzGTfXmeZ&si=dgQVIX\\_GBxyYjTDM](https://youtube.com/playlist?list=PLB7ffAhbDy-lcd4oWWcWsPpEmzGTfXmeZ&si=dgQVIX_GBxyYjTDM)**

# Course Outline

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- **Magnetic confinement fusion (MCF)**
  - Gyro motion, MHD
  - 1D equilibrium (z pinch, theta pinch)
  - Drift: ExB drift, grad B drift, and curvature B drift
  - Tokamak, Stellarator (toroidal field, poloidal field)
  - Magnetic flux surface
  - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
  - Stability (Kink instability, sausage instability, Safety factor Q)
  - Central-solenoid (CS) start-up (discharge) and current drive
  - CS-free current drive: electron cyclotron current drive, bootstrap current.
  - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

# When forces are balanced, the system is in the equilibrium state, or called “Magnetohydrostatics”



- Equilibrium state:

$$\rho_m \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p \equiv 0$$

$$\vec{j} \times \vec{B} = \nabla p$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = \frac{1}{\mu_0} \left[ (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla B^2 \right] = \nabla p$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \left( p + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}$$

Magnetic  
pressure

Magnetic  
tension

← Forces caused by  
curvature of the field lines

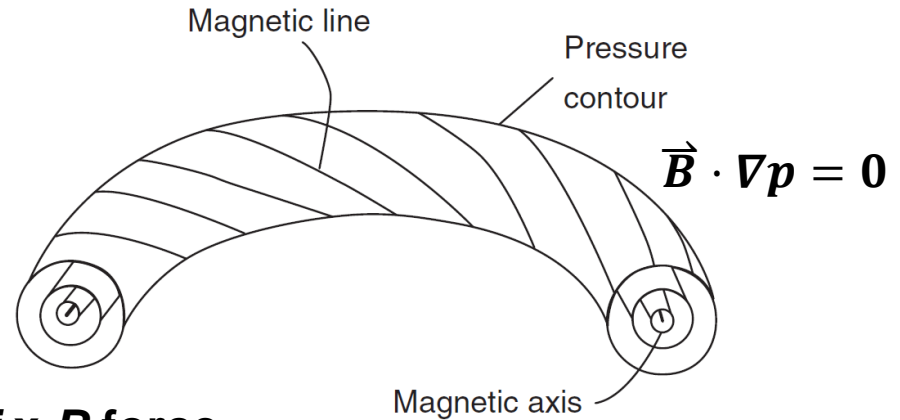
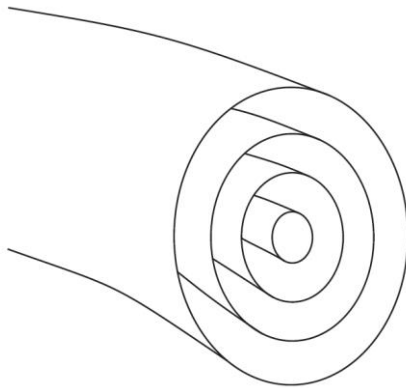
$$\vec{j} \perp \nabla p \quad \vec{B} \perp \nabla p \quad \Rightarrow \quad \vec{j} \cdot \nabla p = 0 \quad \vec{B} \cdot \nabla p = 0$$

- The surfaces with  $p = \text{constant}$  are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).

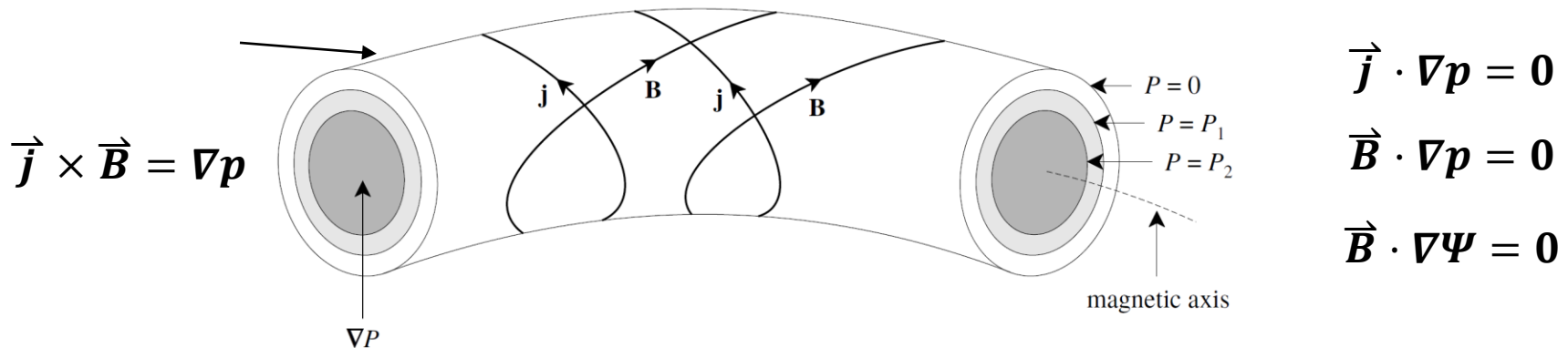
# Magnetic lines lying on pressure contour



- Contours of constant pressure
- Magnetic lines lying on pressure contour



- Pressure gradient is balanced by the  $\vec{j} \times \vec{B}$  force

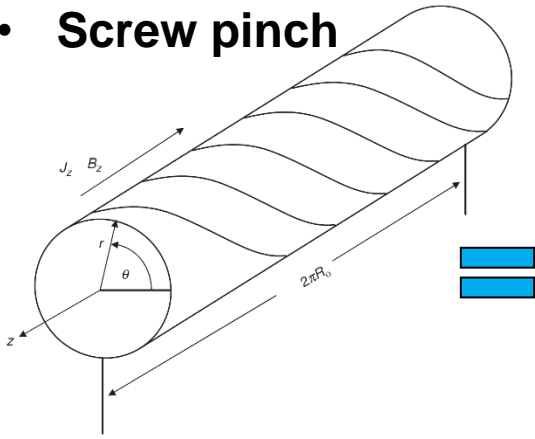


- A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to  $\vec{B}$ .

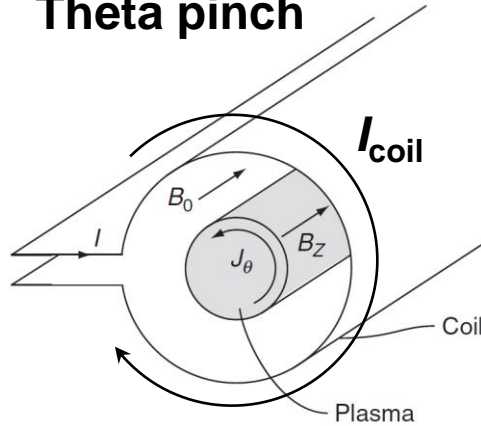
# General screw pinch is flexible with varies range of $\beta$



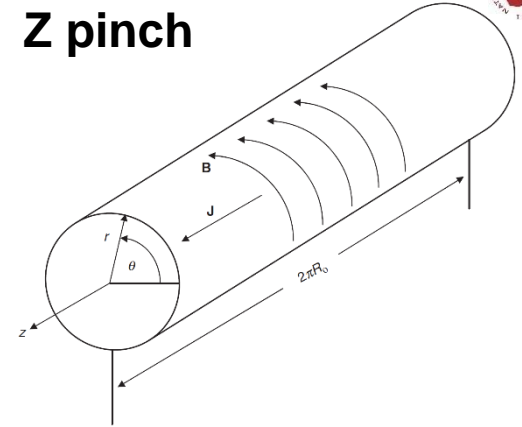
• Screw pinch



• Theta pinch



• Z pinch



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\vec{B} = B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta}$$

$$\vec{B} = B_\theta \hat{\theta}$$

$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

$$p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$\frac{d}{dr} \left( p + \frac{B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

$$\int_0^a \pi r^2 dr \left[ \frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} \right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_0} + \frac{1}{2\mu_0} (B_0^2 - \langle B_z^2 \rangle)$$

$$\beta_t = \frac{2\mu_0 \langle p \rangle}{B_0^2}$$

$$\beta_p = \frac{2\mu_0 \langle p \rangle}{B_{\theta a}^2}$$

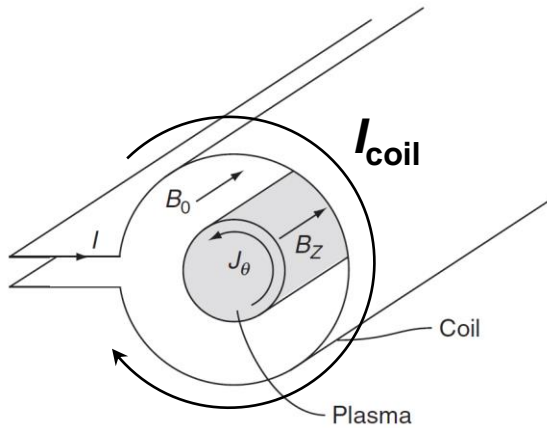
$$\beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_0 \langle p \rangle}{B_0^2 + B_{\theta a}^2}$$

$$0 \leq \langle \beta \rangle \leq 1$$

# Theta pinch is stable while z pinch is unstable



- **Theta pinch**

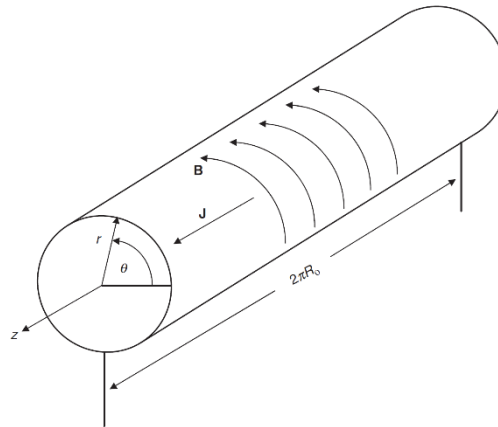


$$\vec{B} = B_z \hat{z}$$

$$q_\theta = \infty$$

**Stable**

- **Z pinch**



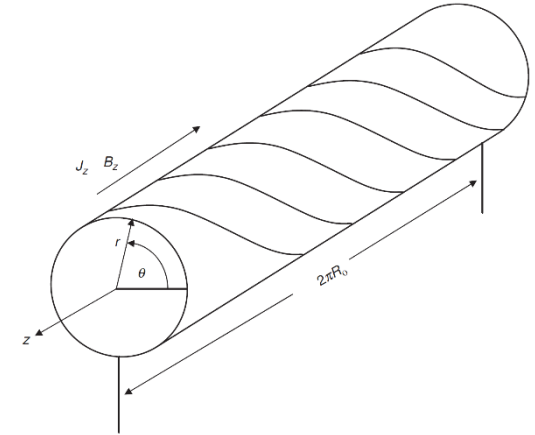
$$\vec{B} = B_\theta \hat{\theta}$$

$$q_z = 0$$

**Unstable**

$$q(r) = \frac{r B_z(r)}{R_0 B_\theta(r)}$$

- **Screw pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

**q can be controlled.**

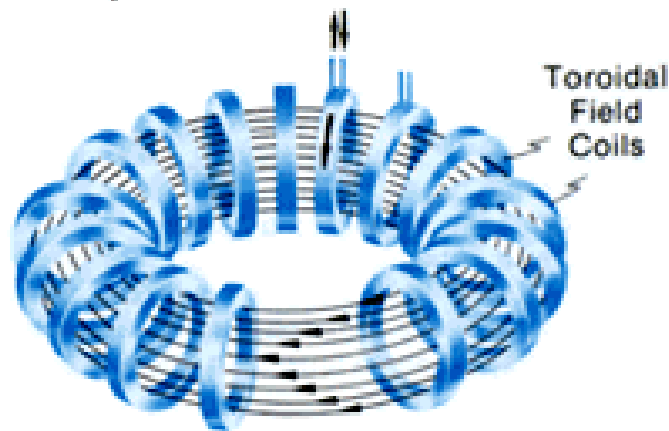
**Stable/Unstable**

# Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field



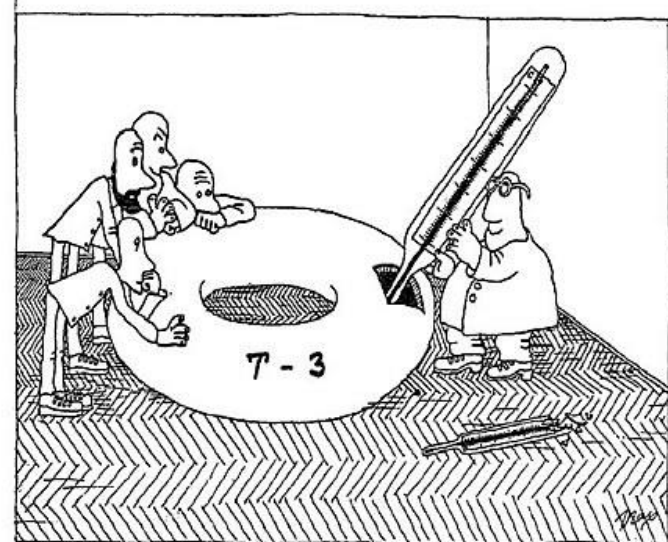
- Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)

Relatively Constant Electric Current



Nature

Constant Toroidal Field



Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

N. J. PEACOCK  
D. C. ROBINSON  
M. J. FORREST  
P. D. WILCOCK  
UKAEA Research Group,  
Culham Laboratory,  
Abingdon, Berkshire

V. V. SANNIKOV  
I. V. Kurchatov Institute,  
Moscow

$$T_e = 100 \sim 1 \text{ keV}$$
$$n_e = 1\text{-}3 \times 10^{13} \text{ cm}^{-3}$$

Electron temperatures of 100 eV up to 1 keV and densities in the range  $1\text{-}3 \times 10^{13} \text{ cm}^{-3}$  have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

<https://www.iter.org/mach/tokamak>

[https://en.wikipedia.org/wiki/Tokamak#cite\\_ref-4](https://en.wikipedia.org/wiki/Tokamak#cite_ref-4)

Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.

