Introduction to Nuclear Fusion as An Energy Source

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Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 6

2025 spring semester

Tuesday 9:00-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=mf1a33a5dab5eb71de9da4380ae888592

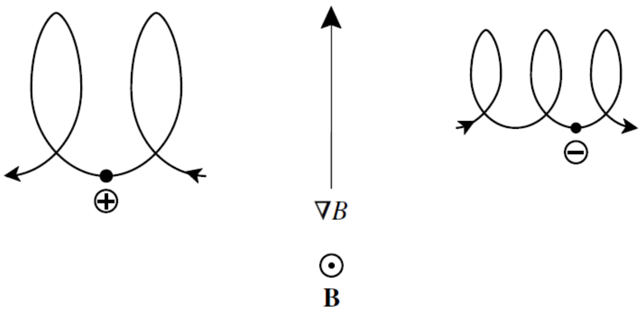
2025/3/25 updated

Note!

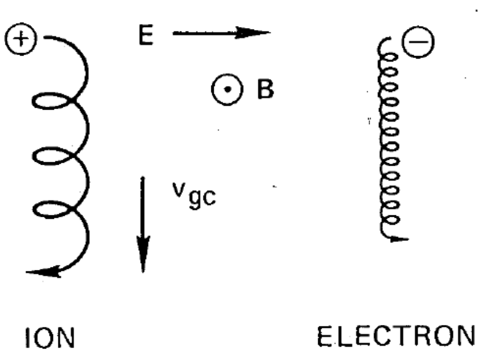
- **Midterm 4/15 (One double-sided A4 cheating sheet is allowed.)**
- **Final exam 6/ 3**

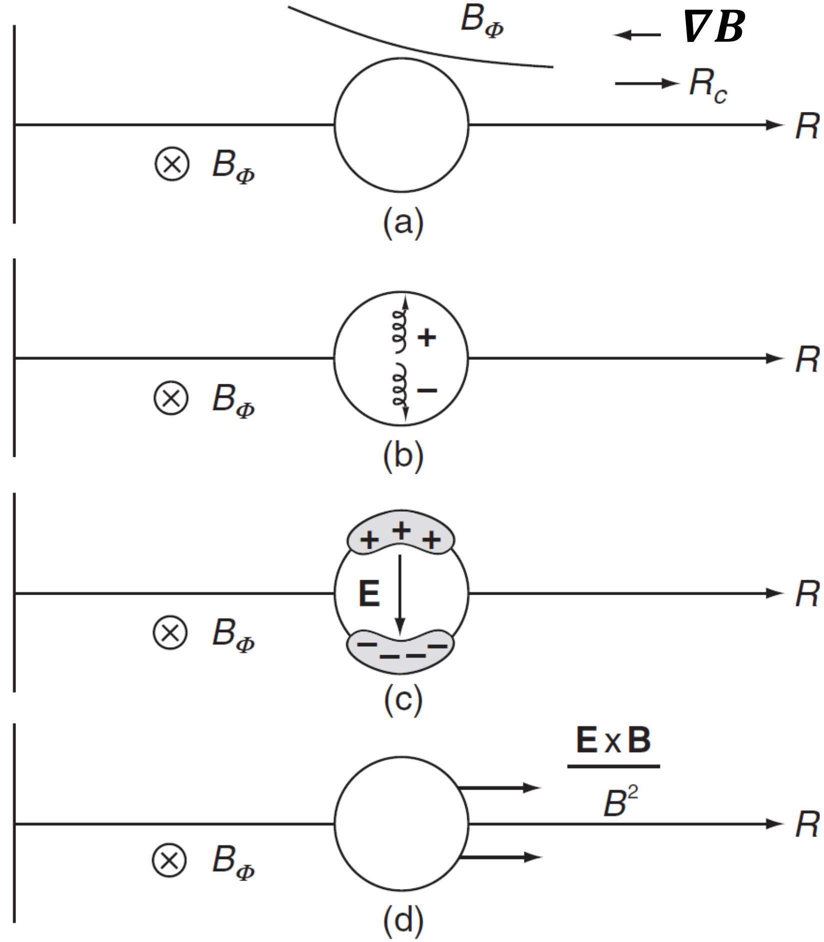
Charged particles drift across field lines