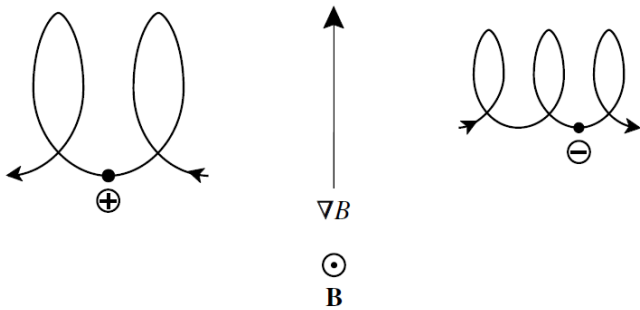
N. J. Peacock, et al., Nature **224**, 488 (1969)



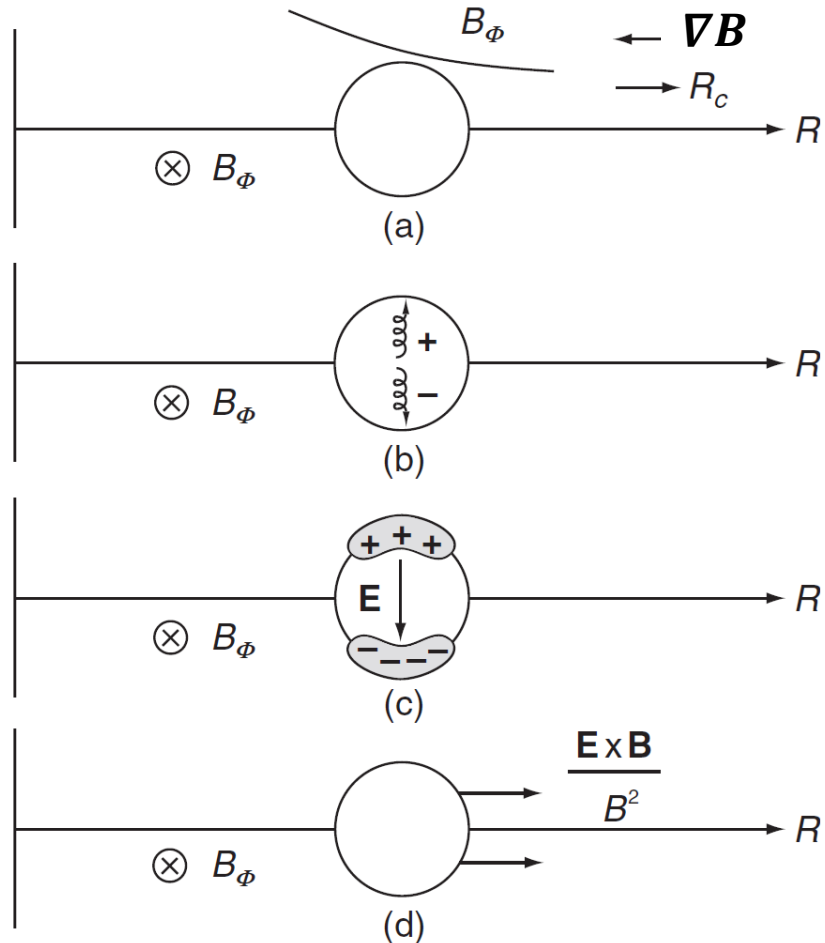
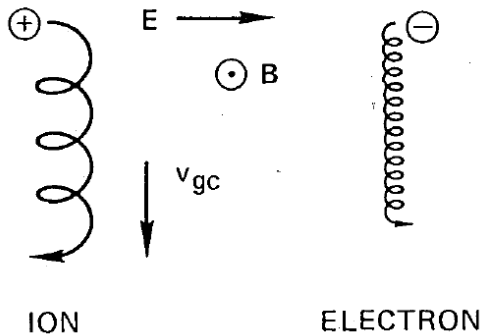
# Charged particles drift across field lines



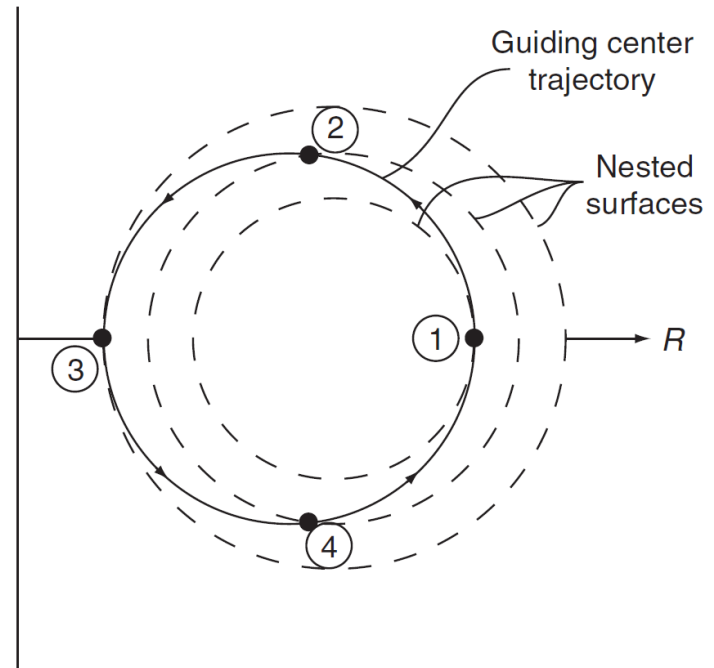
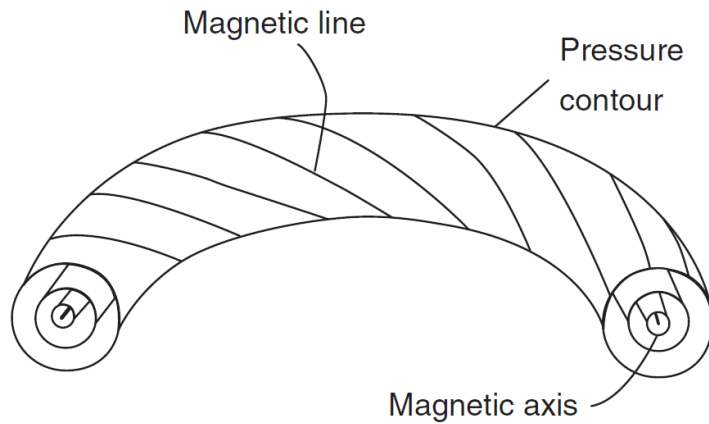
- Grad-B drift**



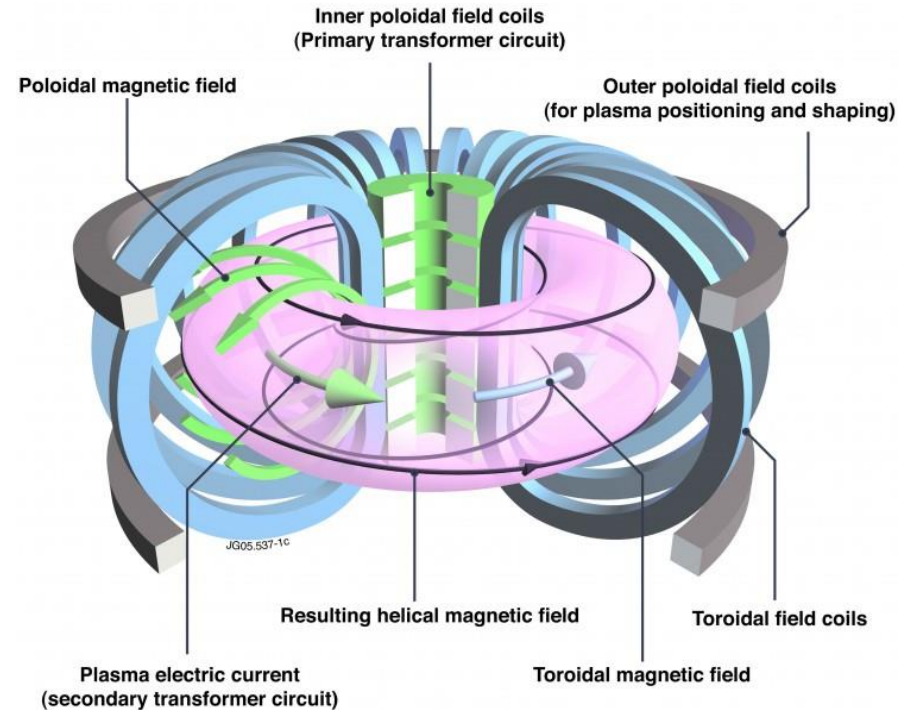
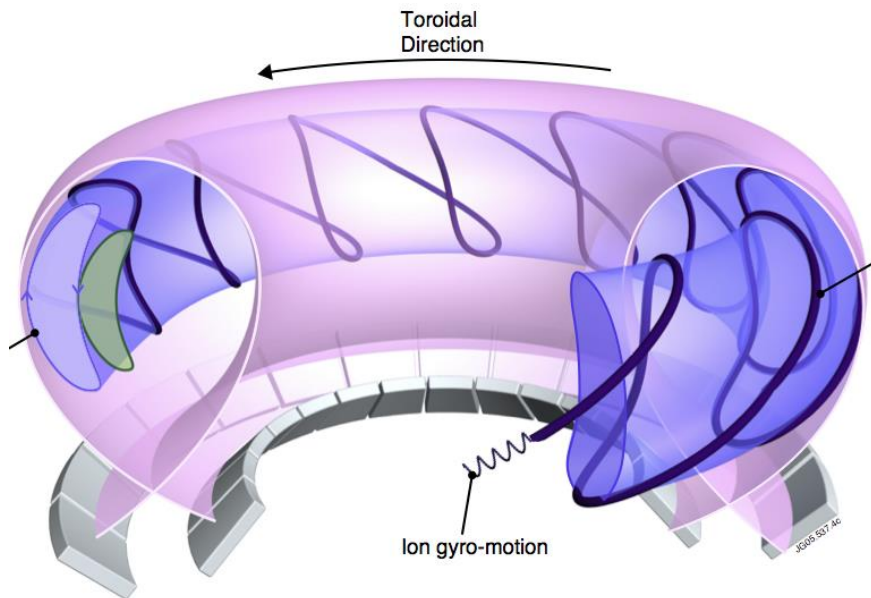
- ExB drift**



# The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field



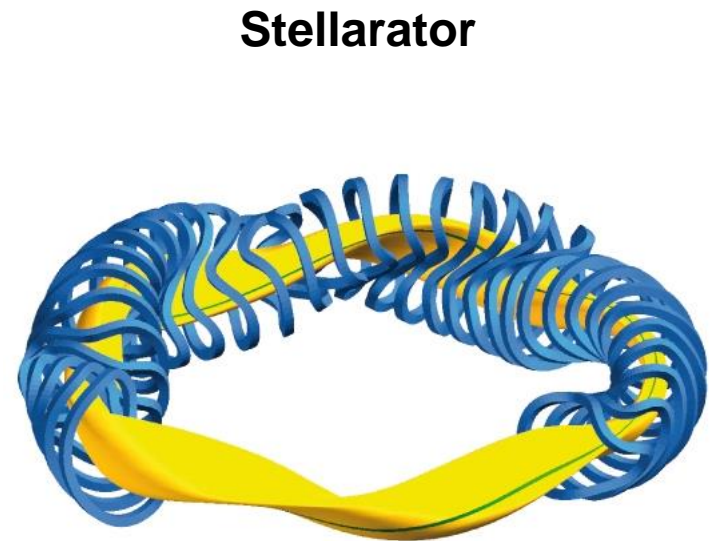
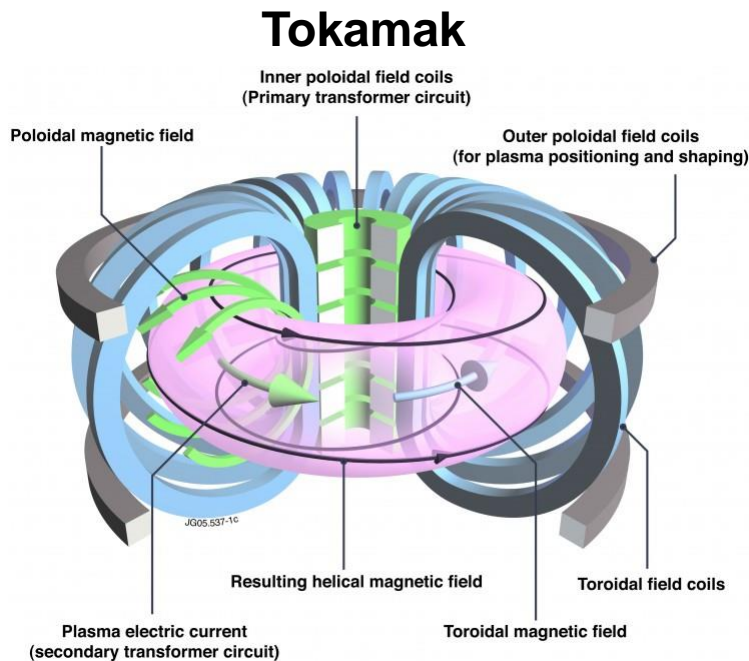
# A poloidal magnetic field is required to reduce the drift across field lines



# A poloidal magnetic field is required to reduce the drift across field lines



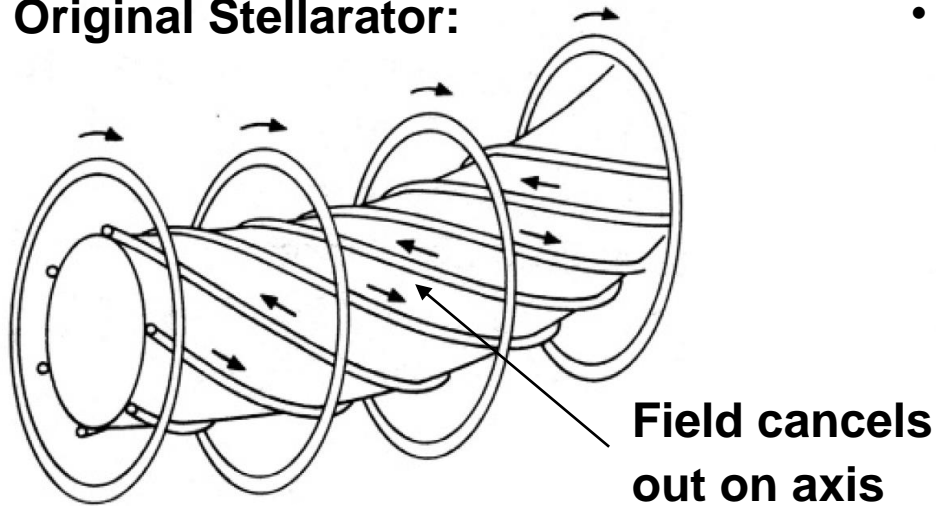
# Stellarator uses twisted coil to generate poloidal magnetic field