- **Grad-B drift**



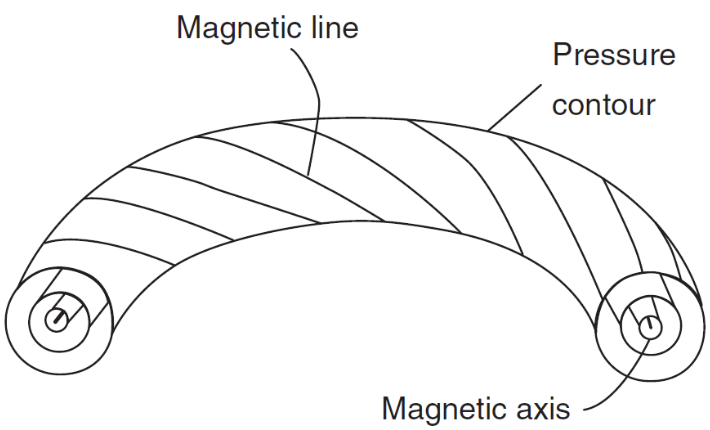
- **ExB drift**

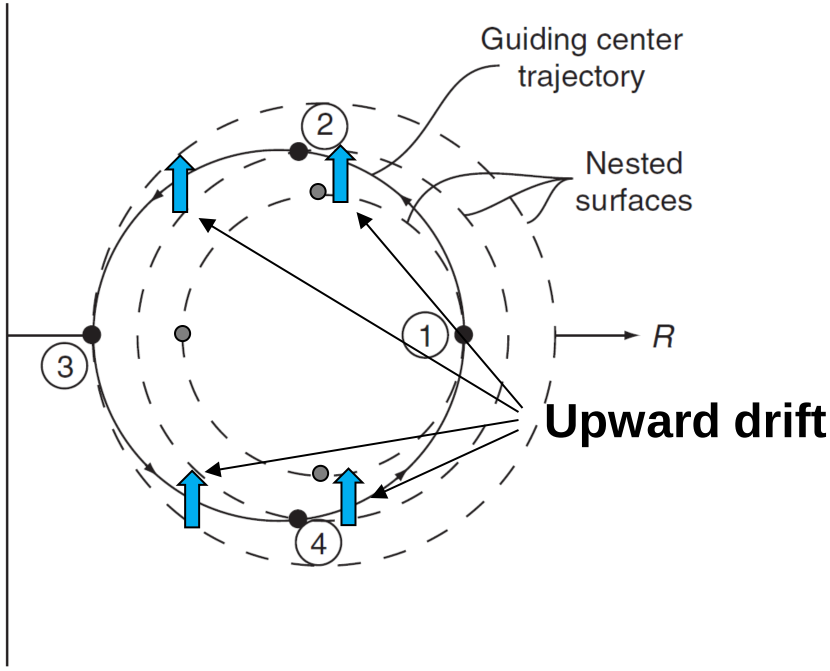




J. P. Freidberg, Ideal Magnetohydrodynamics

The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

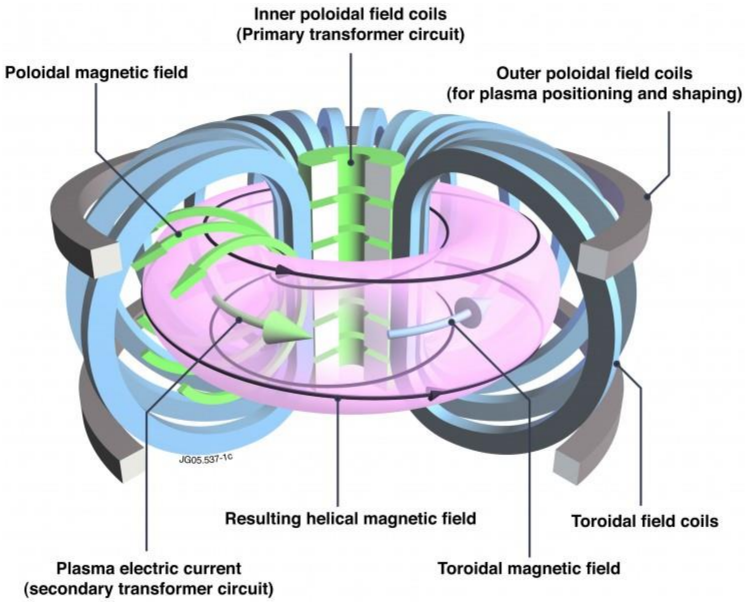




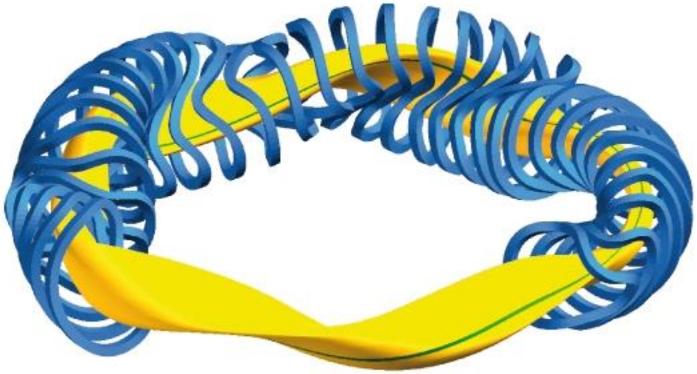
- Points with no drift

Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak

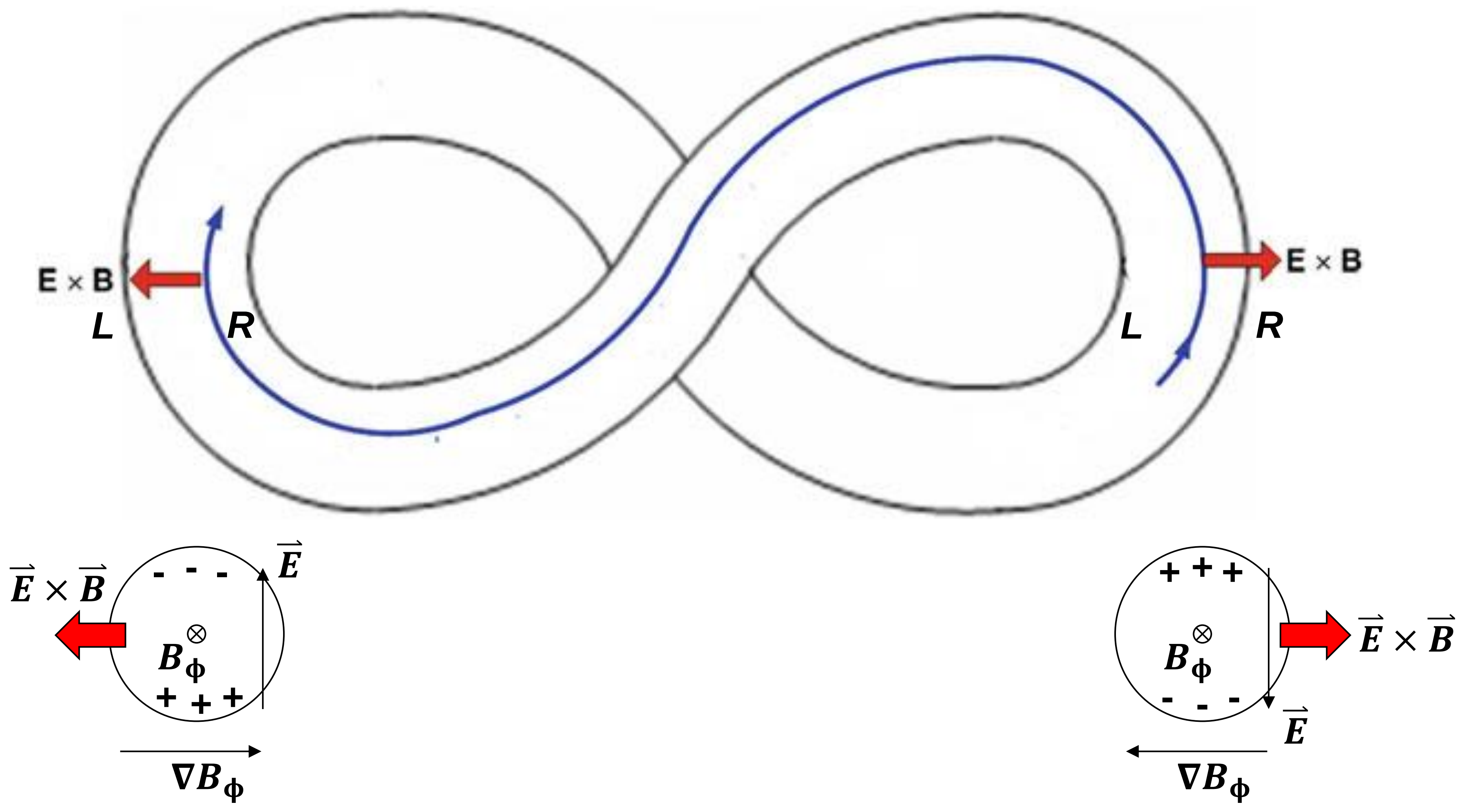


Stellarator

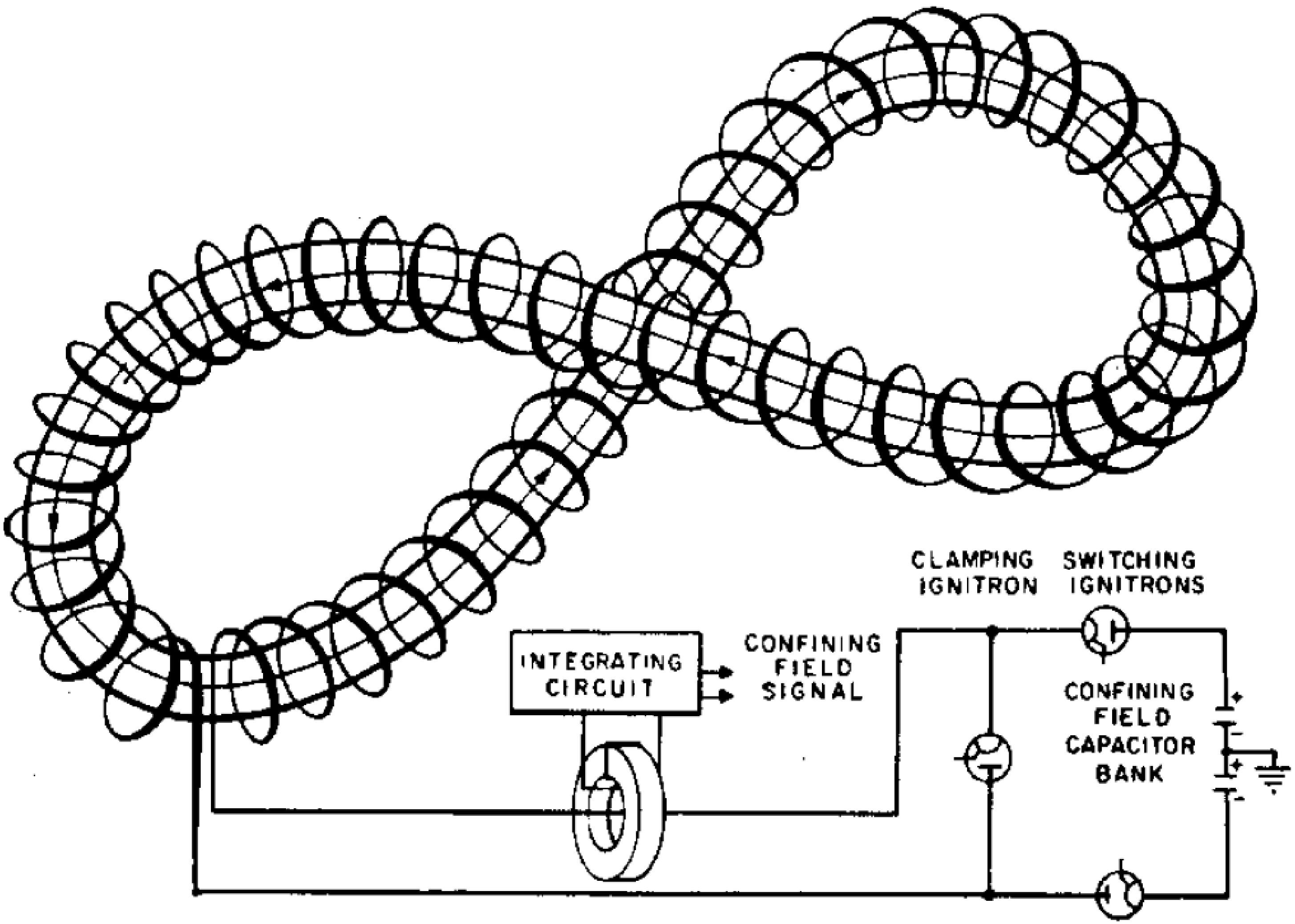


https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

A figure-8 stellarator solved the drift issues

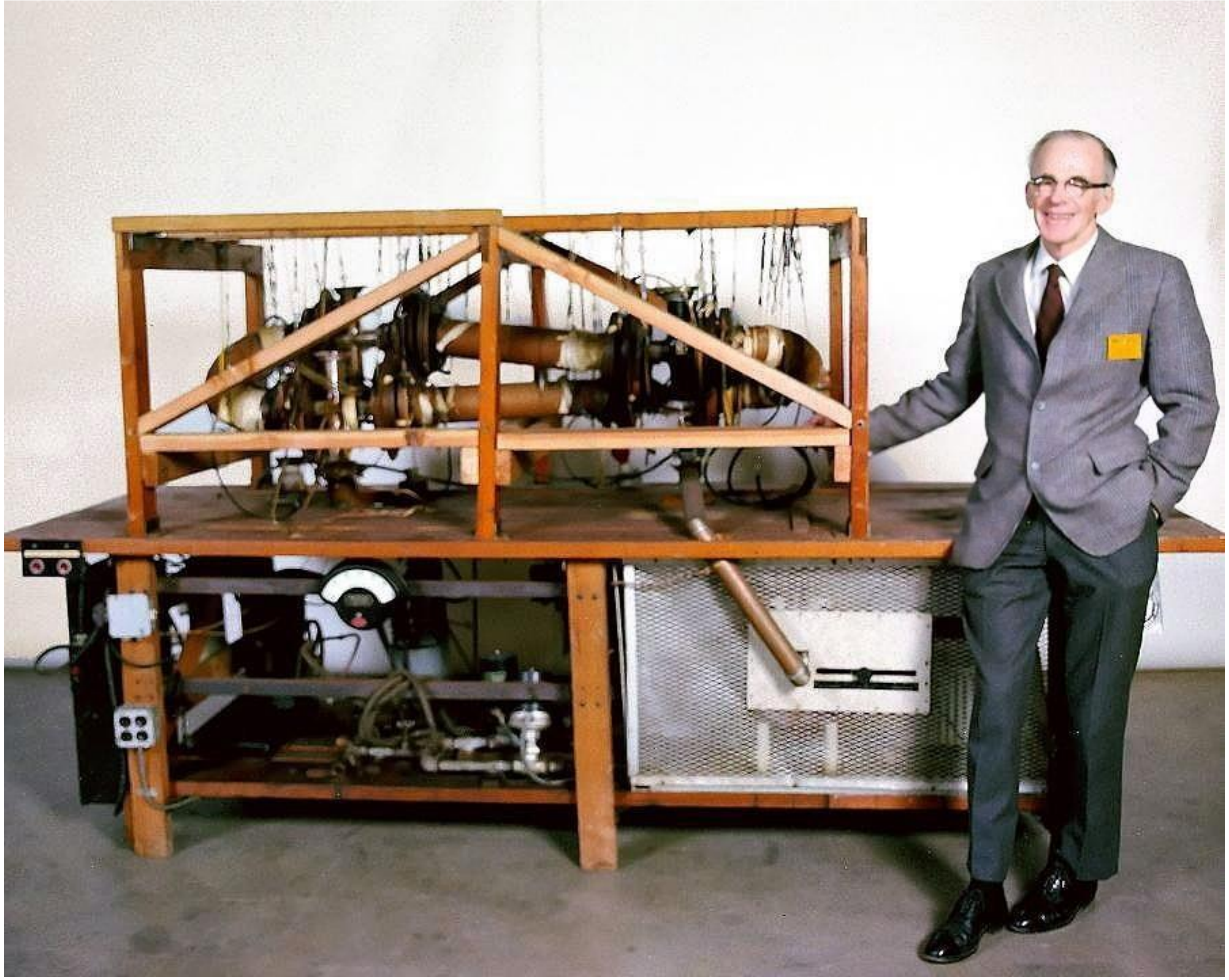


Concept of figure-8 stellarator



T. Coor, et al., Phys. Fluids **1**, 411 (1958)

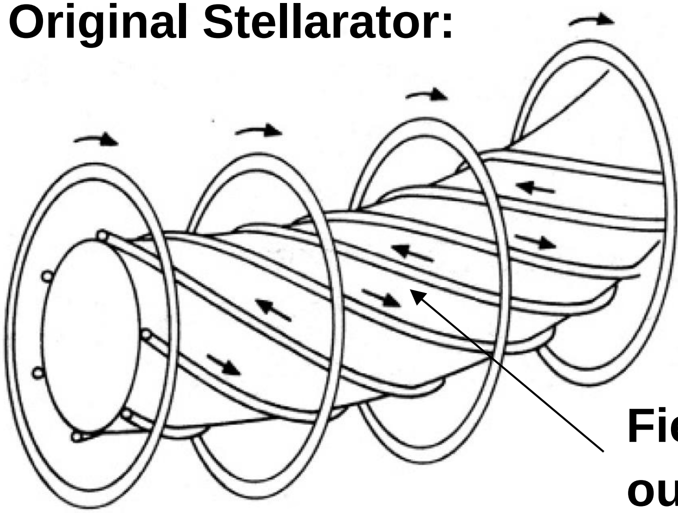
Model A stellarator



https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

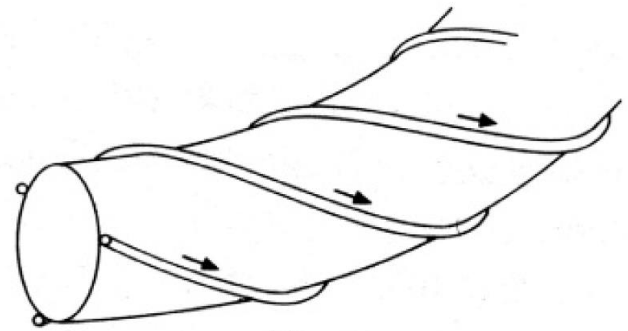
Different types of stellarators

- **Original Stellarator:**

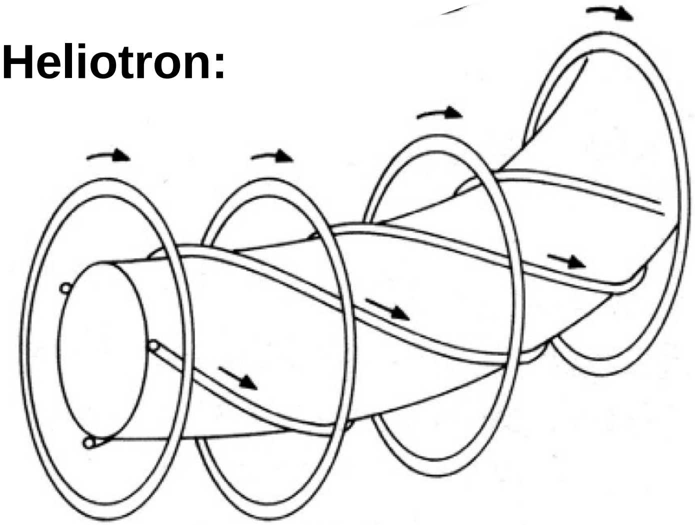


Field cancels out on axis

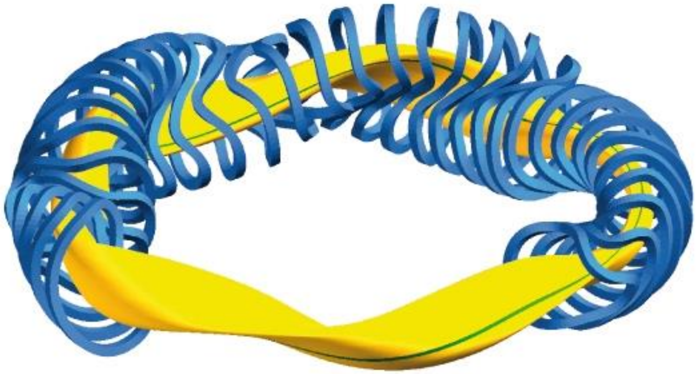
- **Torsatron:**



- **Heliotron:**

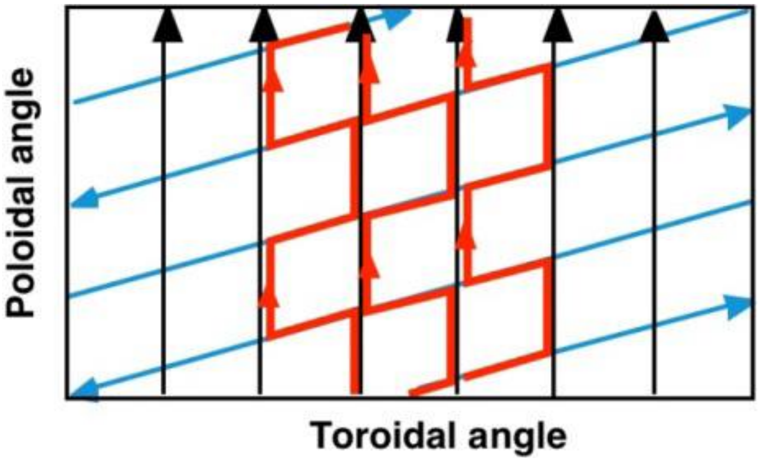


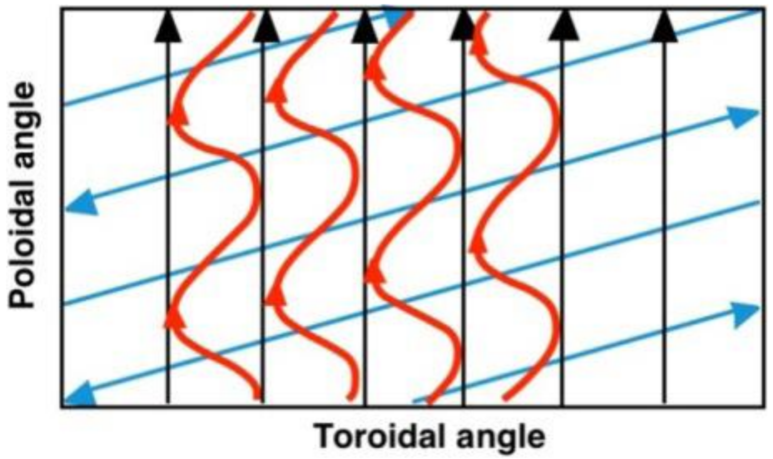
- **Helias:**

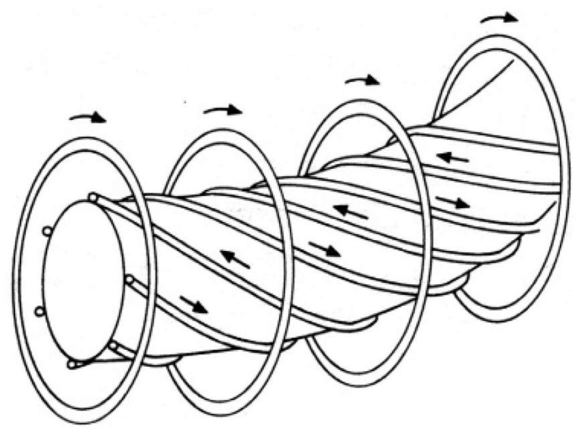


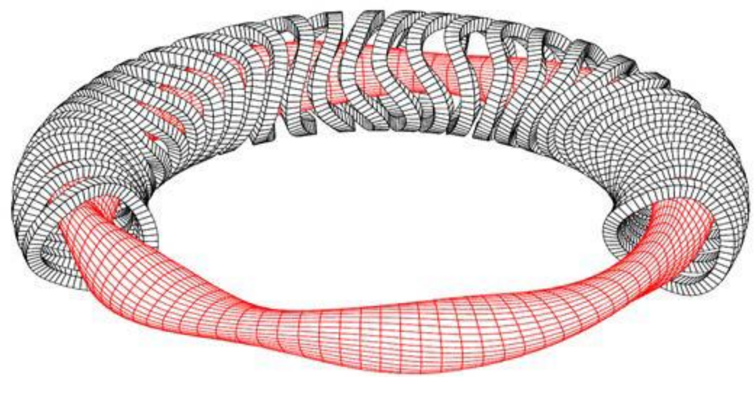