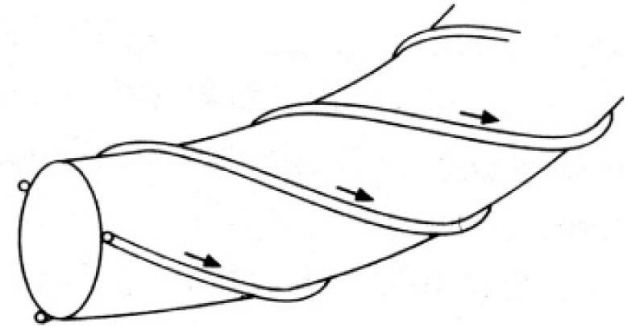
# Different types of stellarators



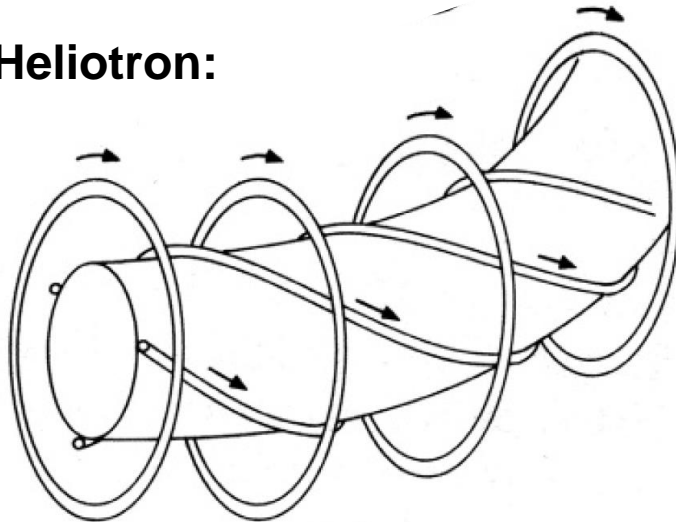
- **Original Stellarator:**



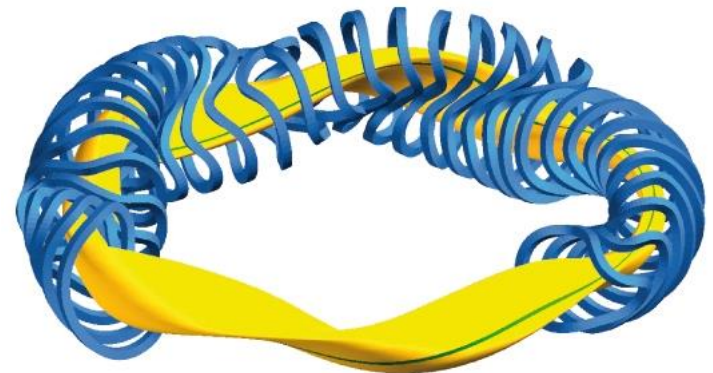
- **Torsatron:**



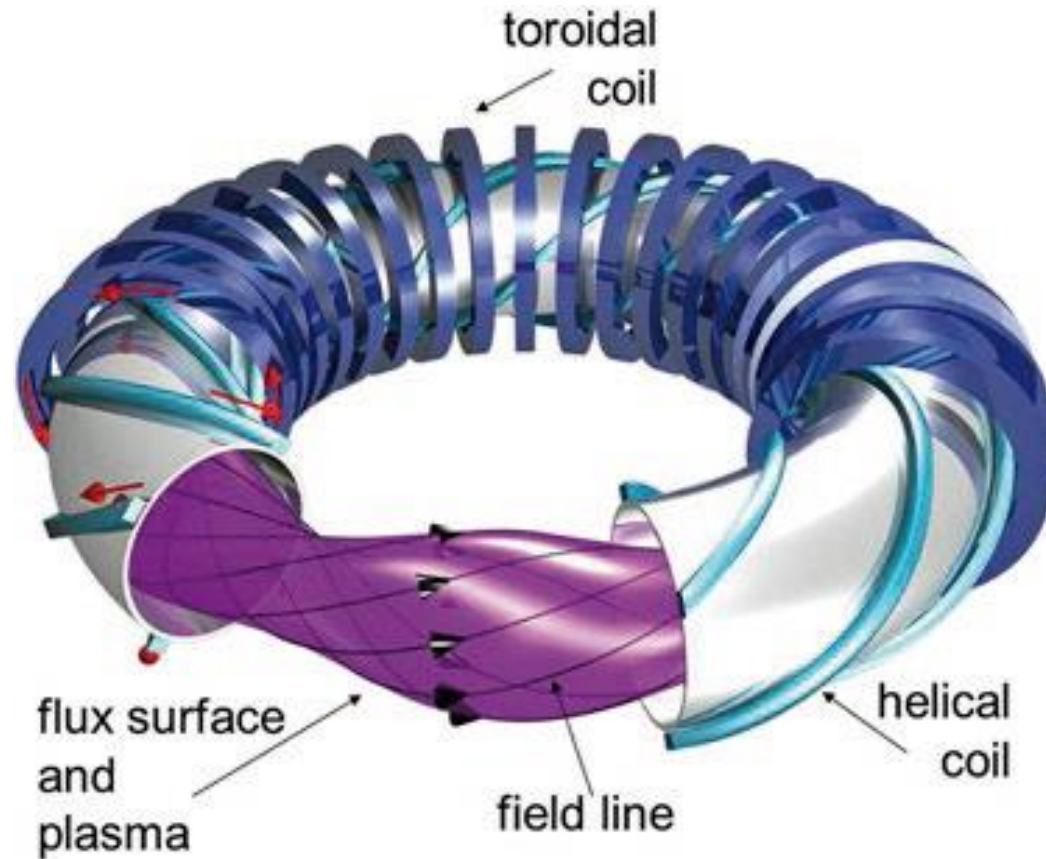
- **Heliotron:**



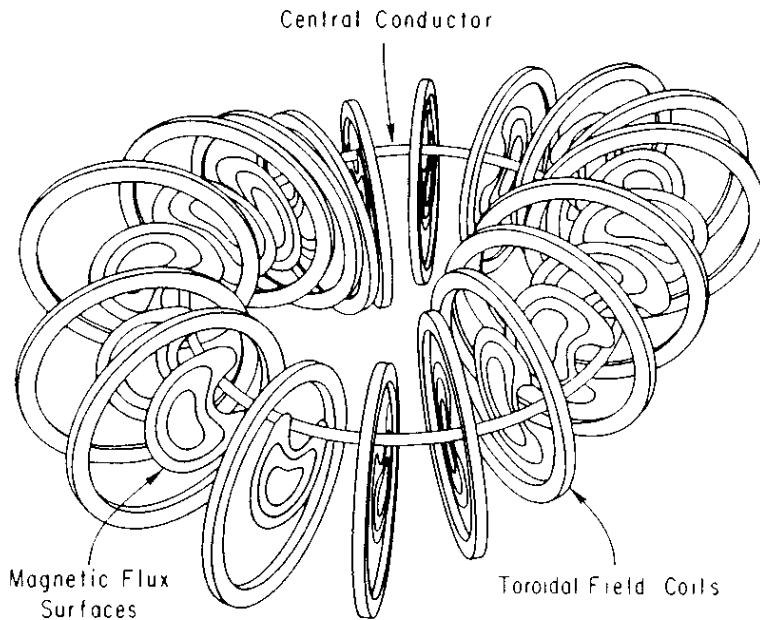
- **Helias:**



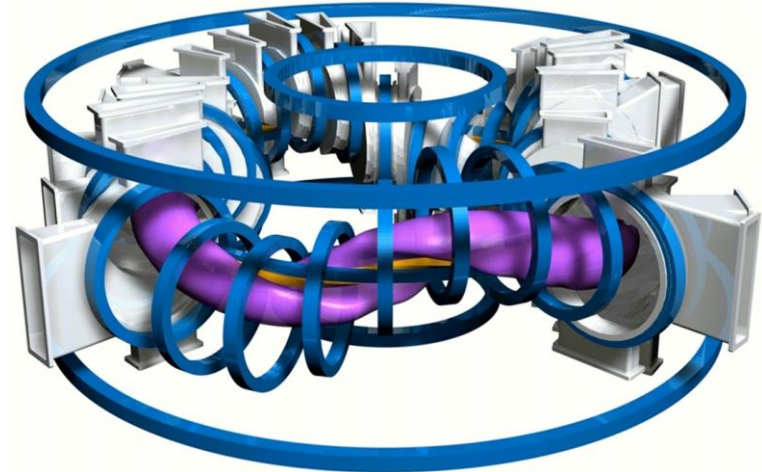
# Twisted magnetic field lines can be provided by toroidal coils with helical coils



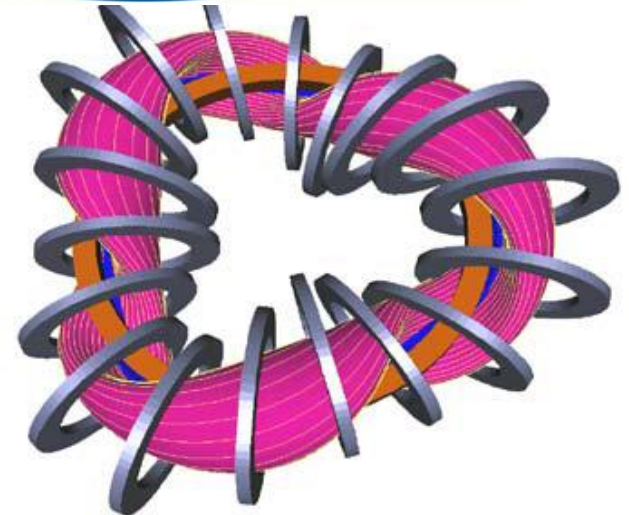
# Heliac (Helical Axis stellarator)



- **TJ-II (Spain's National Fusion Laboratory):**



- **H-1 (Australian Plasma Fusion Research Facility):**



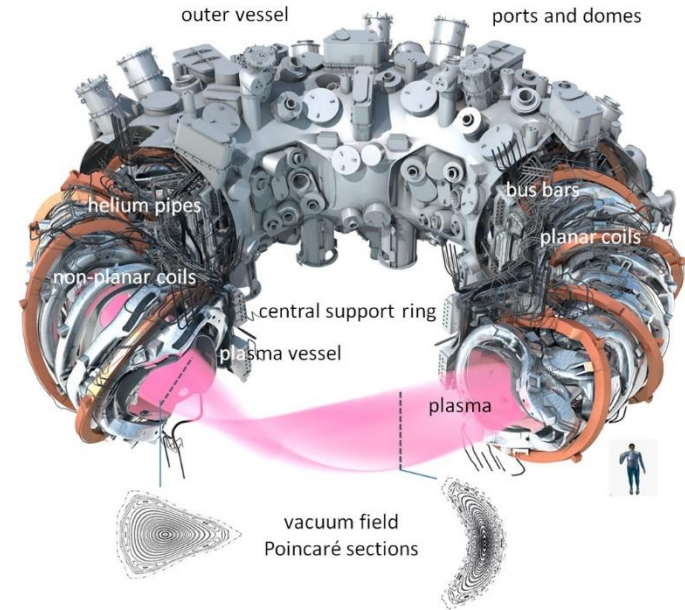
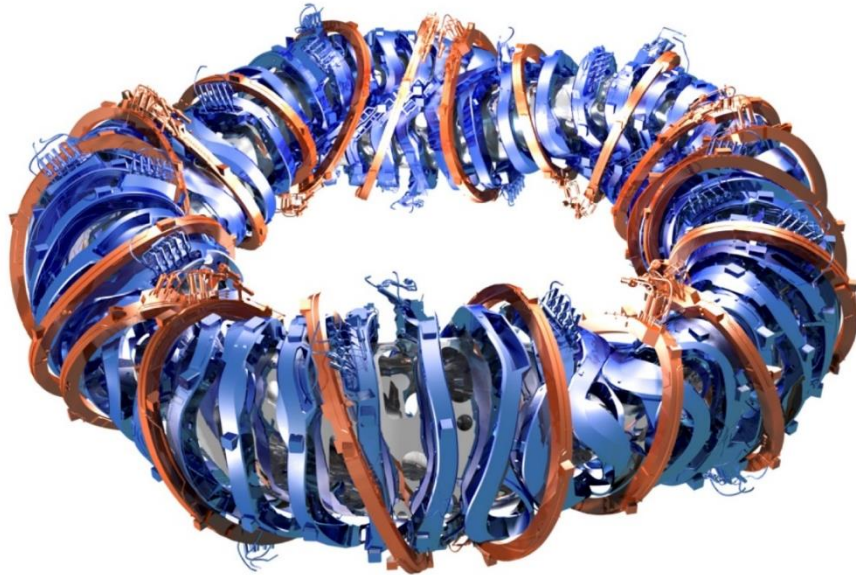
A. H. Boozer, Phys. Plasmas, 5, 1647 (1998)

<https://wiki.fusion.ciemat.es/wiki/TJ-II>

B. D. Blackwell, et. al, 23rd IAEA Fusion Energy Conference, 2010



# Wendelstein 7-X is a (Helias) stellarator built by Max Planck Institute for Plasma Physics (IPP)



- **Wendelstein 7-x is now installing new diverters.**



# Advantages of Stellarator

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- **No need to drive plasma current. It is intrinsically steady state.**
- **With zero net current, one potentially dangerous class of MHD instabilities, the current-driven kink modes, is eliminated.**
- **Magnetic configuration is set by external coils, not by currents in the plasma. Stellarators do not suffer violent disruptions.**
- **Potential for greater range of designs and optimization of fusion performance.**

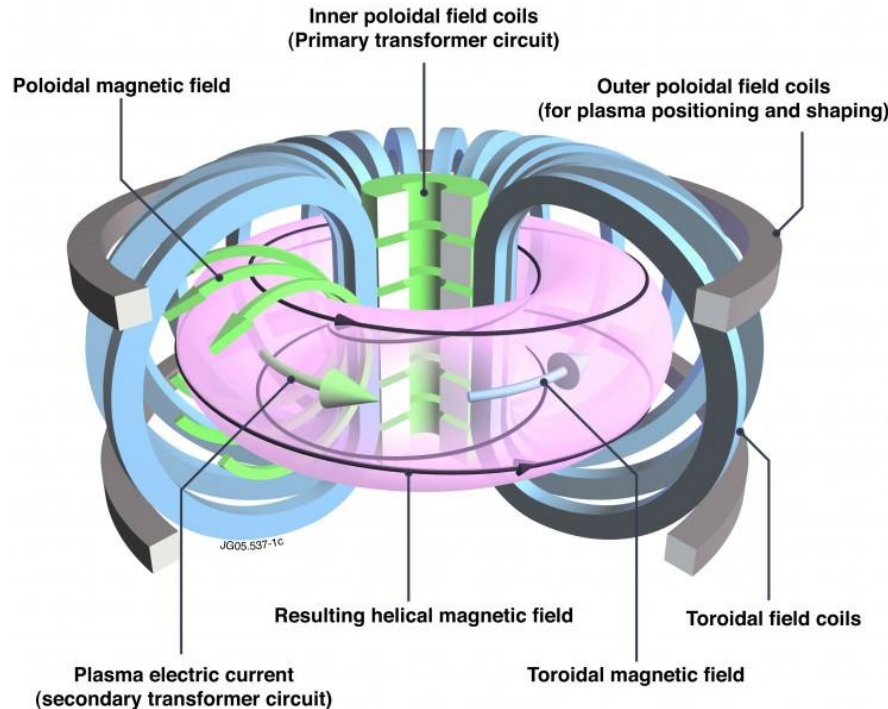
# Disadvantages of Stellarator

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- **Complicated coil configurations. It's difficult to design. The precision requirement is high. It is expensive to build coils for stellarators.**
- **Achieving good particle confinement in stellarators is more difficult than that in tokamaks.**
- **Divertors and heat load geometry in stellarators is more complicated than those in tokamaks.**

# 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \perp \nabla p \quad \vec{B} \perp \nabla p$$



$$\vec{j} \cdot \nabla p = 0 \quad \vec{B} \cdot \nabla p = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \quad \Rightarrow \quad \nabla \cdot \vec{j} = 0$$

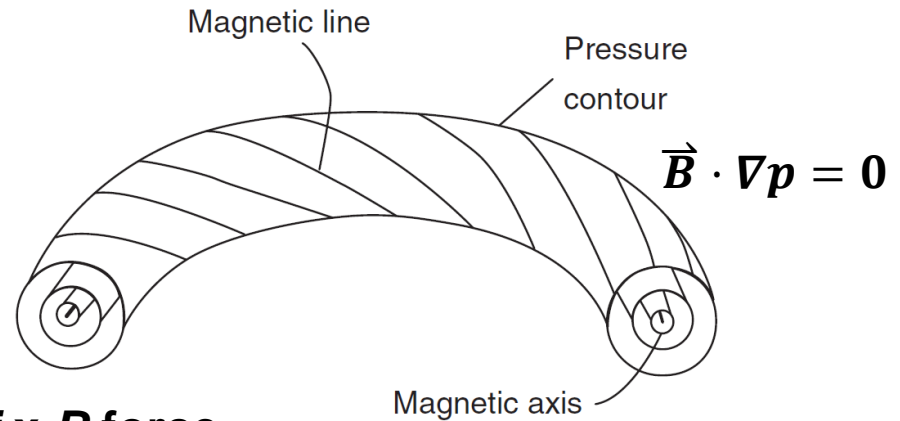
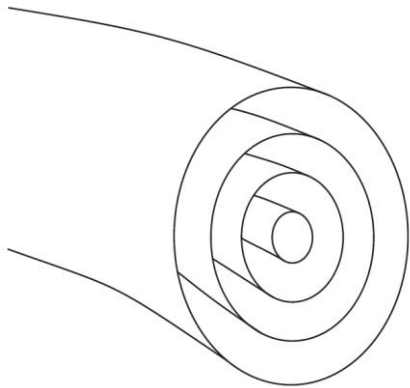
$$\nabla \cdot \vec{B} = 0$$

- The surfaces with  $p = \text{constant}$  are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).