Helical coils with toroidal field coils can be replaced by smoothed twisting coils

- **Superposition of helical windings (blue) and the TF-coils (black) and mapped into the *θ-Φ* plane.**
- **Realization of the smoothed twisting coils**



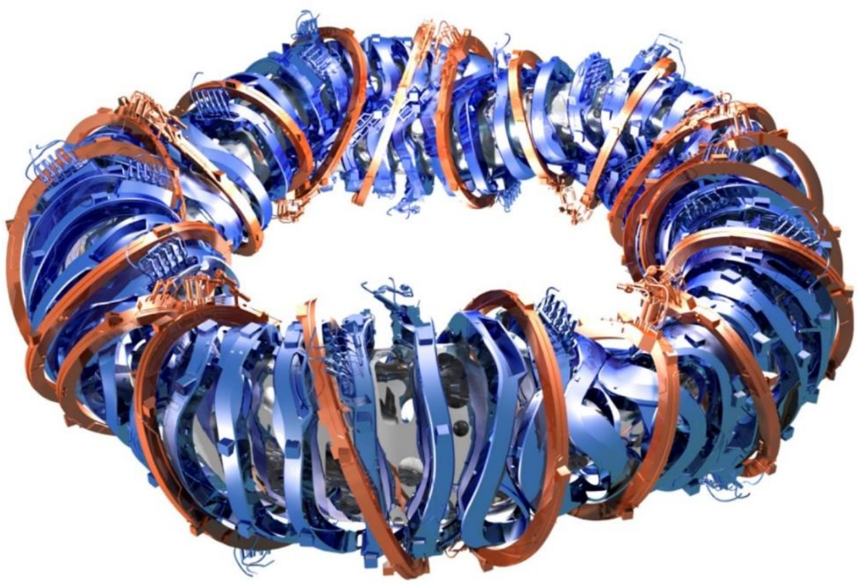


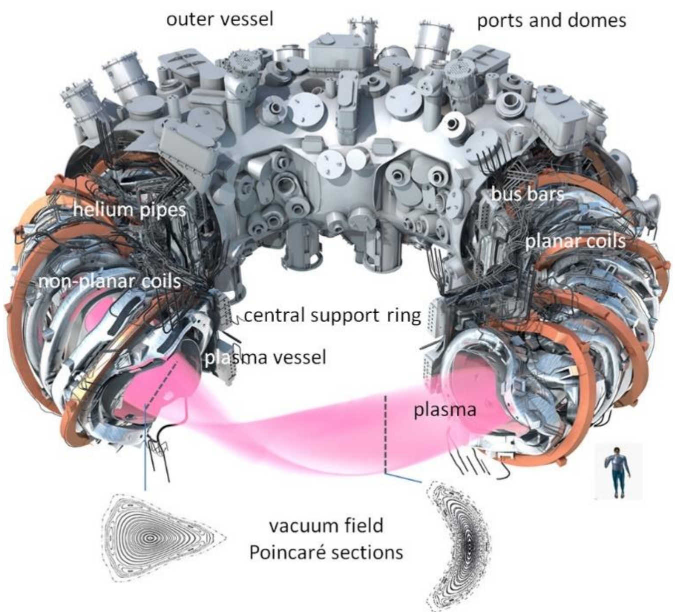




Fusion Physics, by IAEA

Wendelstein 7-X is a (Helias) stellarator built by Max Planck Institute for Plasma Physics (IPP)





- **Wemdelstein 7-x is now installing new diverters.**

Advantages of Stellarator

- **No need to drive plasma current. It is intrinsically steady state.**
- **With zero net current, one potentially dangerous class of MHD instabilities, the current-driven kink modes, is eliminated.**
- **Magnetic configuration is set by external coils, not by currents in the plasma. Stellarators do not suffer violent disruptions.**
- **Potential for greater range of designs and optimization of fusion performance.**

Disadvantages of Stellarator

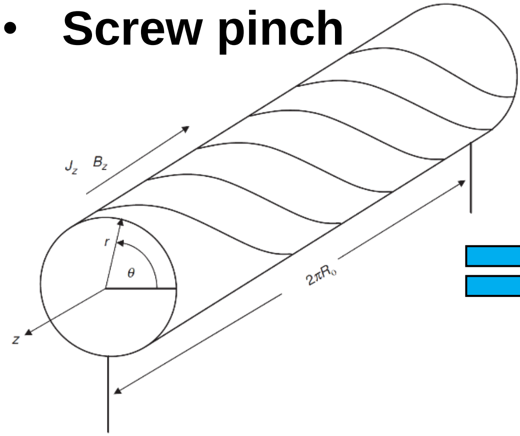
- Complicated coil configurations. It’s difficult to design. The precision requirement is high. It is expensive to build coils for stellarators.
- Achieving good particle confinement in stellarators is more difficult than that in tokamaks.
- Divertors and heat load geometry in stellarators is more complicated than those in tokamaks.