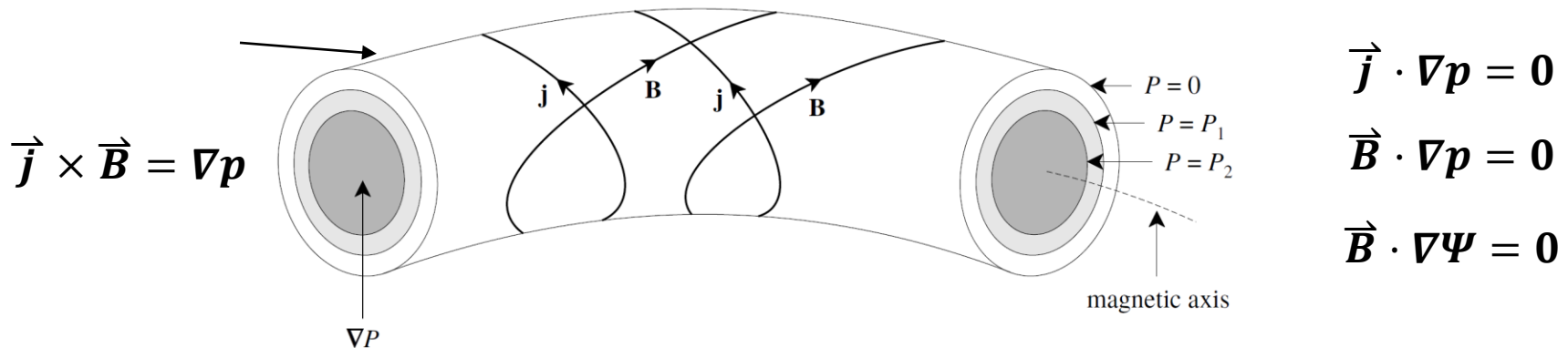
# Magnetic lines lying on pressure contour



- Contours of constant pressure
- Magnetic lines lying on pressure contour



- Pressure gradient is balanced by the  $\vec{j} \times \vec{B}$  force



- A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to  $\vec{B}$ .

# Derivation of Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p \quad \nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \cdot \vec{j} = 0 \quad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = (B_R, B_\phi, B_z) \quad \text{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R} \frac{\partial}{\partial R} (R B_R) + \frac{1}{R} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0$$

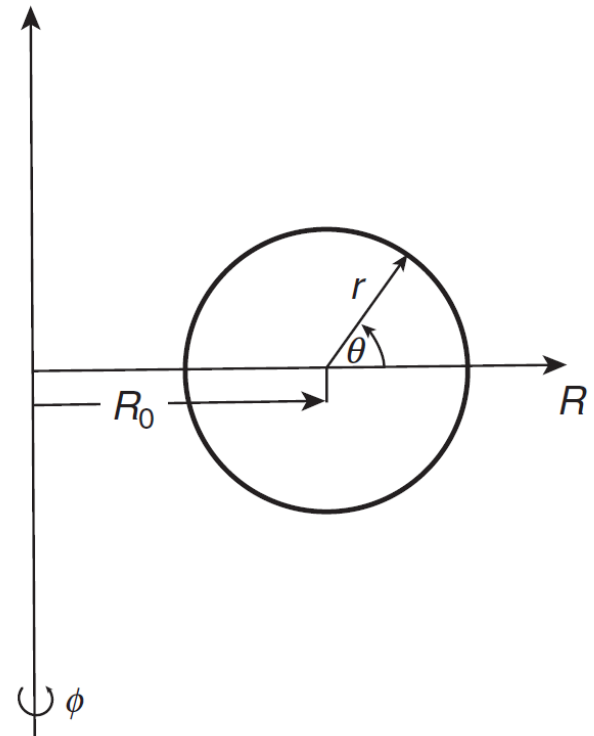
$$\frac{1}{R} \frac{\partial}{\partial R} (R B_R) + \frac{\partial B_z}{\partial z} = 0$$

- Represent the magnetic field using a vector potential  $A$ :

$$\vec{B} = \nabla \times \vec{A} = \hat{R} \left( \frac{1}{R} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\phi} \left( \frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial}{\partial R} (R A_\phi) - \frac{1}{R} \frac{\partial A_R}{\partial \phi} \right)$$

$$= \hat{R} \left( -\frac{\partial A_\phi}{\partial z} \right) + \hat{\phi} \left( \frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial}{\partial R} (R A_\phi) \right)$$

$$\equiv \hat{R} B_R + \hat{\phi} B_\phi + \hat{z} B_z \quad B_R = -\frac{\partial A_\phi}{\partial z} \quad B_z = \frac{1}{R} \frac{\partial}{\partial R} (R A_\phi)$$



# Pressure can be written as a function of flux



$$\frac{1}{R} \frac{\partial}{\partial R} (RB_R) + \frac{\partial B_z}{\partial z} = 0$$

$$B_R = -\frac{\partial A_\phi}{\partial z}$$

$$B_z = \frac{1}{R} \frac{\partial}{\partial R} (RA_\phi)$$

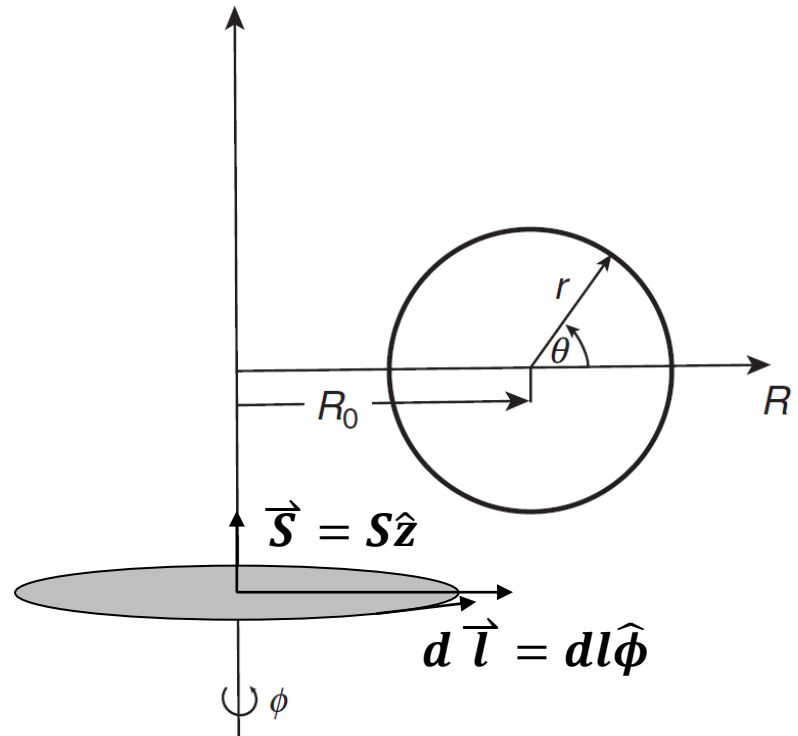
$$\psi \equiv \frac{1}{2\pi} \int \vec{B} \cdot d\vec{S} = \frac{1}{2\pi} \int (\nabla \times \vec{A}) \cdot d\vec{S}$$

$$= \frac{1}{2\pi} \int \vec{A} \cdot 2\pi R \cdot d\vec{l} = \int \vec{A} \cdot \hat{\phi} R dl = RA_\phi$$

$$B_R = -\frac{1}{R} \frac{\partial \psi}{\partial z} \quad B_z = \frac{1}{R} \frac{\partial \psi}{\partial R}$$

$$\vec{B} \cdot \nabla \psi = B_R \frac{\partial \psi}{\partial R} + B_\phi \frac{1}{R} \frac{\partial \psi}{\partial \phi} + B_z \frac{\partial \psi}{\partial z} = B_R \frac{\partial \psi}{\partial R} + B_z \frac{\partial \psi}{\partial z}$$

$$= \left( -\frac{1}{R} \frac{\partial \psi}{\partial z} \right) \frac{\partial \psi}{\partial R} + \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) \frac{\partial \psi}{\partial z} = 0$$



$$\vec{B} \cdot \nabla \psi = 0$$

$$\vec{B} \cdot \nabla p = 0$$

for  $\nabla p \neq 0$ :

$$p = p(\psi)$$

# Derivation of Grad-Shafranov equation



- Let's see the  $\hat{\phi}$  component of the force-balance equation:

$$(\vec{j} \times \vec{B} = \nabla p)_{\phi} \quad j_z B_R - j_R B_z = \frac{1}{R} \frac{\partial p}{\partial \phi} \equiv 0$$

- Ampère's law:

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\begin{aligned} \nabla \times \vec{B} &= \hat{R} \left( \frac{1}{R} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_{\phi}}{\partial z} \right) + \hat{\phi} \left( \frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial}{\partial R} (R B_{\phi}) - \frac{1}{R} \frac{\partial B_R}{\partial \phi} \right) \\ &= \hat{R} \left( -\frac{\partial B_{\phi}}{\partial z} \right) + \hat{\phi} \left( \frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial}{\partial R} (R B_{\phi}) \right) \\ &= \hat{R} \mu_0 j_R + \hat{\phi} \mu_0 j_{\phi} + \hat{z} \mu_0 j_z \end{aligned}$$

$$j_R = -\frac{1}{\mu_0} \frac{\partial B_{\phi}}{\partial z} \quad j_z = \frac{1}{\mu_0} \frac{1}{R} \frac{\partial}{\partial R} (R B_{\phi})$$

$$\frac{B_R}{R} \frac{\partial}{\partial R} (R B_{\phi}) + B_z \frac{\partial B_{\phi}}{\partial z} = 0$$



# Magnetic field can be decomposed into the poloidal component and the toroidal component



$$\frac{B_R}{R} \frac{\partial}{\partial R} (RB_\phi) + B_z \frac{\partial B_\phi}{\partial z} = 0 \quad \Rightarrow \quad B_R \frac{\partial}{\partial R} (RB_\phi) + B_z \frac{\partial}{\partial z} (RB_\phi) = 0$$

$$F \equiv RB_\phi \quad \Rightarrow \quad B_R \frac{\partial F}{\partial R} + B_z \frac{\partial F}{\partial z} = 0 \quad \Rightarrow \quad \vec{B} \cdot \nabla F = 0$$

$$\left( \frac{\partial}{\partial \phi} = 0 \right)$$



$$\vec{B} \cdot \nabla p = 0$$

$$p = p(\psi)$$

$$F = F(\psi)$$

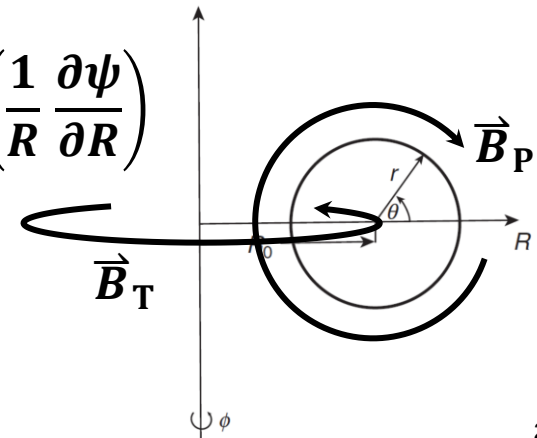
$$B_R = -\frac{\partial A_\phi}{\partial z} = -\frac{1}{R} \frac{\partial \psi}{\partial z}$$

$$B_z = \frac{1}{R} \frac{\partial}{\partial R} (RA_\phi) = \frac{1}{R} \frac{\partial \psi}{\partial R} \quad (\psi = RA_\phi)$$

$$B_\phi = \frac{F(\psi)}{R}$$

$$\vec{B} = \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z = \hat{R} \left( -\frac{1}{R} \frac{\partial \psi}{\partial z} \right) + \hat{\phi} \left( \frac{F(\psi)}{R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right)$$

$$\equiv \left( \frac{\nabla \psi}{R} \right) \times \hat{\phi} + \frac{F(\psi)}{R} \hat{\phi}$$



Poloidal  
component  $\vec{B}_P$

Toroidal  
component  $\vec{B}_T$

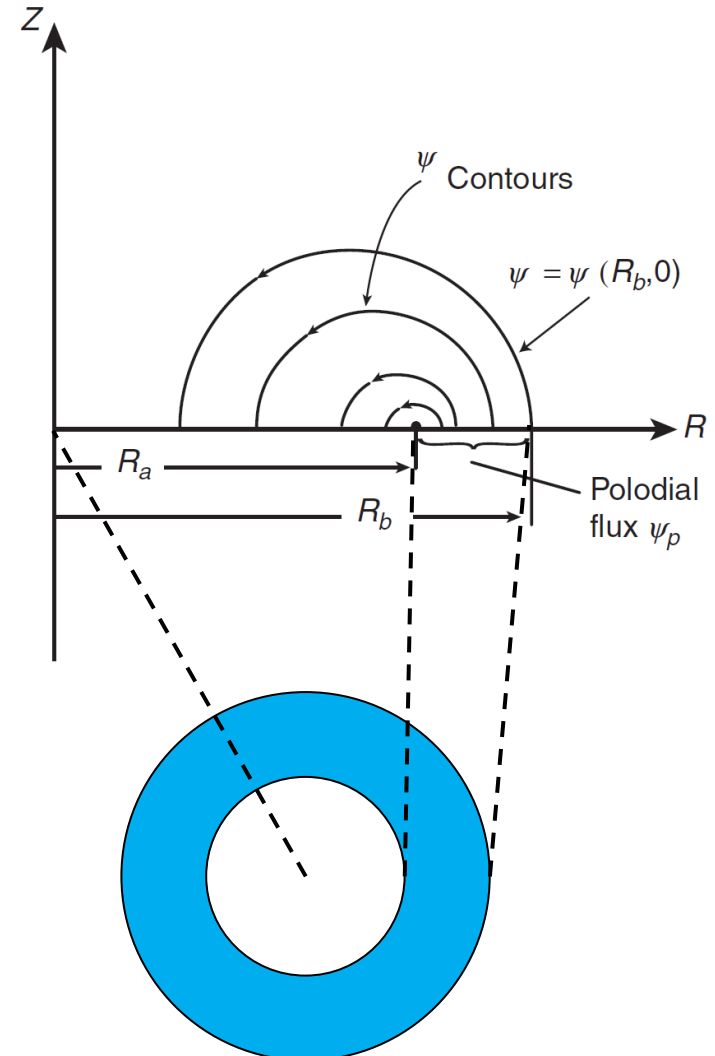
# Arbitrary integration constant associated with flux can be chosen such that flux equals to zero on the field axis



- The poloidal flux of the area of a washer-shaped surface lying in the  $z = 0$  plane from  $R = R_a$  to an arbitrary  $\psi$  contour defined by  $\psi = \psi(R_b, 0)$ :

$$\begin{aligned}\psi_P &\equiv \frac{1}{2\pi} \int \vec{B}_P \cdot d\vec{S} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{R_a}^{R_b} dR R B_z(R, 0) \\ &= \psi(R_b, 0) - \psi(R_a, 0) \\ &\equiv \psi(R_b, 0)\end{aligned}$$

where  $\psi(R_a, 0) \equiv 0$  is chosen.



# Derivation of Grad-Shafranov equation



- Let's see the  $\hat{R}$  component of the force-balance equation:

$$(\vec{j} \times \vec{B} = \nabla p)_R \quad j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial R}$$

$$B_R = -\frac{1}{R} \frac{\partial \psi}{\partial z}$$

$$B_z = \frac{1}{R} \frac{\partial \psi}{\partial R}$$

$$B_\phi = \frac{F(\psi)}{R}$$

- Ampère's law:

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \times \vec{B} = \hat{R} \left( -\frac{\partial B_\phi}{\partial z} \right) + \hat{\phi} \left( \frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right) + \hat{z} \left( \frac{1}{R} \frac{\partial}{\partial R} (R B_\phi) \right) = \hat{R} \mu_0 j_R + \hat{\phi} \mu_0 j_\phi + \hat{z} \mu_0 j_z$$

$$\mu_0 j_\phi = \frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} = \frac{\partial}{\partial z} \left( -\frac{1}{R} \frac{\partial \psi}{\partial z} \right) - \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) = -\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{R} \frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R^2} \frac{\partial \psi}{\partial R}$$

$$\equiv -\frac{1}{R} \Delta^* \psi \quad \text{where } \Delta^* \psi \equiv \frac{\partial^2 \psi}{\partial z^2} + R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right)$$

$$\mu_0 j_z = \frac{1}{R} \frac{\partial}{\partial R} (R B_\phi) = \frac{1}{R} \frac{\partial F}{\partial R} = \frac{1}{R} \frac{dF}{d\psi} \frac{\partial \psi}{\partial R}$$

# Derivation of Grad-Shafranov equation



$$j_{\phi} B_z - j_z B_{\phi} = \frac{\partial p}{\partial R}$$

$$B_{\phi} = \frac{F}{R}$$

$$B_z = \frac{1}{R} \frac{\partial \psi}{\partial R}$$

$$j_{\phi} = -\frac{1}{\mu_0 R} \Delta^* \psi$$

$$j_z = \frac{1}{\mu_0 R} \frac{dF}{d\psi} \frac{\partial \psi}{\partial R}$$

$$\frac{\partial p}{\partial R} = \frac{dp}{d\psi} \frac{\partial \psi}{\partial R}$$

$$-\frac{1}{\mu_0 R} \Delta^* \psi \frac{1}{R} \frac{\partial \psi}{\partial R} - \frac{1}{\mu_0 R} \frac{dF}{d\psi} \frac{\partial \psi}{\partial R} \frac{F}{R} = \frac{dp}{d\psi} \frac{\partial \psi}{\partial R}$$

$$-\Delta^* \psi \frac{1}{\mu_0} \frac{1}{R^2} - \frac{1}{\mu_0} \frac{F}{R^2} \frac{dF}{d\psi} = \frac{dp}{d\psi}$$

**Grad – Shafranov equation:  $\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi}$**

where  $\Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right)$        $\vec{B} = \left( \frac{\nabla \psi}{R} \right) \times \hat{\phi} + \frac{F(\psi)}{R} \hat{\phi}$

# Derivation of Grad-Shafranov equation



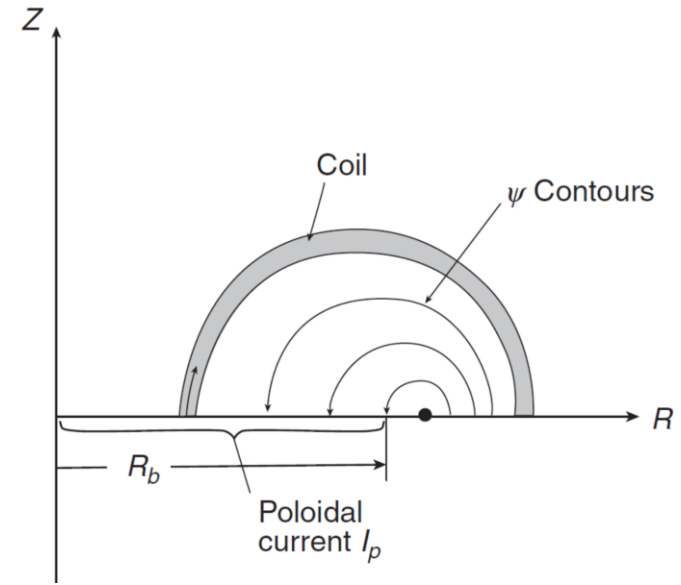
$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi} \quad \text{where } \Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right) \quad \vec{B} = \left( \frac{\nabla \psi}{R} \right) \times \hat{\phi} + \frac{F(\psi)}{R} \hat{\phi}$$

$$\mu_0 j_\phi = -\frac{1}{R} \Delta^* \psi \quad \mu_0 j_z = \frac{1}{R} \frac{\partial F}{\partial R} \quad F \equiv RB_\phi$$

$$\mu_0 j_R = -\frac{\partial B_\phi}{\partial z} = -\frac{1}{R} \frac{\partial}{\partial z} (RB_\phi) = -\frac{1}{R} \frac{\partial F}{\partial z}$$

$$\begin{aligned} \mu_0 \vec{j} &= \hat{R} \mu_0 j_R + \hat{\phi} \mu_0 j_\phi + \hat{z} \mu_0 j_z = \hat{R} \left( -\frac{1}{R} \frac{\partial F}{\partial z} \right) + \hat{\phi} \left( -\frac{1}{R} \Delta^* \psi \right) + \hat{z} \left( \frac{1}{R} \frac{\partial F}{\partial R} \right) \\ &\equiv \left( \frac{\nabla F}{R} \right) \times \hat{\phi} + \left( -\frac{1}{R} \Delta^* \psi \right) \hat{\phi} \end{aligned}$$

$$\begin{aligned} I_P &= \int \vec{j}_P \cdot d\vec{S} = - \int_0^{2\pi} d\phi \int_0^{R_b} dR R j_z(R, 0) \\ &= -2\pi \int_0^{R_b} dR R \frac{1}{R} \frac{\partial F(R, 0)}{\partial R} = -2\pi F(\psi) \end{aligned}$$



# Example of the analytical solution of the Grad-Shafranov equation



$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi}$$

• For  $\mu_0 \frac{dp}{d\psi} = -C_2$        $\frac{1}{2} \frac{dF^2}{d\psi} = C_1$

$$\psi(R, z) = -\frac{C_1}{2} z^2 + \frac{C_2}{8} R^4 + C_3 + C_4 R^2 + C_5 (R^4 - 4R^2 z^2)$$

$$C_1 = 1$$

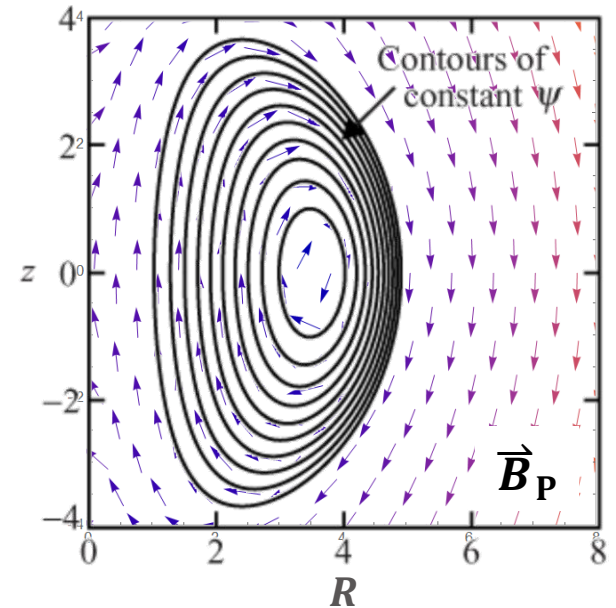
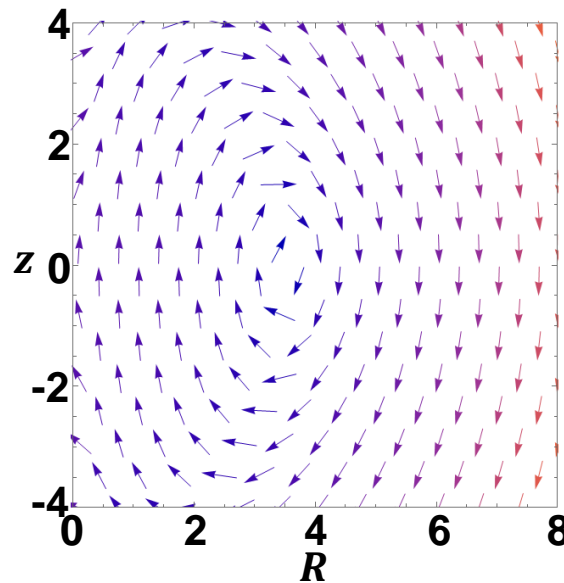
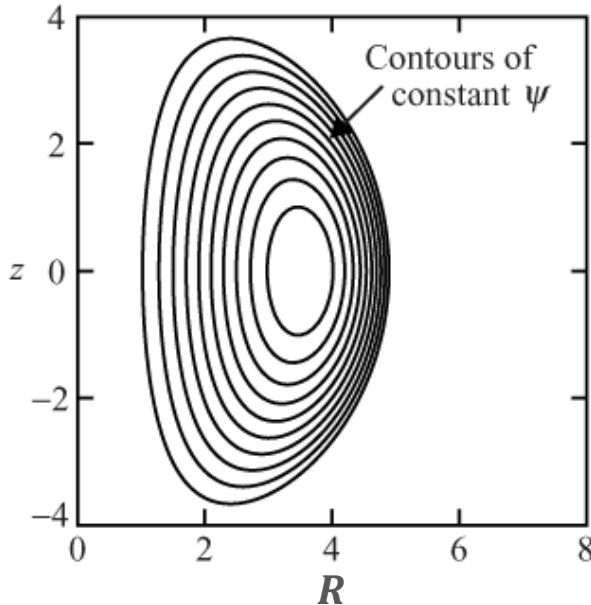
$$C_2 = -8$$

$$C_3 = -20$$

$$C_4 = 20$$

$$C_5 = 0.2$$

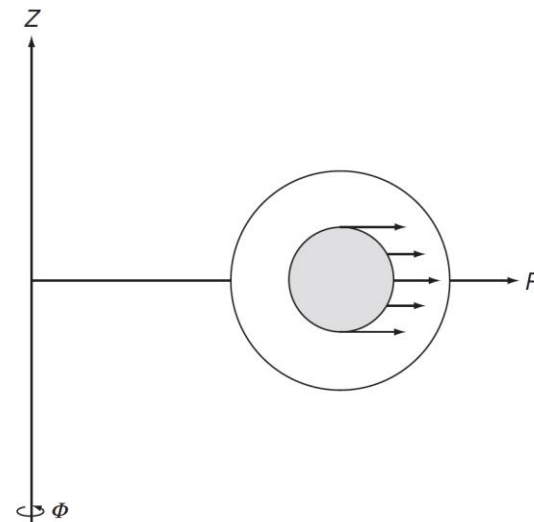
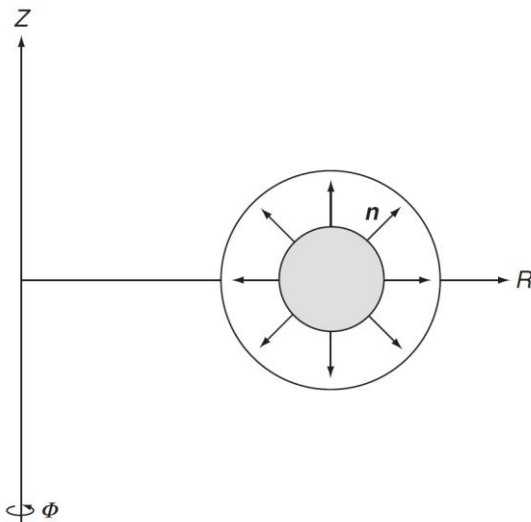
$$B_R(R, z) = -\frac{1}{R} (-C_1 z - 8C_5 R^2 z) \quad B_z(R, z) = \frac{1}{R} \left( \frac{C_2}{2} R^3 + 2C_4 R + C_5 (4R^3 - 8Rz^2) \right)$$



# Magnetically confined toroidal equilibrium



1. Radial pressure balance in the poloidal plan needs to be provided so that the pressure contours form closed nested surfaces. Both toroidal and poloidal fields can readily accomplish this task.
  2. The radially outward expansion force inherent in all toroidal geometries needs to be balanced without sacrificing stability.
- Forces associated with toroidal force balance are usually than those corresponding to radial pressure balance. However, they are more difficult to compensate.