Course Outline

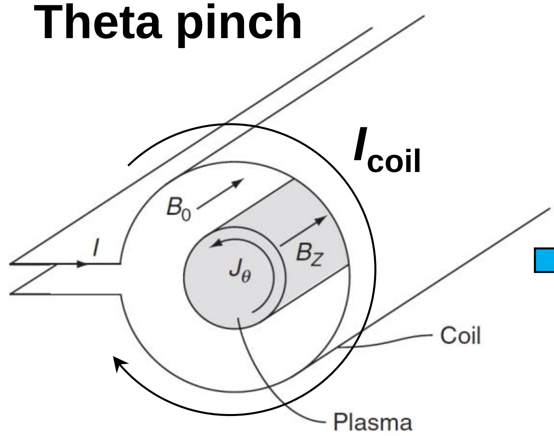
- Magnetic confinement fusion (MCF)
  - Gyro motion, MHD
  - 1D equilibrium (z pinch, theta pinch)
  - Drift: ExB drift, grad B drift, and curvature B drift
  - Tokamak, Stellarator (toroidal field, poloidal field)
  - Magnetic flux surface
  - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
  - Stability (Kink instability, sausage instability, Safety factor Q)
  - Central-solenoid (CS) start-up (discharge) and current drive
  - CS-free current drive: electron cyclotron current drive, bootstrap current.
  - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

General screw pinch is flexible with varies range of $\boldsymbol{\beta}$

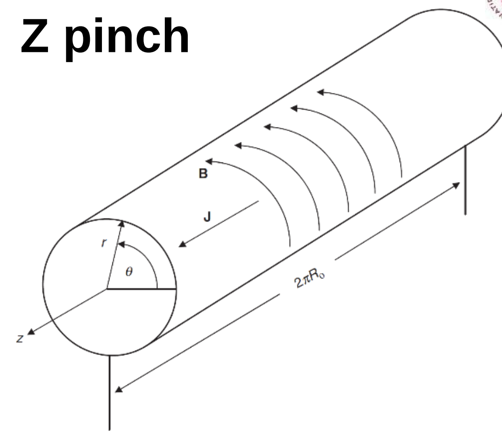
- **Screw pinch**



- **Theta pinch**



- **Z pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\vec{B} = B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta}$$

$$\vec{B} = B_\theta \hat{\theta}$$
$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

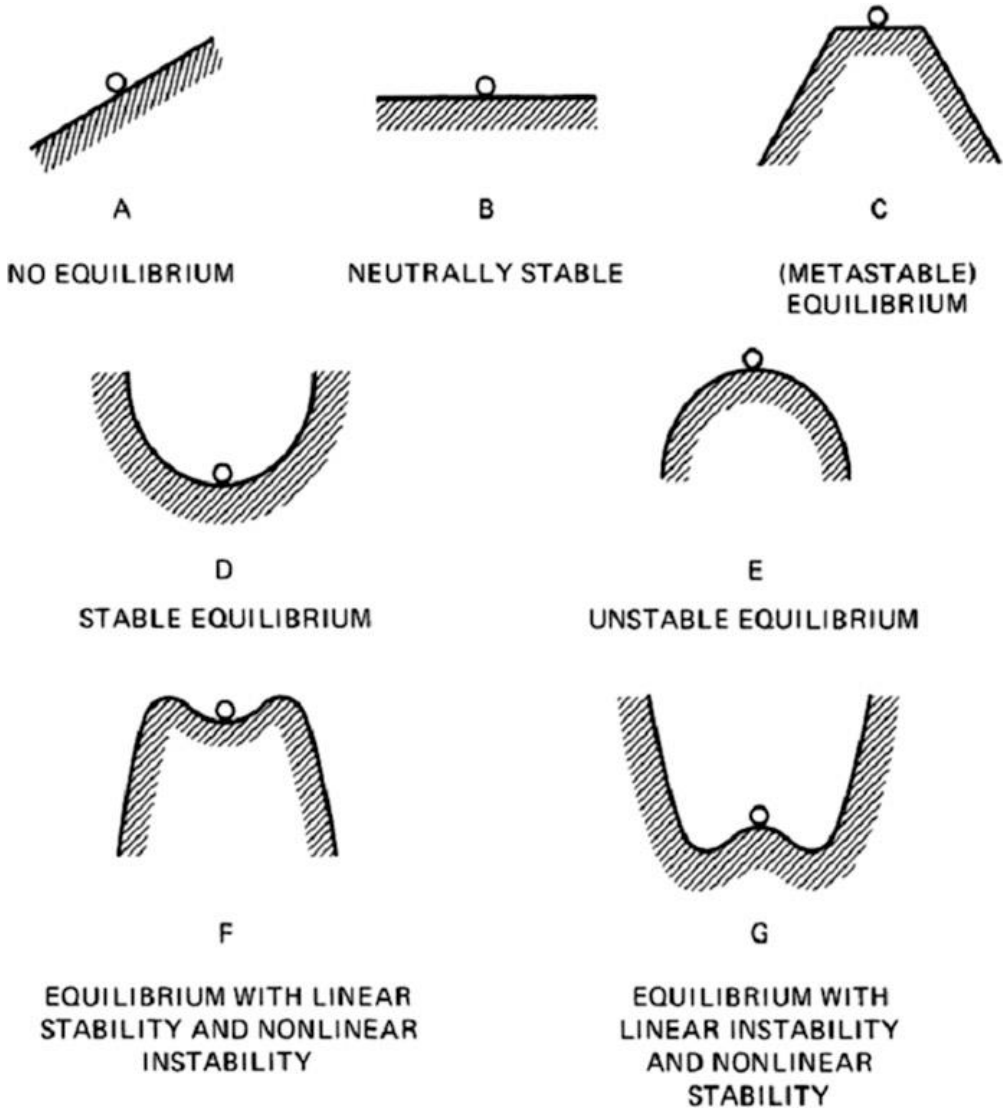
$$\int_0^a \pi r^2 dr \left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2 \rangle\right)$$

$$\beta_t = \frac{2\mu_o \langle p \rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o \langle p \rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_o \langle p \rangle}{B_o^2 + {B_{\theta a}}^2}$$

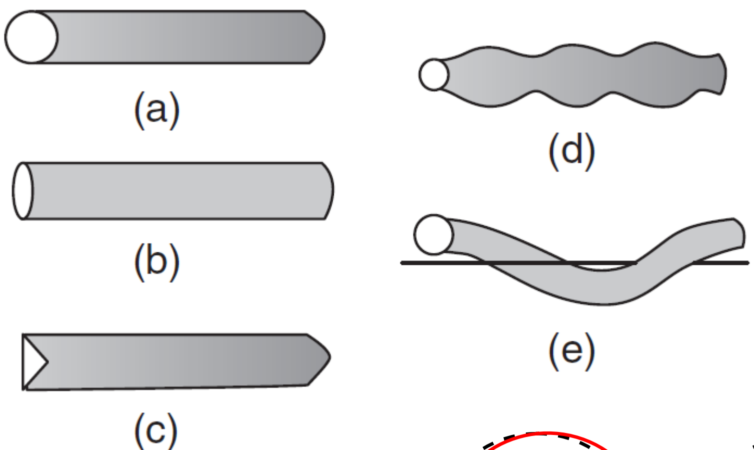
$$\boxed{0 \leq \langle \beta \rangle \leq 1}$$

An equilibrium state may not be stable

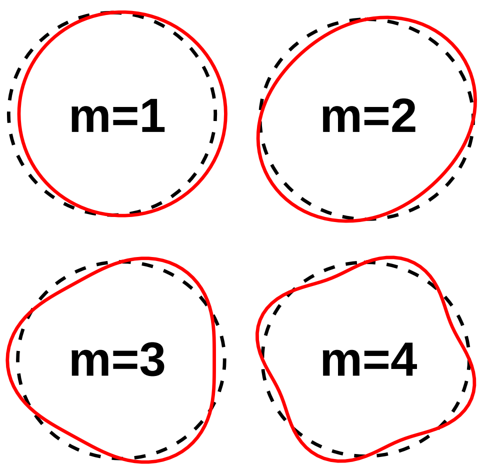


A cylindrical plasma column may not be stable

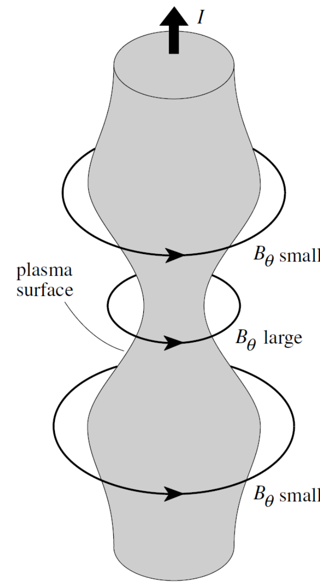
- **Instabilities of theta pinch**



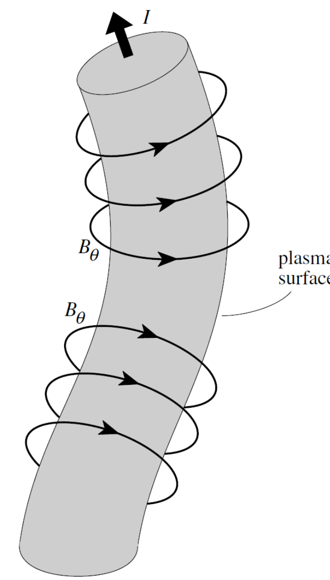
(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0



- **Instabilities of z pinch**



Sausage instability (m=0)

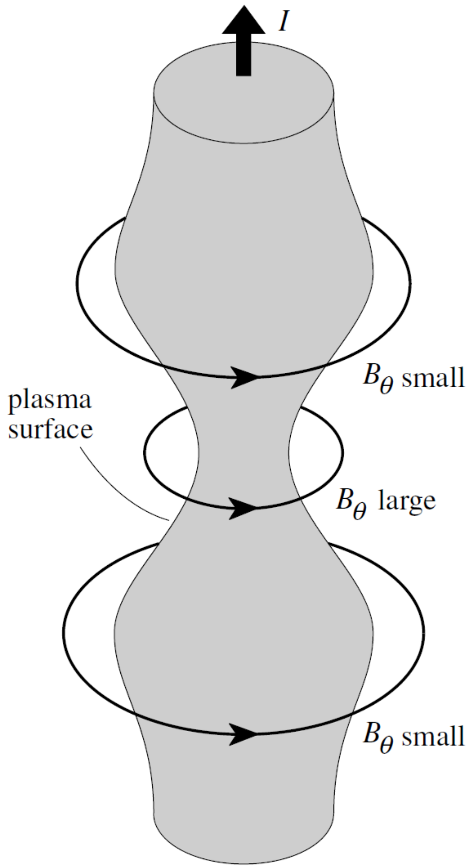


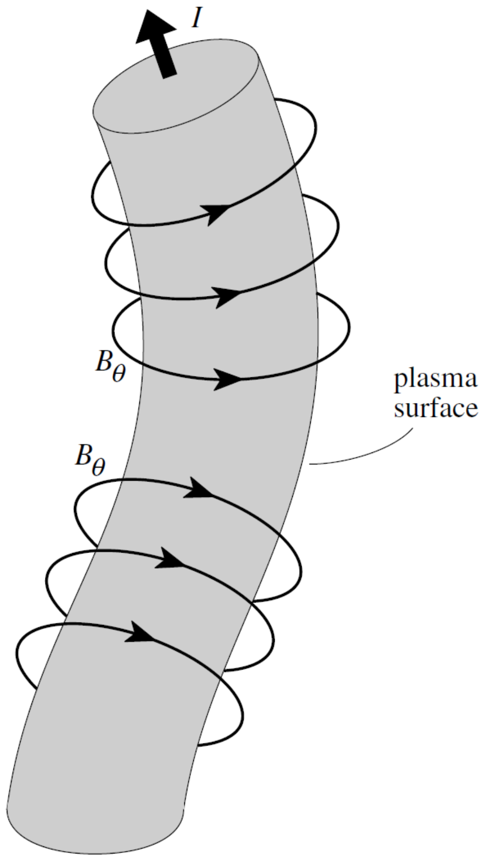
Kink instability (m=1)

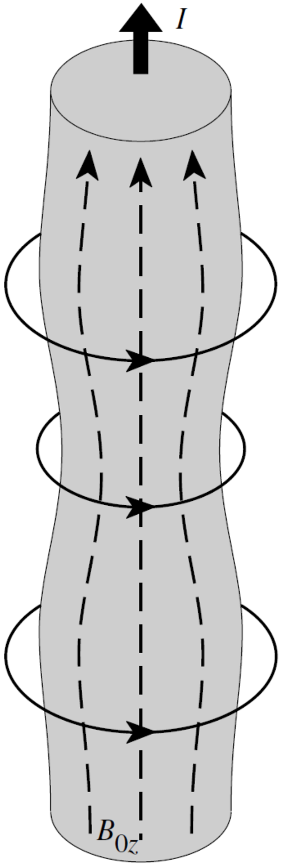
$$\zeta(\vec{r}) = \zeta(r) exp(im\theta + ikz)$$