# Toroidal configuration with a purely poloidal magnetic field

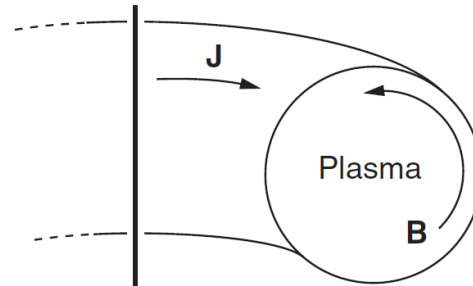
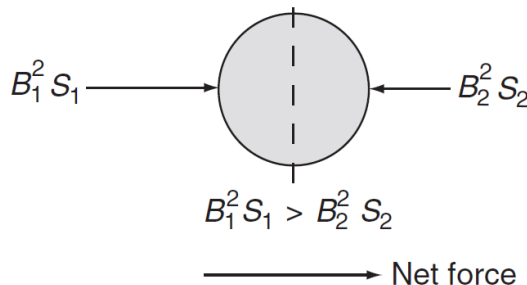
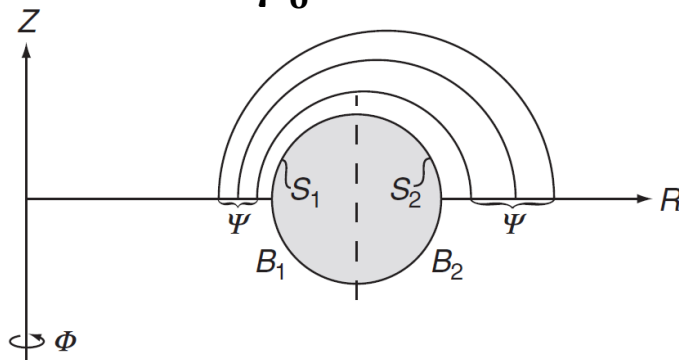


- Hoop force:

$$\psi_1 = \psi_2 \equiv \psi$$

$$S_1 < S_2 \quad B_1 > B_2$$

$$\vec{F}_{H,R} \propto \hat{e}_R \frac{B_1^2 S_1 - B_2^2 S_2}{2\mu_0} > 0$$

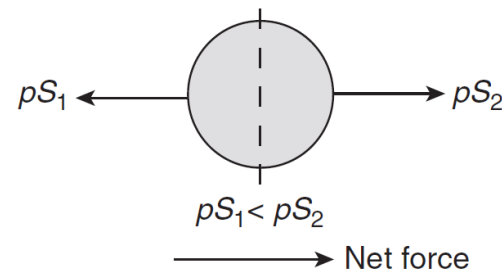
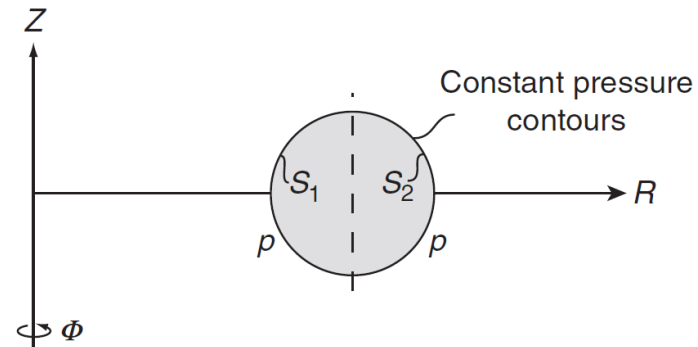


- Tire tube force

$$p_1 = p_2 \equiv p$$

$$S_1 < S_2$$

$$\vec{F}_{T,R} \propto -\hat{e}_R (pS_1 - pS_2) > 0$$

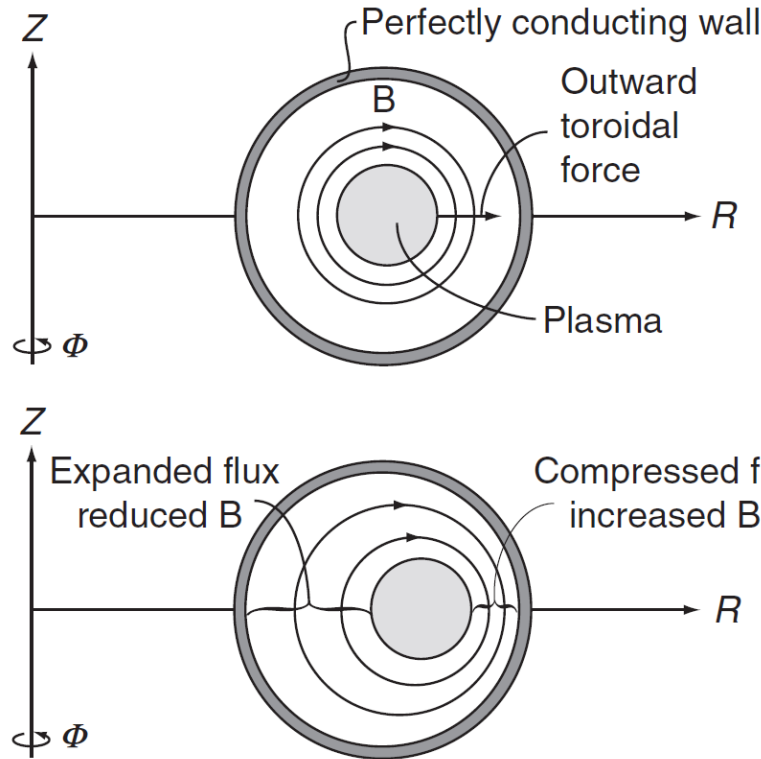




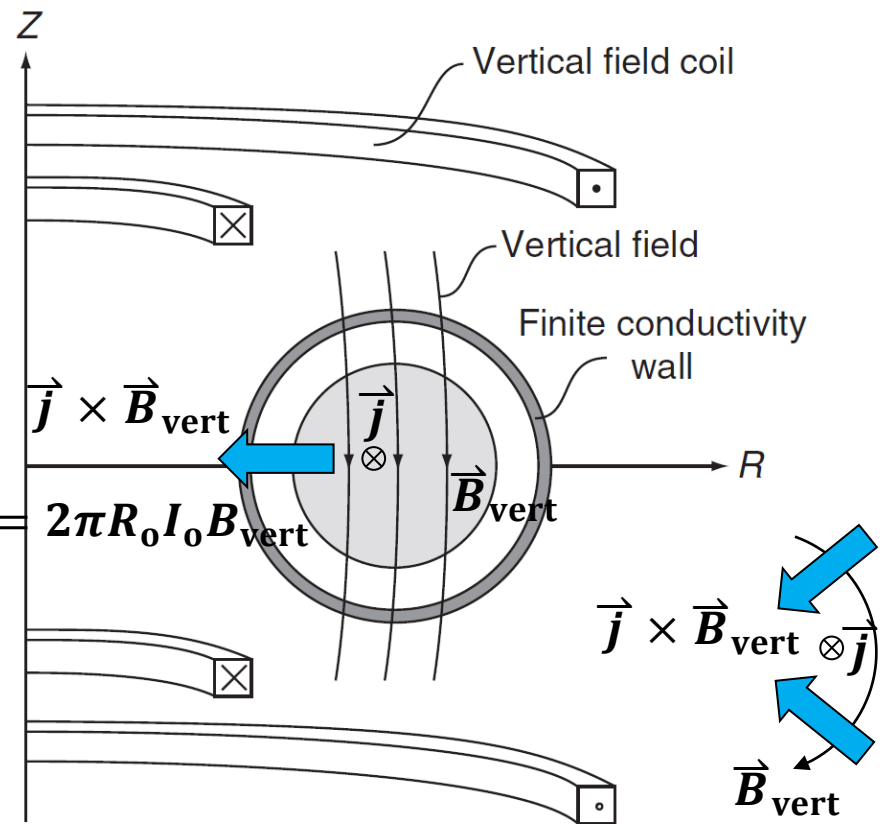
# The outward force can be compensated by either a perfectly conducting shell or externally applied vertical field



- Perfectly conducting shell

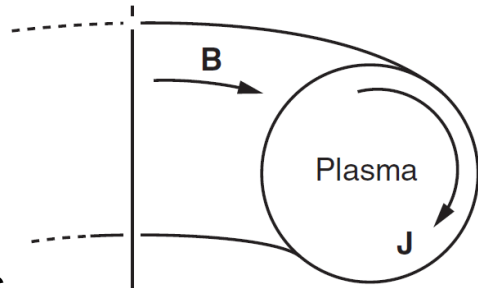


- Externally applied vertical field

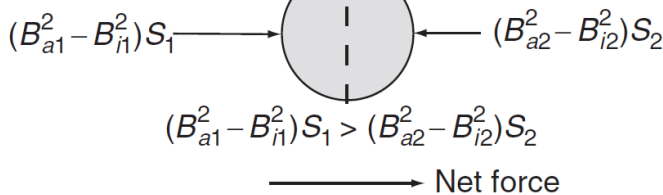
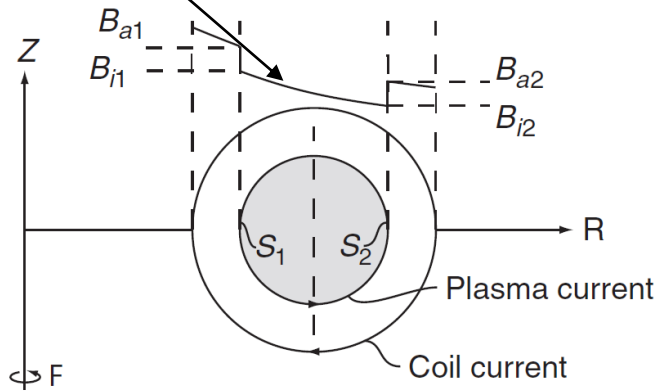


- With a finite conductivity wall, flux can only remain compressed for about a skin time.
- This configuration develops disastrous MHD instabilities (z pinch).

# Toroidal configuration with a purely toroidal magnetic field, stable but NOT balanced

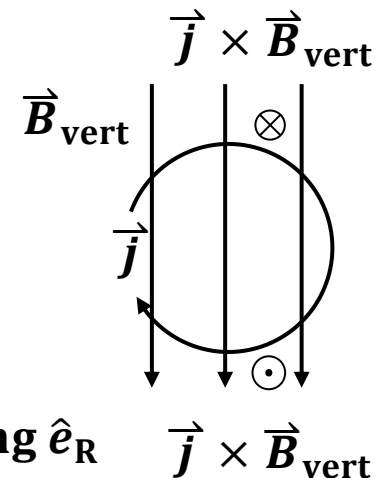
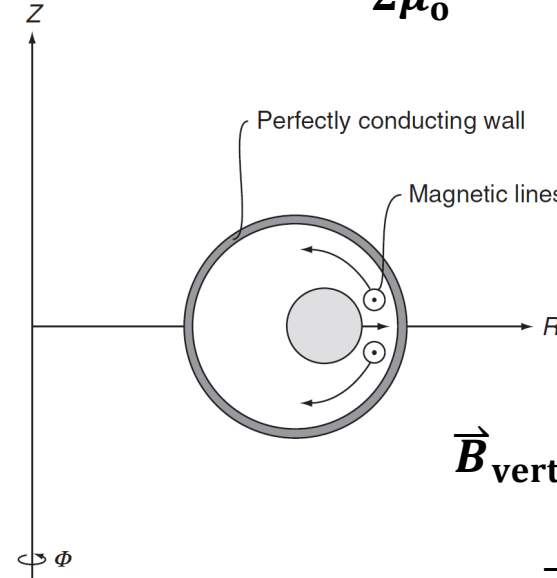


Plasma is diamagnetic



$$\vec{B} = B_\phi \hat{e}_\phi \quad B_\phi = B_0 \frac{R_0}{R} \quad B_0 = \frac{\mu_0 I_c}{2\pi R_0}$$

$$\vec{F}_R \propto \hat{e}_R \frac{(B_{a1}^2 - B_{i1}^2)S_1 - (B_{a2}^2 - B_{i2}^2)S_2}{2\mu_0} > 0$$

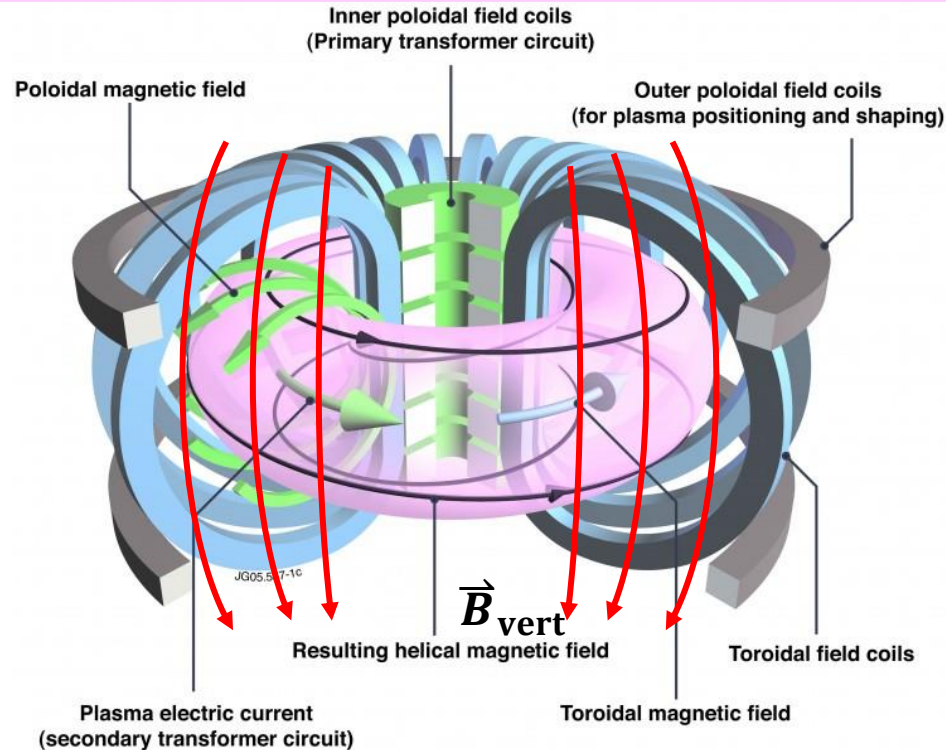


$$\vec{j} \times \vec{B}_{\text{vert}}$$

is not pointed along  $\hat{e}_R$

$$\vec{j} \times \vec{B}_{\text{vert}}$$

# Coils in a tokamak



- **Toroidal field coils (in poloidal direction) – generate toroidal field for confinement.**
- **Poloidal field coils – generate vertical field for plasma positioning and shaping.**
- **Central solenoid – for breakdown and generating plasma current (in toroidal direction) and thus generating poloidal field for confinement.**

# Plasma condition can be obtained by solving Grad-Shafranov equation



$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi} \quad \text{where } \Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right)$$

- The usual strategy to solve the Grad-Shafranov equation:
  1. Specify two free functions, the plasma pressure  $p = p(\psi)$  and the toroidal field function  $F = F(\psi)$ .
  2. Solve the equation with specified boundary conditions to determine the flux function  $\psi(R, z)$ .
  3. Calculation the magnetic field using the following equations:

$$B_R = -\frac{1}{R} \frac{\partial \psi}{\partial z} \quad B_\phi = \frac{F(\psi)}{R} \quad B_z = \frac{1}{R} \frac{\partial \psi}{\partial R}$$

4. The pressure profile can then be obtained from  $p = p(\psi(R, z))$ .

# Application of solving Grad-Shafranov equation for designing a tokamak



- Given  $I_{\text{plasma}}$ ,  $p(\psi)$ ,  $I(\psi)$ ,  $I_{\text{coils}}$ , free boundary of plasma, perfect conductor as the chamber.
- Given  $I_{\text{plasma}}$ ,  $p(\psi)$ ,  $I(\psi)$ ,  $I_{\text{coils}}$ , free boundary of plasma, insulator chamber.
- Given  $I_{\text{plasma}}$ ,  $p(\psi)$ ,  $I(\psi)$ ,  $I_{\text{coils}}$ , free boundary of plasma, chamber with eddy current.
- Given  $I_{\text{plasma}}$ ,  $p(\psi)$ ,  $I(\psi)$ , fixed boundary of plasma. Then, use  $I_{\text{coils}}$ , free boundary of plasma and match the plasma shape calculated in the fixed boundary condition.

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi} \quad \text{where } \Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right)$$

$$I_p = -2\pi F(\psi)$$

$$\mu_0 \vec{j} = \left( \frac{\nabla F}{R} \right) \times \hat{\phi} + \left( -\frac{1}{R} \Delta^* \psi \right) \hat{\phi} \quad \vec{B} = \left( \frac{\nabla \psi}{R} \right) \times \hat{\phi} + \frac{F(\psi)}{R} \hat{\phi}$$

# Application of solving Grad-Shafranov equation for reconstructing a tokamak equilibrium state



- **Measure**

- boundary conditions, including  $\psi$ ,  $B$ , etc., on the wall (using flux loop and B-dot probe).
- Pressure.
- Plasma current (using Rogowski coil).