A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**
- **Kink instability**



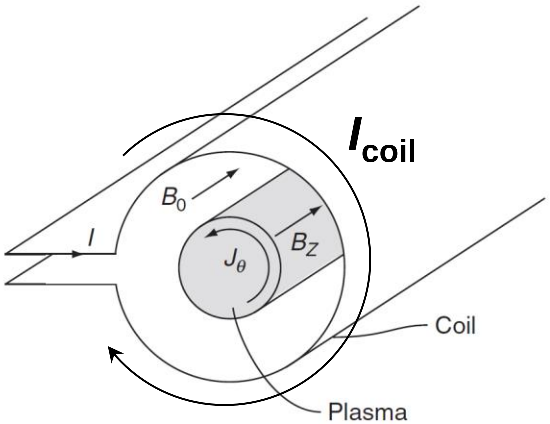




- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_o B_\theta(r)}$ **Kruskal–Shafranov limit**

Theta pinch is stable while z pinch is unstable

- **Theta pinch**

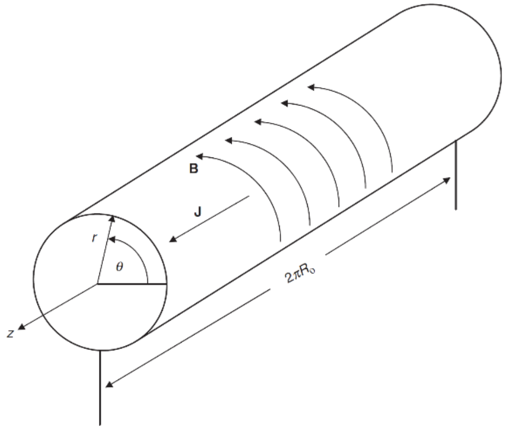


$$\vec{B} = B_z\hat{z}$$

$$q_\theta = \infty$$

Stable

- **Z pinch**

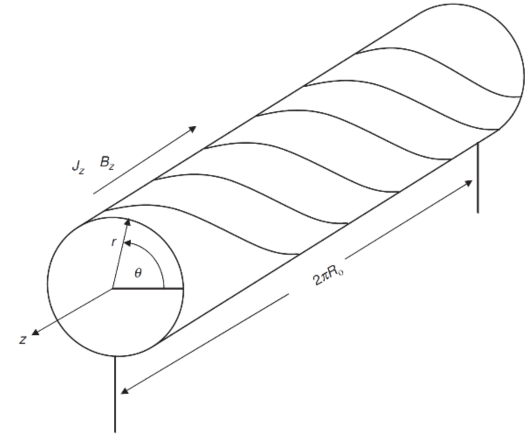


$$\vec{B} = B_\theta\hat{\theta}$$

$$q_z = 0$$

Unstable

- **Screw pinch**



$$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$$

$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

q **can be controlled.**

Stable/Unstable

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

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Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

J. P. Freidberg, Ideal Magnetohydrodynamics

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"

- **Equilibrium state:**

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} \right] = \vec{j} \times \vec{B} - \nabla p \equiv 0$$

$$\vec{j} \times \vec{B} = \nabla p$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{\mu_o}\left[(\vec{B} \cdot \vec{\nabla})\,\vec{B} - \frac{1}{2}\vec{\nabla} B^2 \right] = \nabla p$$

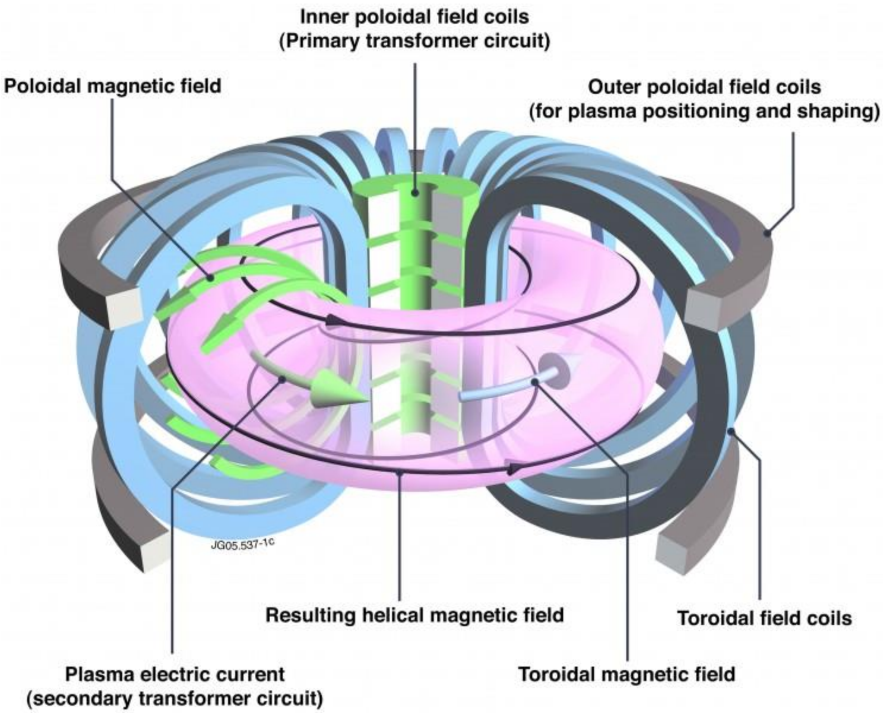
$$\nabla \left(P + \frac{B^2}{2\mu_o} \right) = \frac{1}{\mu_o}(\vec{B} \cdot \vec{\nabla})\,\vec{B}$$

Magnetic pressure **Magnetic tension** ← **Forces caused by curvature of the field lines**

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p \quad \Rightarrow \quad \vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p$$

$$\Downarrow$$

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p$$

$$\Downarrow$$

$$\vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j} \quad \Rightarrow \quad \nabla \cdot \vec{j} = 0$$

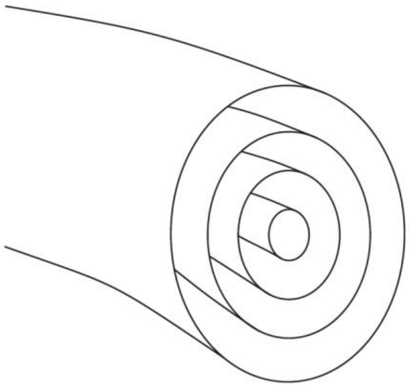
$$\nabla \cdot \vec{B} = 0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

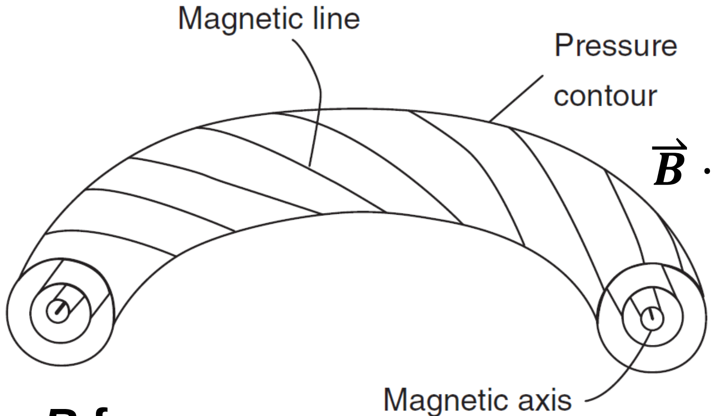
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

Magnetic lines lying on pressure contour

- **Contours of constant pressure**



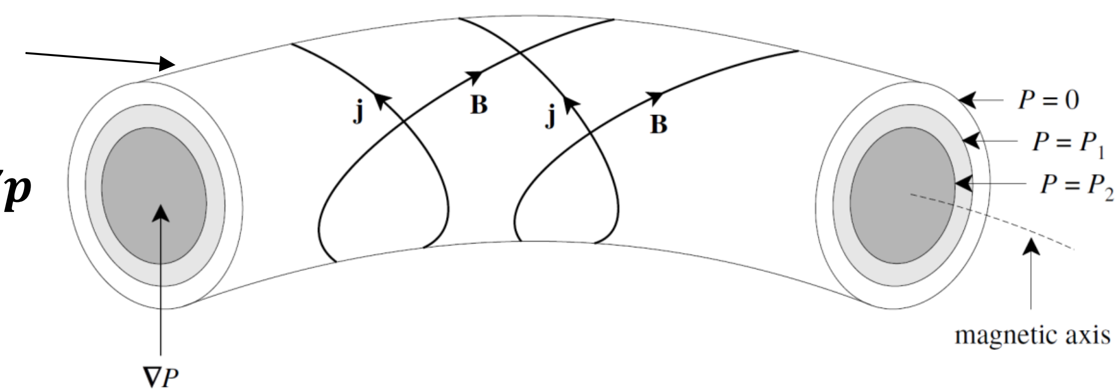
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Derivation of Grad-Shafranov equation

$$\vec{j} \times \vec{B} = \nabla p \qquad \nabla \times \vec{B} = \mu_o \vec{j}$$

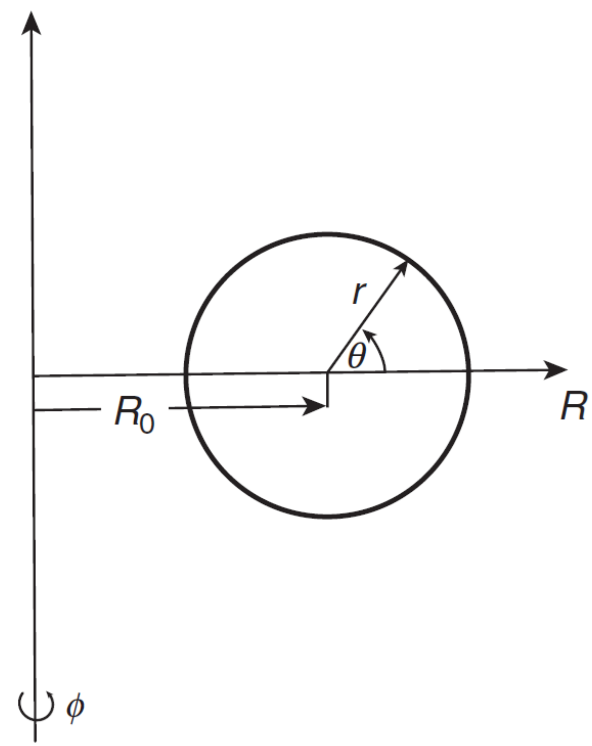
$$\nabla \cdot \vec{j} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = \left(B_R, B_\phi, B_z\right) \qquad \textbf{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\cancel{\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$



- **Represent the magnetic field using a vector potential *A*:**

$$\vec{B} = \nabla \times \vec{A} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\cancel{\frac{\partial A_R}{\partial \phi}}\right)$$

$$= \hat{R}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\right)$$

$$\equiv \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z$$

$$B_R = -\frac{\partial A_\phi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