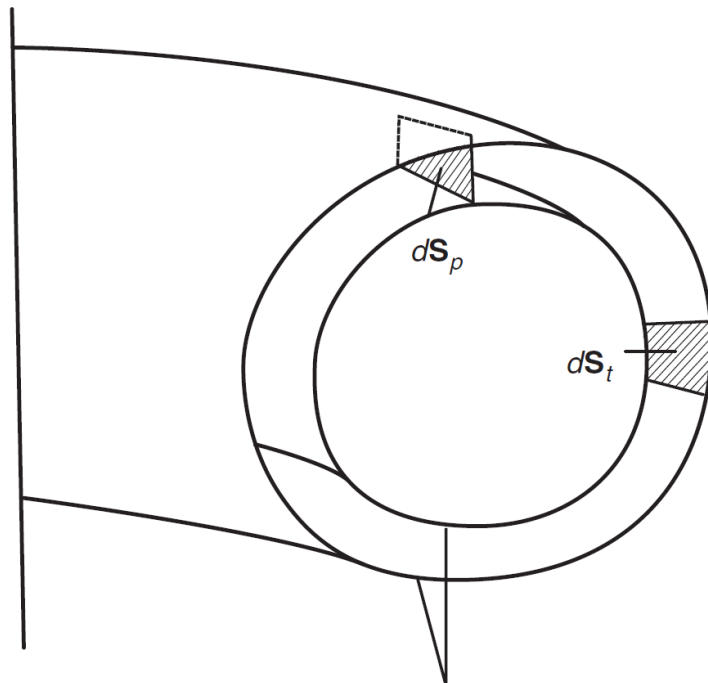
- Reconstruct  $\psi(r,z)$ ,  $j$ ,  $p(\psi)$ ,  $I(\psi)$ , etc.

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi} \quad \text{where } \Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right)$$

$$I_p = -2\pi F(\psi)$$

$$\mu_0 \vec{j} = \left( \frac{\nabla F}{R} \right) \times \hat{\phi} + \left( -\frac{1}{R} \Delta^* \psi \right) \hat{\phi} \quad \vec{B} = \left( \frac{\nabla \psi}{R} \right) \times \hat{\phi} + \frac{F(\psi)}{R} \hat{\phi}$$

# Fluxes and currents



Two neighboring flux surfaces

- **Poloidal flux:**  $\psi_p = \int \vec{B} \cdot d\vec{S}_p$   
 $\psi_p = \psi_p(p)$
- **Toroidal flux:**  $\psi_t = \int \vec{B} \cdot d\vec{S}_t$
- **Poloidal current:**  $I_p = \int \vec{j} \cdot d\vec{S}_p$
- **Toroidal current:**  $I_t = \int \vec{j} \cdot d\vec{S}_t$

# Normalized plasma pressure, $\beta$



$$\beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{2\mu_0 \langle p \rangle}{B^2}$$

- Plasma pressure:  $\langle p \rangle = \frac{1}{V_p} \int p d\vec{r}$

- Magnetic pressure:  $P_B = \frac{B^2}{2\mu_0}$

$$B^2 = B_t^2 + B_p^2 = B_0^2 + \left(\frac{\mu_0 I_p}{2\pi a}\right)^2 \frac{2}{1 + \kappa^2}$$

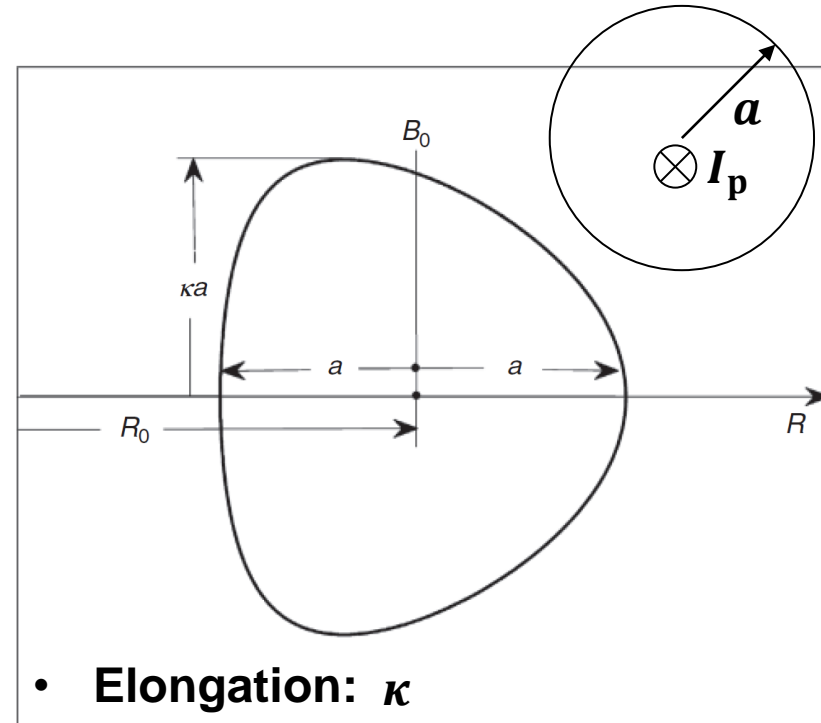
$$B_t^2 = B_0^2 \quad B_0 = B @ R = R_0$$

$$B_p^2 = \left(\frac{\mu_0 I_p}{2\pi a}\right)^2 = \left(\frac{\mu_0 I_p}{C_p}\right)^2$$

$$C_p \approx 2\pi a \sqrt{\frac{1 + \kappa^2}{2}}$$

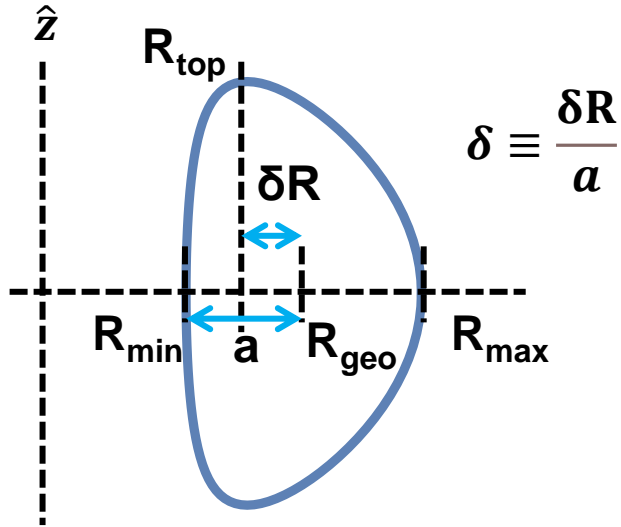
$$\beta_t = \frac{2\mu_0 \langle p \rangle}{B_0^2} \quad \beta_p = \frac{4\pi^2 a^2 (1 + \kappa^2) p}{\mu_0 I_p^2}$$

$$\frac{1}{\beta} = \frac{1}{\beta_t} + \frac{1}{\beta_p}$$





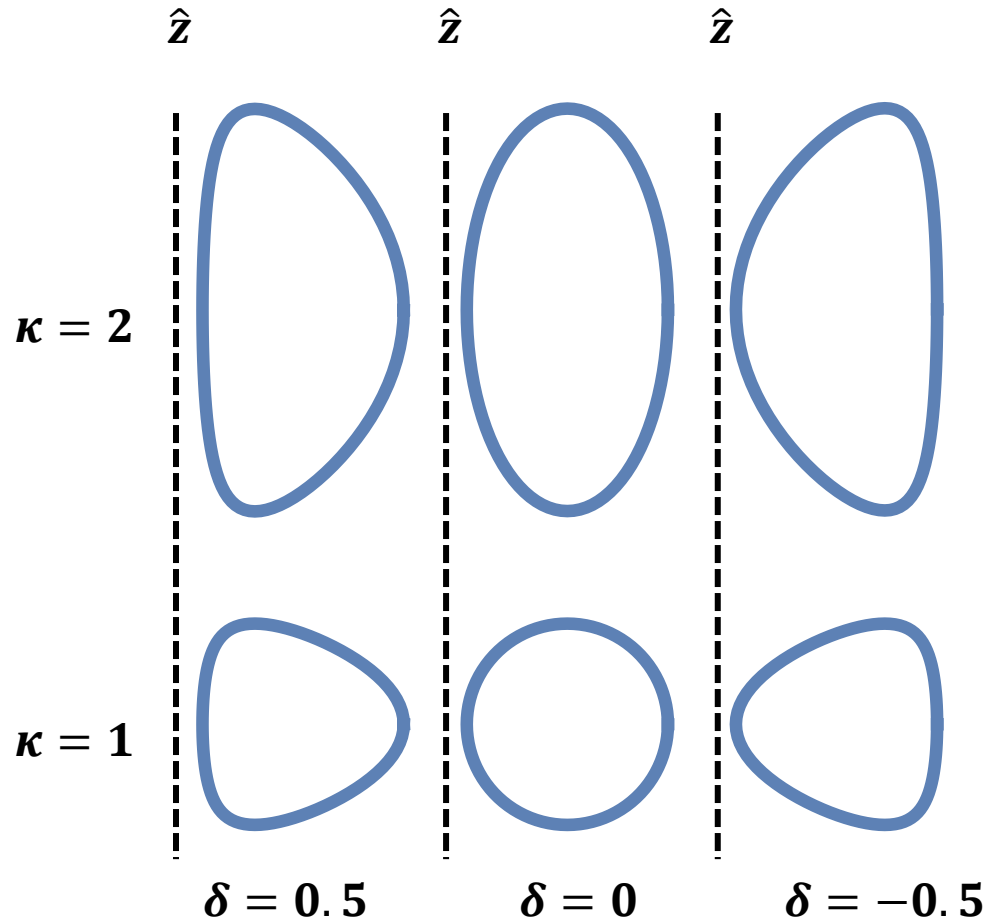
# Different poloidal shapes



$$r = R + a \cos(\theta + \delta \sin(\theta))$$

$$z = a \kappa \sin(\theta)$$

- Aspect ratio:  $\frac{R}{a}$
- Elongation:  $\kappa$
- Triangularity:  $\delta$

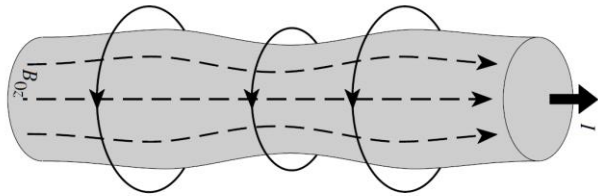


# Safety factor

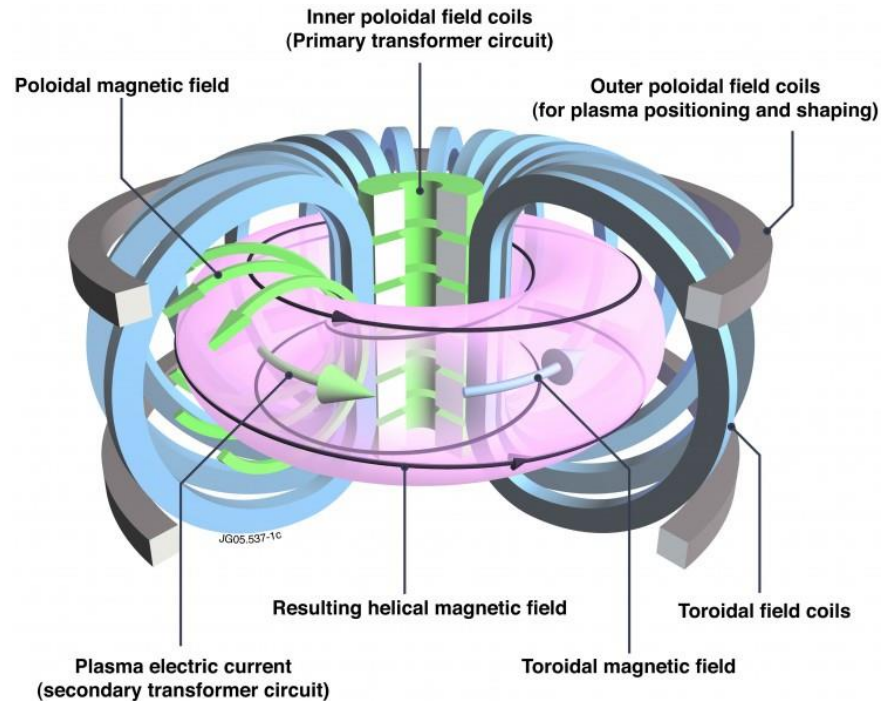


- Kink Safety Factor:

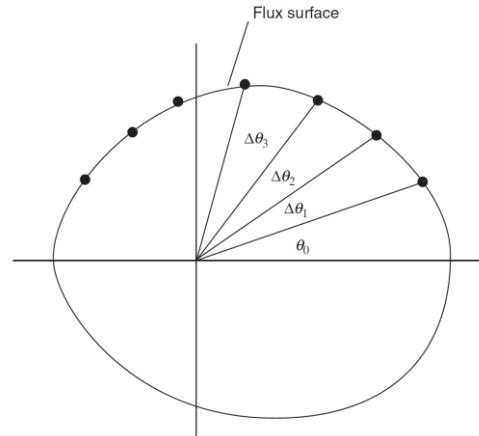
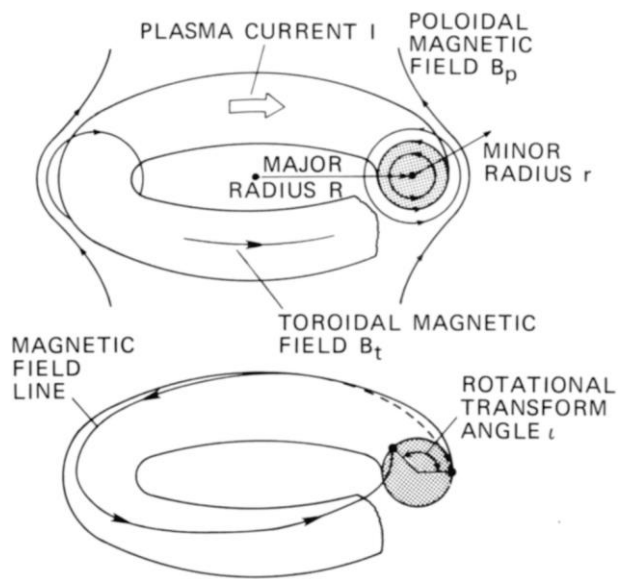
$$q^*(r) = \frac{aB_o}{R_oB_p} = \frac{2\pi a^2 \kappa B_o}{\mu_o R_o I_o}$$



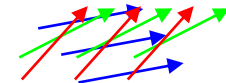
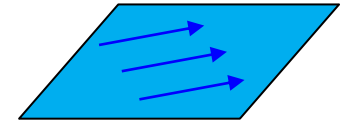
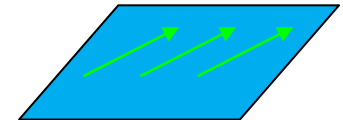
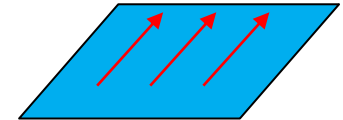
$$q(r) = \frac{rB_z(r)}{R_oB_\theta(r)}$$



# Safety factor



- **Shear:**



- **Rotational transform:** 
$$\iota \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum \Delta\theta_n$$
- **MHD safety factor:** 
$$q(V) \equiv \frac{2\pi}{\iota(V)} = \frac{d\psi_t/dV}{d\psi_p/dV}$$
  $\psi_t = \psi_t(V) \quad \psi_p = \psi_p(V)$
- **Shear:** 
$$s(V) \equiv 2 \frac{V}{q} \frac{dq}{dV}$$
 
$$\iota(V) \equiv 2\pi \left( \frac{d\psi_p/dV}{d\psi_t/dV} \right)$$

# Magnetic well



$$\widehat{W} = 2 \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \left\langle \frac{B^2}{2} \right\rangle$$

$$= 2 \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \left\langle \mu_0 p + \frac{B^2}{2} \right\rangle$$

- A magnetic well is a quantity that measures plasma stability against short perpendicular wavelength modes driven by the plasma pressure gradient.