Pressure can be written as a function of flux

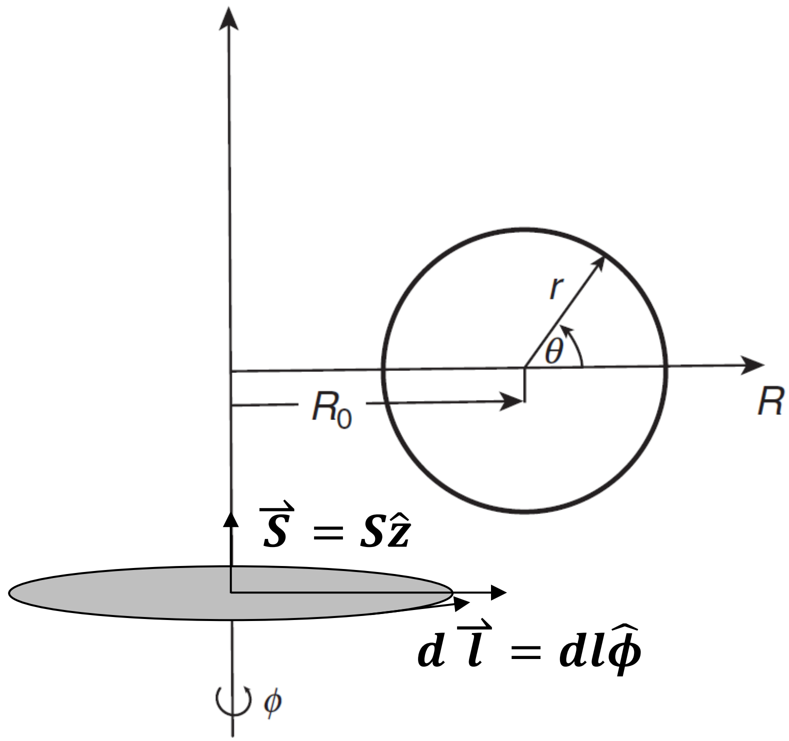
$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$

$$B_R = -\frac{\partial A_\phi}{\partial z}$$

$$B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

$$\psi \equiv \frac{1}{2\pi}\int \vec{B}\cdot d\vec{S} = \frac{1}{2\pi}\int (\nabla\times\vec{A})\cdot d\vec{S}$$

$$= \frac{1}{2\pi}\int \vec{A}\, 2\pi R \cdot d\vec{l} = \int \vec{A}\cdot\hat{\phi} R\, dl = RA_\phi$$



$$\vec{S} = S\hat{z}$$

$$d\vec{l} = dl\hat{\phi}$$

$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}\cdot\nabla\psi = B_R\frac{\partial\psi}{\partial R} + B_\phi\frac{1}{R}\cancel{\frac{\partial\psi}{\partial\phi}} + B_z\frac{\partial\psi}{\partial z} = B_R\frac{\partial\psi}{\partial R} + B_z\frac{\partial\psi}{\partial z}$$

$$= \left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)\frac{\partial\psi}{\partial R} + \left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\frac{\partial\psi}{\partial z} = 0$$

$$\vec{B}\cdot\nabla\psi = 0$$

$$\vec{B}\cdot\nabla p = 0$$

for $\nabla p \neq 0$:

$$p = p(\psi)$$

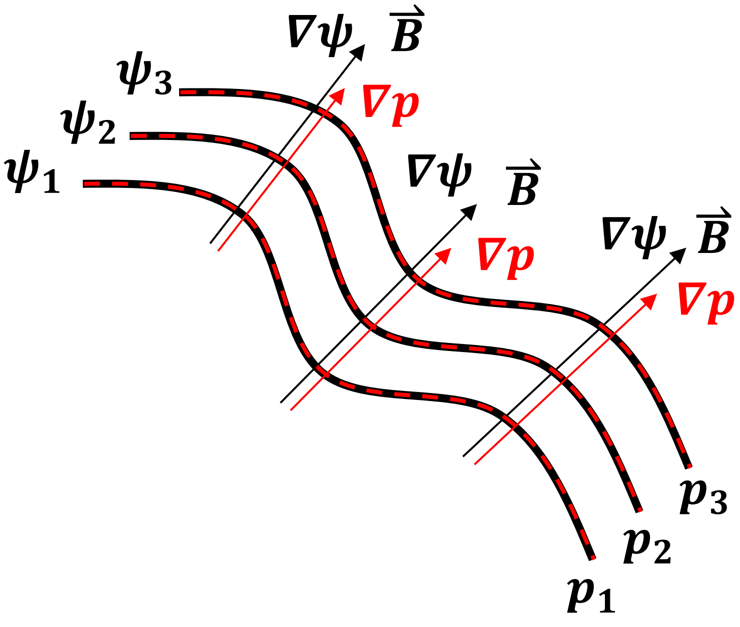
Pressure can be written as a function of flux

$$\vec{B} \cdot \nabla\psi = 0$$

$$\vec{B} \cdot \nabla p = 0$$

for $\nabla p \neq 0$:

$$p = p(\psi)$$



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{\phi}$ component of the force-balance equation:**

$$\left(\vec{j} \times \vec{B} = \nabla p\right)_{\phi} \qquad j_z B_R - j_R B_z = \frac{1}{R}\frac{\partial p}{\partial \phi} \equiv 0$$

- **Ampére's law:**

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \times \vec{B} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial B_z}{\partial \phi}} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi) - \frac{1}{R}\cancel{\frac{\partial B_R}{\partial \phi}}\right)$$

$$= \hat{R}\left(-\frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi)\right)$$

$$= \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z$$

$$j_R = -\frac{1}{\mu_o}\frac{\partial B_\phi}{\partial z} \qquad j_z = \frac{1}{\mu_o}\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi)$$

$$\frac{B_R}{R}\frac{\partial}{\partial R}(RB_\phi) + B_z\frac{\partial B_\phi}{\partial z} = 0$$

Magnetic field can be decomposed into the poloidal component and the toroidal component

$$\frac{B_R}{R}\frac{\partial}{\partial R}(RB_\phi)+B_z\frac{\partial B_\phi}{\partial z}=0 \quad \Rightarrow \quad B_R\frac{\partial}{\partial R}(RB_\phi)+B_z\frac{\partial}{\partial z}(RB_\phi)=0$$

$$F\equiv RB_\phi \quad \Rightarrow \quad B_R\frac{\partial F}{\partial R}+B_z\frac{\partial F}{\partial z}=0 \quad \Rightarrow \quad \vec{B}\cdot\nabla F=0$$

$$\left(\frac{\partial}{\partial\phi}=0\right)$$

$$\Downarrow$$

$$F=F(\psi)$$

$$\vec{B}\cdot\nabla p=0$$

$$p=p(\psi)$$

$$B_R=-\frac{\partial A_\phi}{\partial z}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

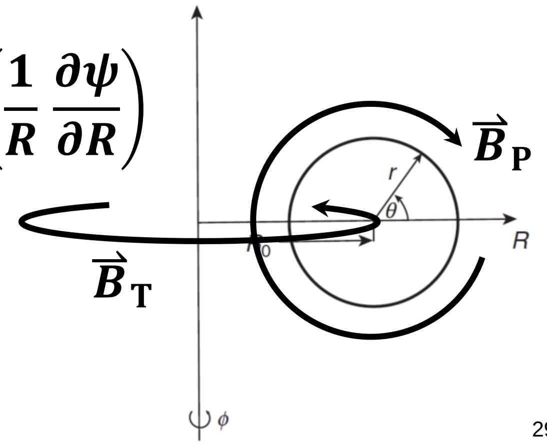
$$B_z=\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)=\frac{1}{R}\frac{\partial\psi}{\partial R} \qquad (\psi=RA_\phi)$$

$$B_\phi=\frac{F(\psi)}{R}$$

$$\vec{B}=\hat{R}B_R+\hat{\phi}B_\phi+\hat{z}B_z=\hat{R}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)+\hat{\phi}\left(\frac{F(\psi)}{R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

$$\equiv\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}+\frac{F(\psi)}{R}\hat{\phi}$$

Poloidal component $\vec{B}_P$ Toroidal component $\vec{B}_T$



Arbitrary integration constant associated with flux can be chosen such that flux equals to zero on the field axis

- **The poloidal flux of the area of a washer-shaped surface lying in the *z* = 0 plane from *R* = R_a to an arbitrary ψ contour defined by $\psi = \psi(R_b, 0)$:**

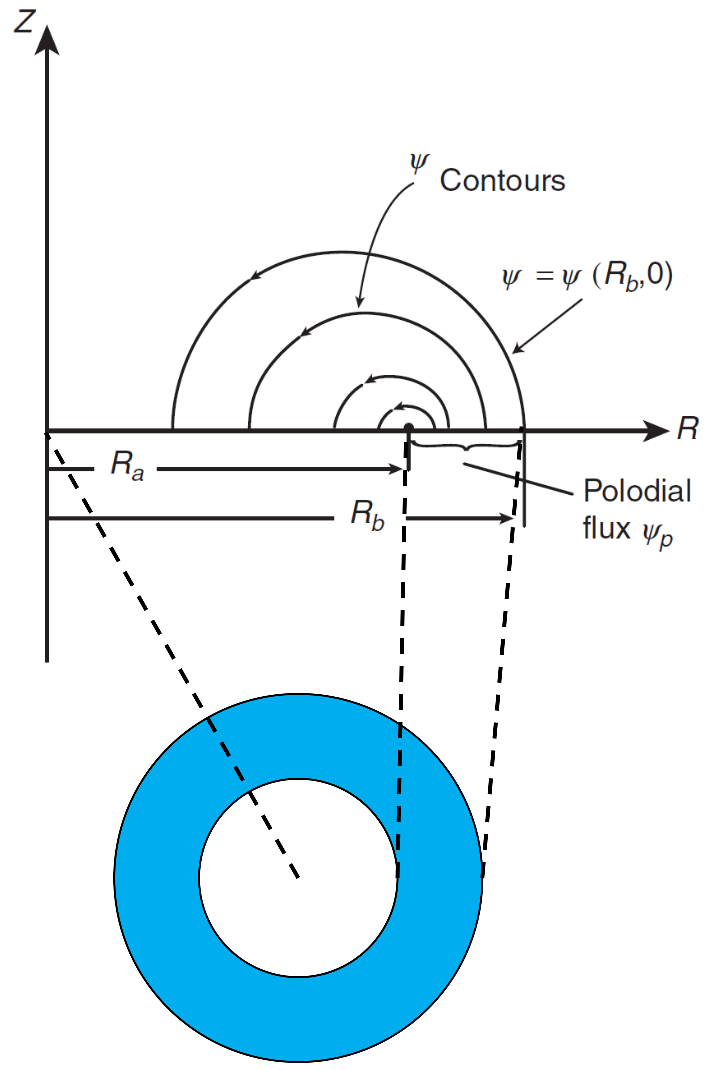
$$\psi_P \equiv \frac{1}{2\pi}\int \vec{B}_P \cdot d\,\vec{S}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{R_a}^{R_b} dR R B_z(R, 0)$$

$$= \psi(R_b, 0) - \psi(R_a, 0)$$

$$\equiv \psi(R_b, 0)$$

where $\psi(R_a, 0) \equiv 0$ is chosen.



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{R}$ component of the force-balance equation:**

$$\left(\vec{j} \times \vec{B} = \nabla p\right)_{\mathrm{R}} \qquad j_{\phi} B_{\mathrm{z}} - j_{\mathrm{z}} B_{\phi} = \frac{\partial p}{\partial R}$$

$$B_{\mathrm{R}} = -\frac{1}{R}\frac{\partial \psi}{\partial z}$$

$$B_{\mathrm{z}} = \frac{1}{R}\frac{\partial \psi}{\partial R}$$

$$B_{\phi} = \frac{F(\psi)}{R}$$

- **Ampére's law:**

$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j}$$

$$\nabla \times \vec{B} = \hat{R}\left(-\frac{\partial B_{\phi}}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_{\mathrm{R}}}{\partial z} - \frac{\partial B_{\mathrm{z}}}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R B_{\phi}\right)\right) = \hat{R}\mu_{\mathrm{o}} j_{\mathrm{R}} + \hat{\phi}\mu_{\mathrm{o}} j_{\phi} + \hat{z}\mu_{\mathrm{o}} j_{z}$$

$$\mu_{\mathrm{o}} j_{\phi} = \frac{\partial B_{\mathrm{R}}}{\partial z} - \frac{\partial B_{\mathrm{z}}}{\partial R} = \frac{\partial}{\partial z}\left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) = -\frac{1}{R}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{R}\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial \psi}{\partial R}$$

$$\equiv -\frac{1}{R}\Delta^* \psi \qquad \text{where } \Delta^* \psi \equiv \frac{\partial^2 \psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) = R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2}\right)$$

$$\mu_{\mathrm{o}} j_{\mathrm{z}} = \frac{1}{R}\frac{\partial}{\partial R}\left(R B_{\phi}\right) = \frac{1}{R}\frac{\partial F}{\partial R} = \frac{1}{R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

Derivation of Grad-Shafranov equation

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial R}$$

$$B_\phi = \frac{F}{R} \qquad j_\phi = -\frac{1}{\mu_o R}\Delta^* \psi \qquad \frac{\partial p}{\partial R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$B_z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad j_z = \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\frac{1}{\mu_o R}\Delta^* \psi \frac{1}{R}\frac{\partial \psi}{\partial R} - \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}\frac{F}{R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\Delta^* \psi \frac{1}{\mu_o}\frac{1}{R^2} - \frac{1}{\mu_o}\frac{F}{R^2}\frac{dF}{d\psi} = \frac{dp}{d\psi}$$

$$\boxed{\mathbf{Grad-Shafranov\ equation:}\ \Delta^* \psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}}$$

$$\text{where } \Delta^* \psi = R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2}\right) \qquad \vec{B} = \left(\frac{\nabla \psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Derivation of Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \quad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \quad \vec{B} = \left(\frac{\nabla\psi}{R}\right)\times\hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

$$\mu_o j_\phi = -\frac{1}{R}\Delta^*\psi \qquad \mu_o j_z = \frac{1}{R}\frac{\partial F}{\partial R} \qquad F \equiv RB_\phi$$

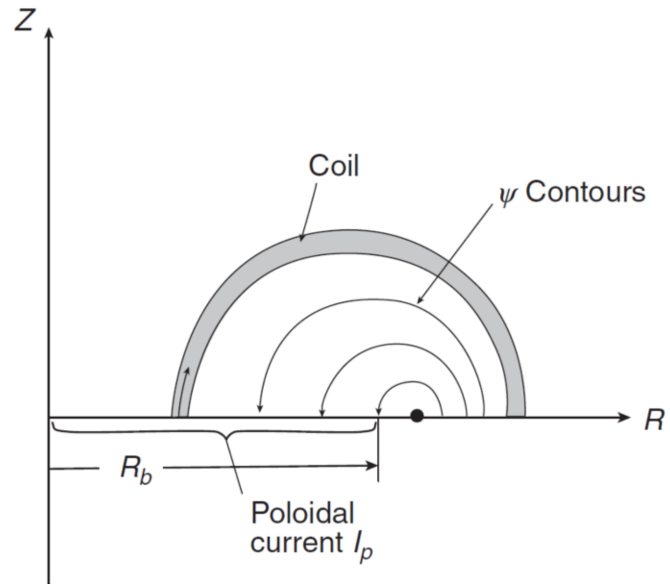
$$\mu_o j_R = -\frac{\partial B_\phi}{\partial z} = -\frac{1}{R}\frac{\partial}{\partial z}(RB_\phi) = -\frac{1}{R}\frac{\partial F}{\partial z}$$

$$\mu_o \vec{j} = \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z = \hat{R}\left(-\frac{1}{R}\frac{\partial F}{\partial z}\right) + \hat{\phi}\left(-\frac{1}{R}\Delta^*\psi\right) + \hat{z}\left(\frac{1}{R}\frac{\partial F}{\partial R}\right)$$

$$\equiv \left(\frac{\nabla F}{R}\right)\times\hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi}$$

$$I_P = \int \vec{j}_P\cdot d\vec{S} = -\int_0^{2\pi} d\phi \int_0^{R_b} dR R j_z(R,0)$$

$$= -2\pi\int_0^{R_b} dR R\frac{1}{R}\frac{\partial F(R,0)}{\partial R} = -2\pi F(\psi)$$



Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right) \qquad F = RB_\phi = -\frac{I_P}{2\pi}$$

- **The usual strategy to solve the Grad-Shafranov equation:**
  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

     $$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Example of the analytical solution of the Grad-Shafranov equation

$$\Delta^{*}\psi=-\mu_o R^2\frac{dp}{d\psi}-\frac{1}{2}\frac{dF^2}{d\psi}$$

- For $\mu_o\frac{dp}{d\psi}=-C_2$ $\quad \frac{1}{2}\frac{dF^2}{d\psi}=C_1$

$$\psi(R,z)=-\frac{C_1}{2}z^2+\frac{C_2}{8}R^4+C_3+C_4R^2+C_5\left(R^4-4R^2z^2\right)$$

$$B_R(R,z)=-\frac{1}{R}\left(-C_1z-8C_5R^2z\right)\quad B_z(R,z)=\frac{1}{R}\left(\frac{C_2}{2}R^3+2C_4R+C_5\left(4R^3-8Rz^2\right)\right)$$

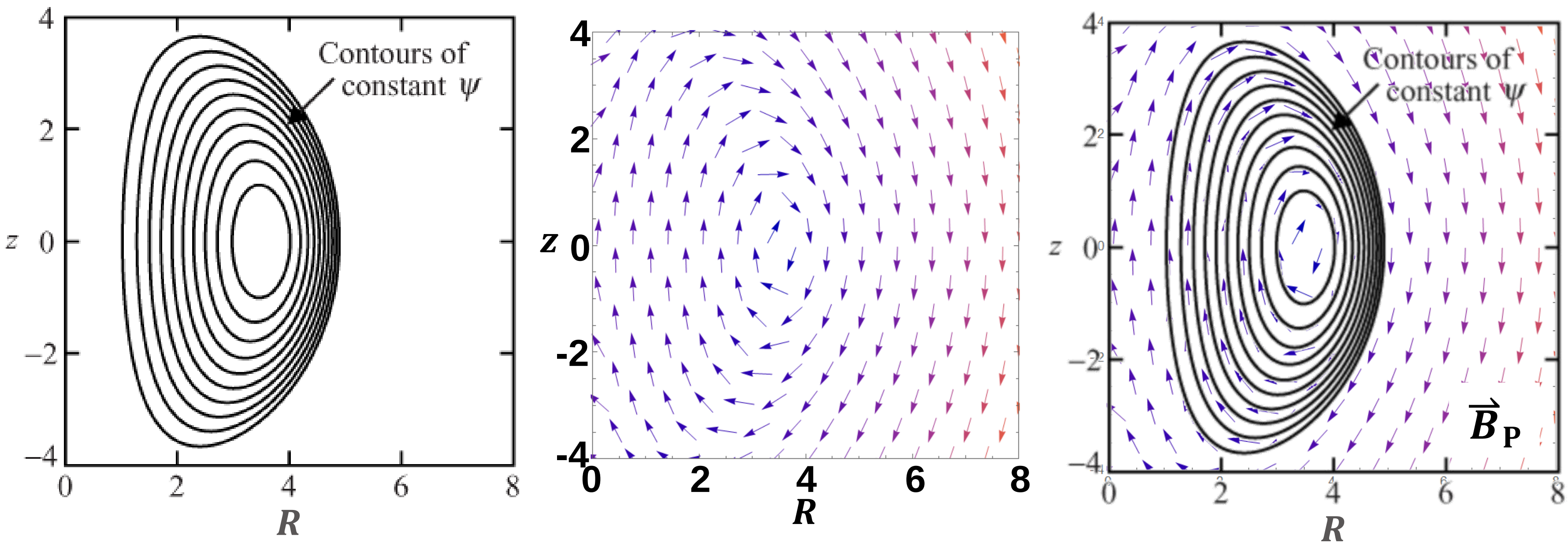
$C_1=1$

$C_2=-8$

$C_3=-20$

$C_4=20$

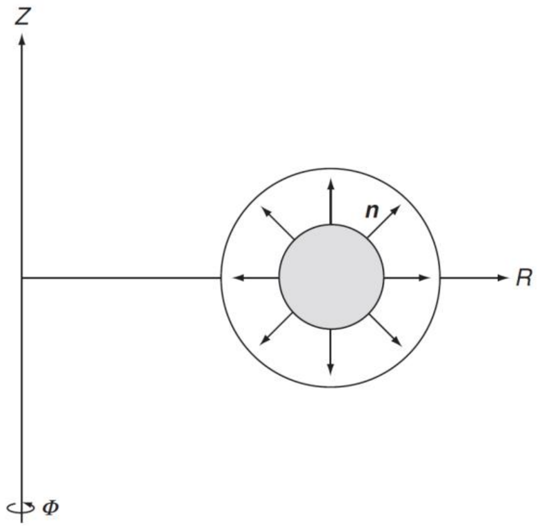
$C_5=0.2$

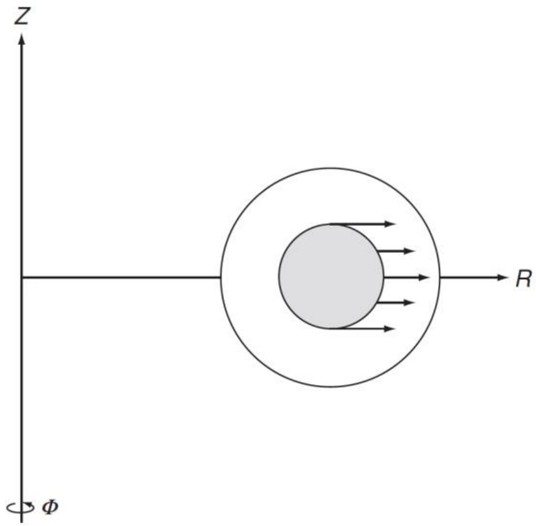


Magnetically confined toroidal equilibrium

1. Radial pressure balance in the poloidal plan needs to be provided so that the pressure contours form closed nested surfaces. Both toroidal and poloidal fields can readily accomplish this task.
2. The radially outward expansion force inherent in all toroidal geometries needs to be balanced without sacrificing stability.

- Forces associated with toroidal force balance are usually than those corresponding to radial pressure balance. However, they are more difficult to compensate.