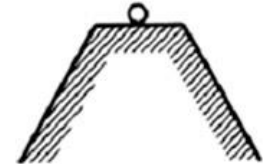
A

NO EQUILIBRIUM



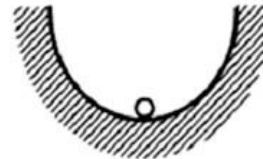
B

NEUTRALLY STABLE



C

(METASTABLE) EQUILIBRIUM



D

STABLE EQUILIBRIUM



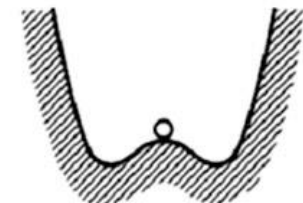
E

UNSTABLE EQUILIBRIUM



F

EQUILIBRIUM WITH LINEAR STABILITY AND NONLINEAR INSTABILITY



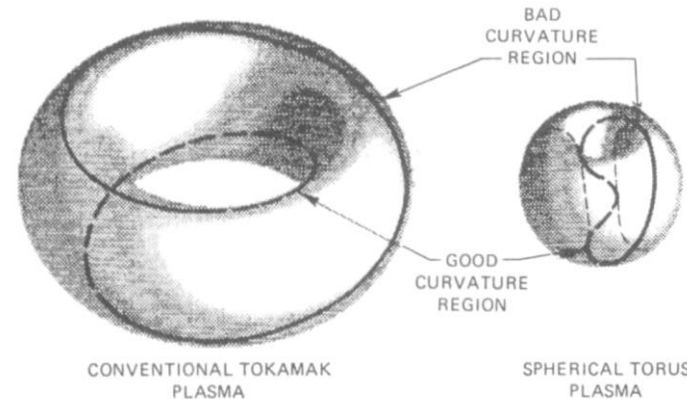
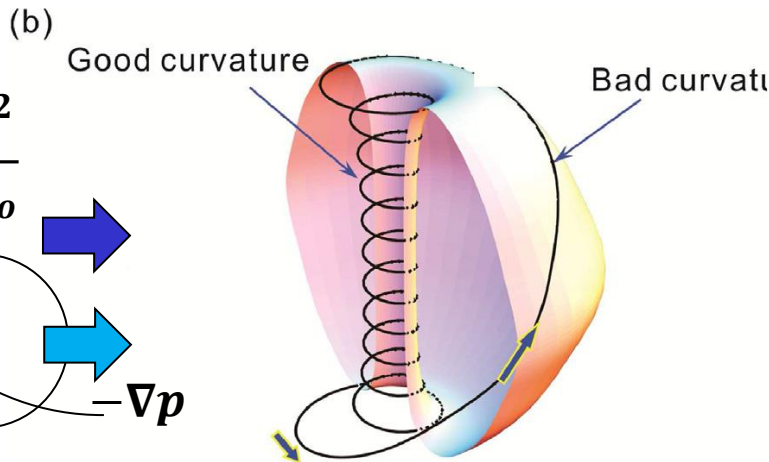
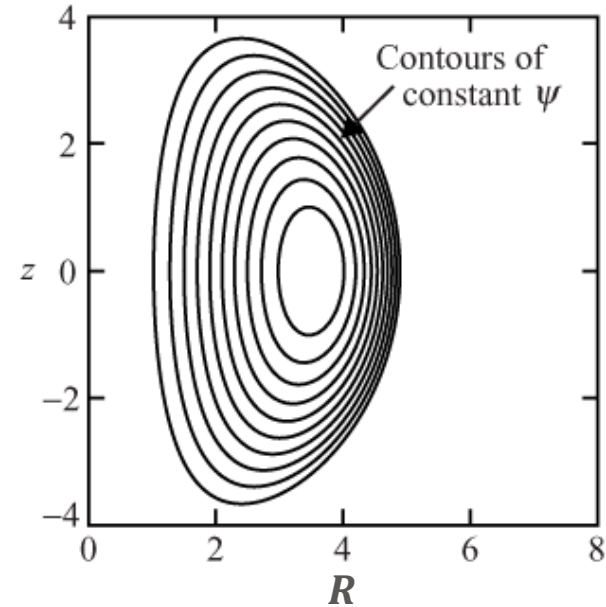
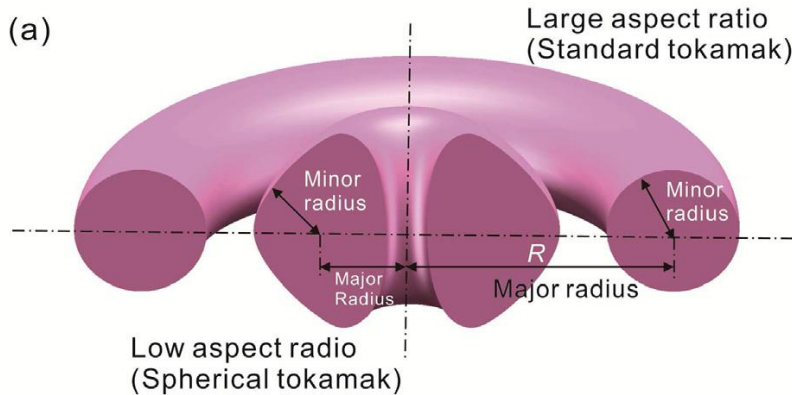
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EQUILIBRIUM WITH LINEAR INSTABILITY AND NONLINEAR STABILITY

# The Spherical tokamak



- Aspect ratio  $R_0/a \sim 1.6$



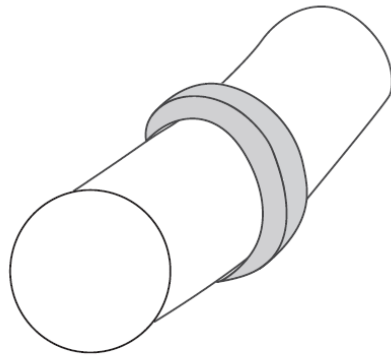
# The Spherical tokamak

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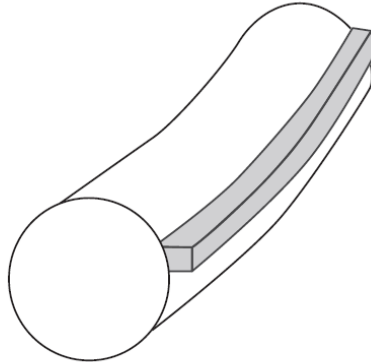


- **Aspect ratio  $R_0/a \sim 1.6$**
- **Advantages:**
  - **Higher  $\beta_t$  limit.**
  - **A compact design almost spherical in appearance.**
- **Challenges:**
  - **Minimum space is given in the center of the torus to accommodate the toroidal field coils.**
  - **With a very compact design the technology associated with the construction and maintenance of the device may be more difficult than for a “normal” tokamak.**
  - **Large currents will have to be driven noninductively, a costly and physically difficult requirement.**

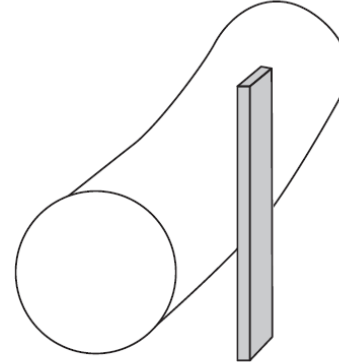
# Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma



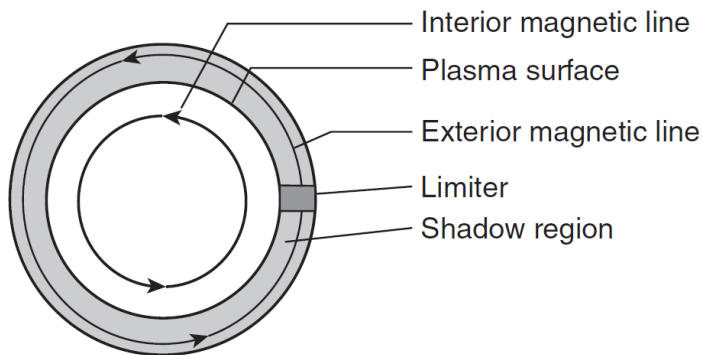
Poloidal limiter



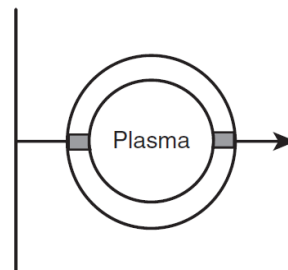
Toroidal limiter



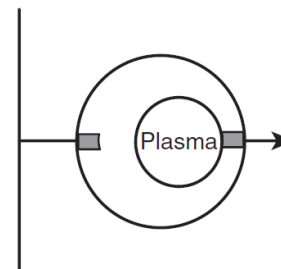
Rail limiter



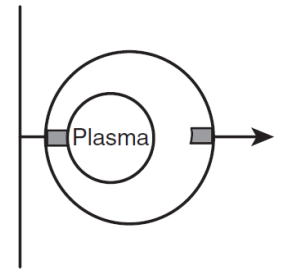
- **Vertical field is correct.**



- **Vertical field is too small.**



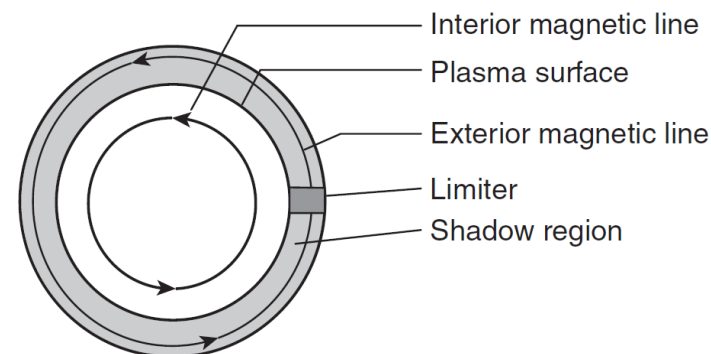
- **Vertical field is too large.**



# Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma

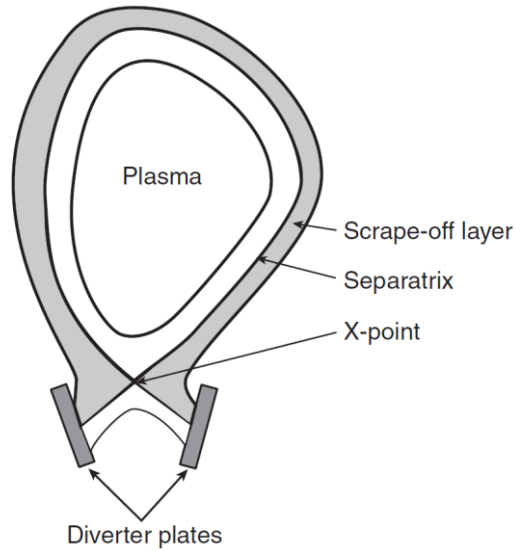


- A mechanical limiter is a robust piece of material, often made of tungsten, molybdenum, or graphite placed inside the vacuum chamber.
- Some of the particles of the limiter surface may escape. Neutral particles can penetrate some distance into the plasma before being ionized.
- The high-z impurities can lead to significant additional energy loss in the plasma through radiation.
- In ignition experiments and fusion reactors, the bombardment is more intense and extends over longer periods of time. In addition, if the impurity level is too high, it may not be possible to achieve a high enough temperature to ignite.

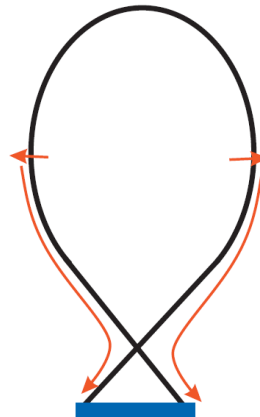




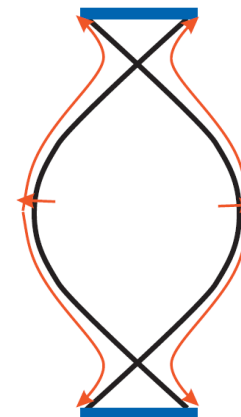
# The magnetic divertor – guide a narrower layer of magnetic lines away from the edge of the plasma



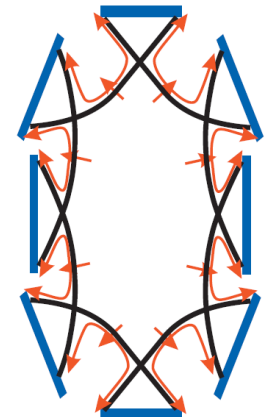
- **Single-null poloidal-field divertors for tokamak**



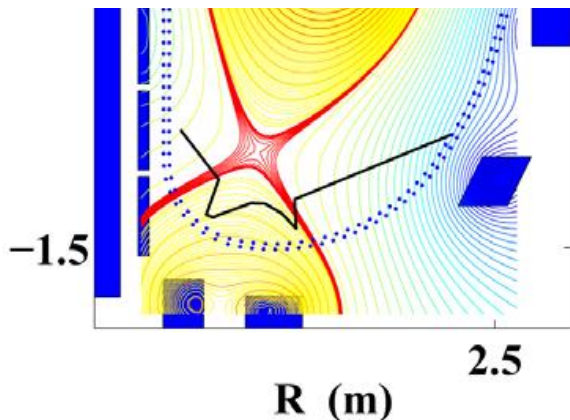
- **Double-null poloidal-field divertors for tokamak**



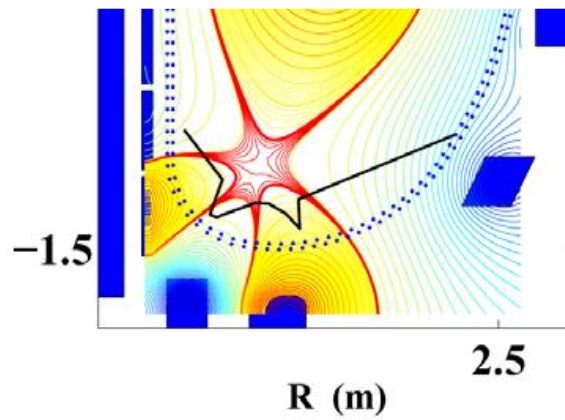
- **Island divertor for stellarators**



- **Standard**



- **Snowflake**



Y. Feng, Nucl. Fusion, **46**, 807 (2006)  
 L Xue *et al*, Plasma Phys. Control. Fusion **58**, 055005 (2016)

# Pros and cons of a divertor



- **Advantages:**
  - The collector plate is remote from the plasma. There is space available to spread out the magnetic lines.
  - A lower intensity of particles and energy bombard the collector plate leading to a longer replacement time.
  - It is more difficult for impurities to migrate into the plasma.
  - There are longer distance distances to travel and if a neutral particle becomes ionized before or during the time it crosses the divertor layer on its way toward the plasma, its parallel motion then carries it back to the collector plate.
  - The larger divertor chamber provides more access to pump out impurities.
  - The plasma edge is not in direct contact with a solid material such as a limiter.
- **Disadvantages:** larger and more complex system and more expensive.