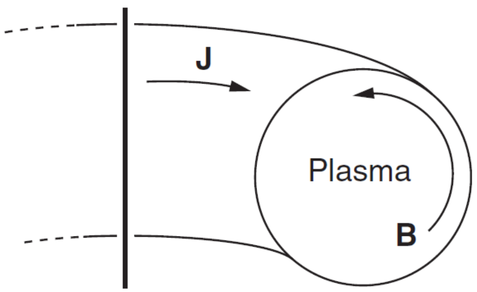
Toroidal configuration with a purely poloidal magnetic field

- **Hoop force:**

$$\psi_1 = \psi_2 \equiv \psi$$

$$S_1 < S_2 \quad B_1 > B_2$$

$$\vec{F}_{\mathrm{H,R}} \propto \hat{e}_{\mathrm{R}} \frac{{B_1}^2 S_1 - {B_2}^2 S_2}{2\mu_o} > 0$$

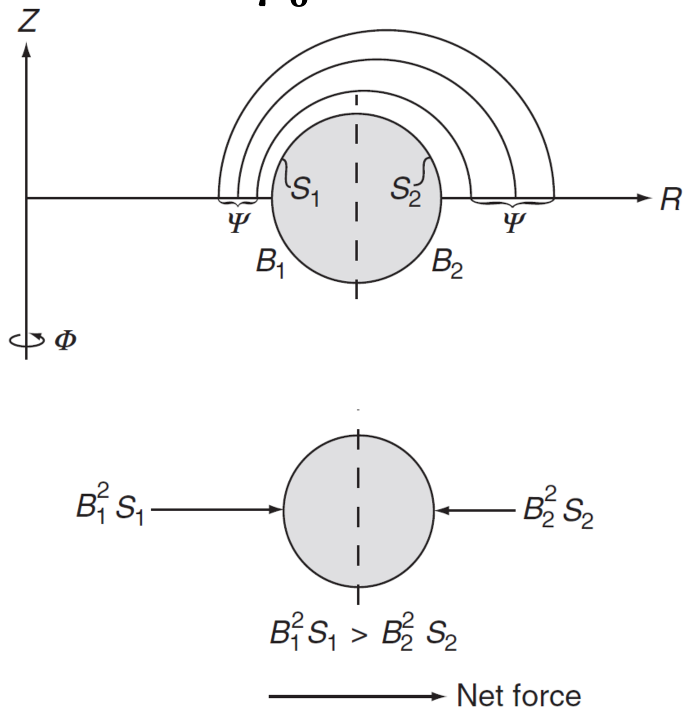


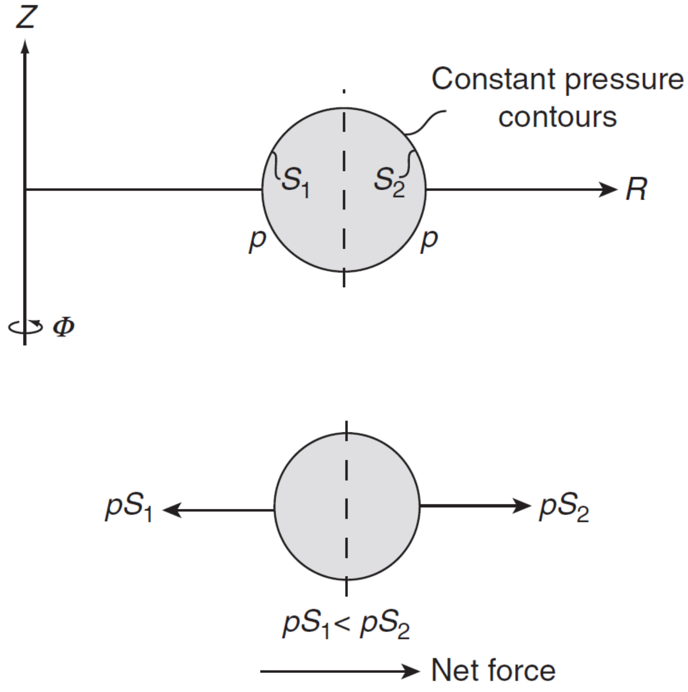
- **Tire tube force**

$$p_1 = p_2 \equiv p$$

$$S_1 < S_2$$

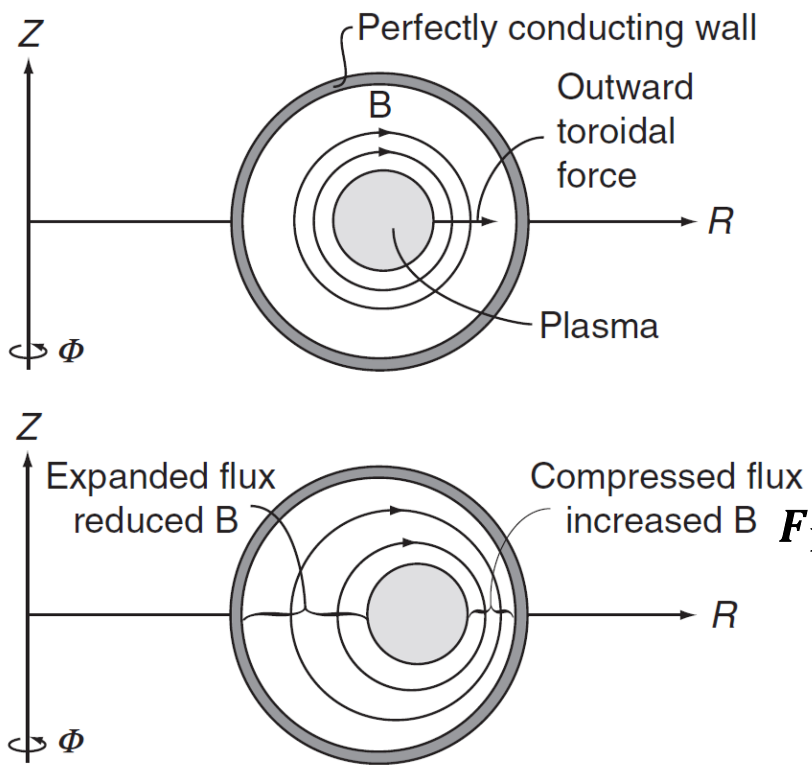
$$\vec{F}_{\mathrm{T,R}} \propto -\hat{e}_{\mathrm{R}}(pS_1 - \mathrm{p}S_2) > 0$$



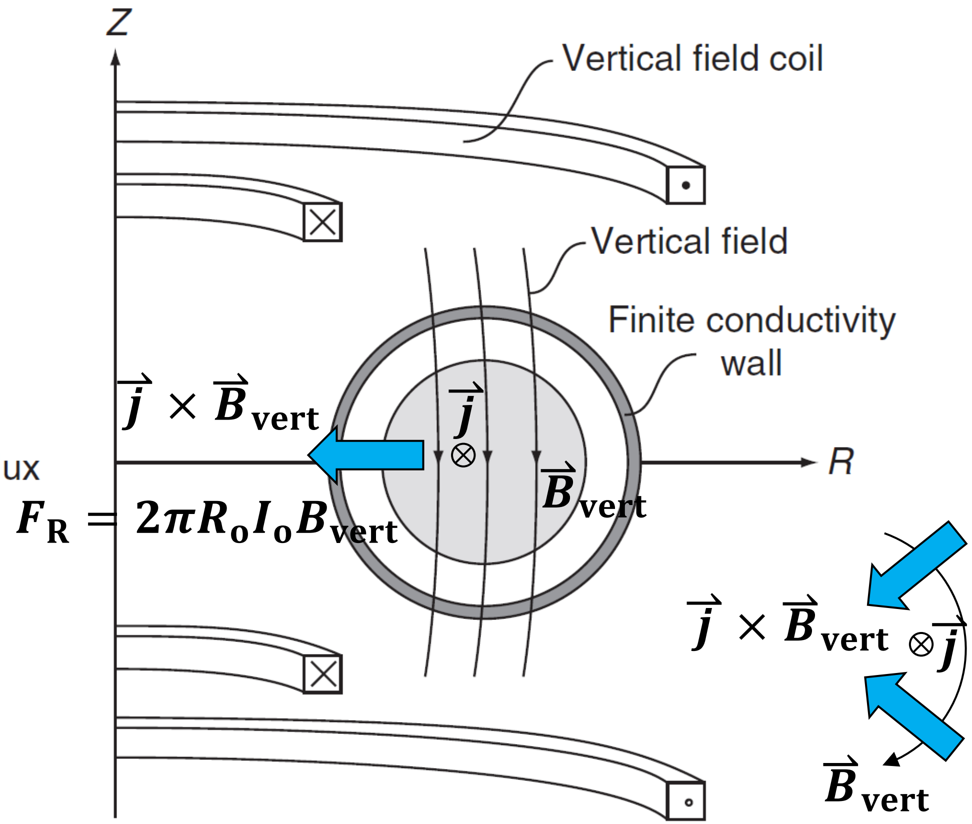


The outward force can be compensated by either a perfectly conducting shell or externally applied vertical field

- **Perfectly conducting shell**



- **Externally applied vertical field**

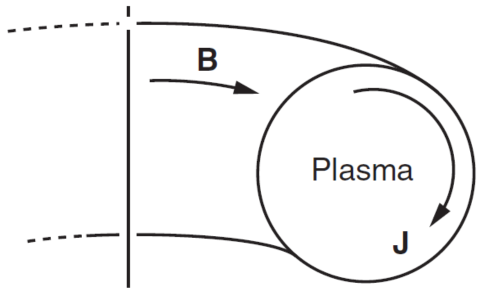


- **With a finite conductivity wall, flux can only remain compressed for about a skin time.**
- **This configuration develops disastrous MHD instabilities (z pinch).**

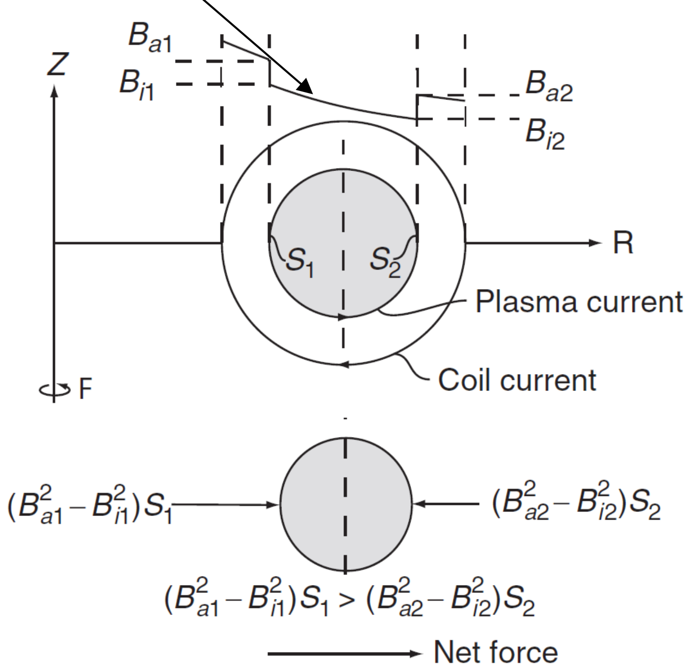
Toroidal configuration with a purely toroidal magnetic field, stable but NOT balanced

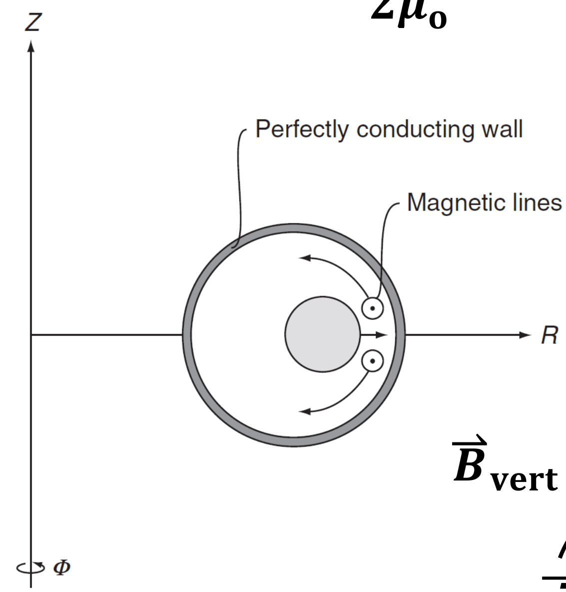
$$\vec{B} = B_\phi \hat{e}_\phi \qquad B_\phi = B_o \frac{R_o}{R} \qquad B_o = \frac{\mu_o I_c}{2\pi R_o}$$

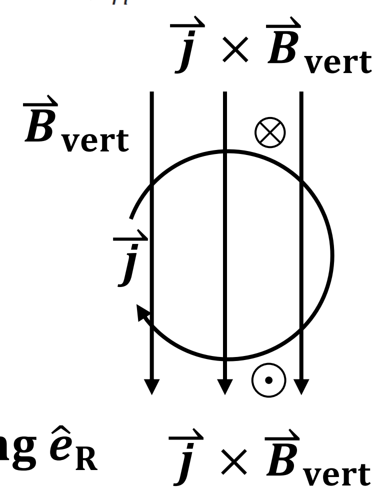
$$\vec{F}_R \propto \hat{e}_R \frac{\left(B_{a1}{}^2 - B_{i1}{}^2\right)S_1 - \left(B_{a2}{}^2 - B_{i2}{}^2\right)S_2}{2\mu_o} > 0$$



Plasma is diamagnetic



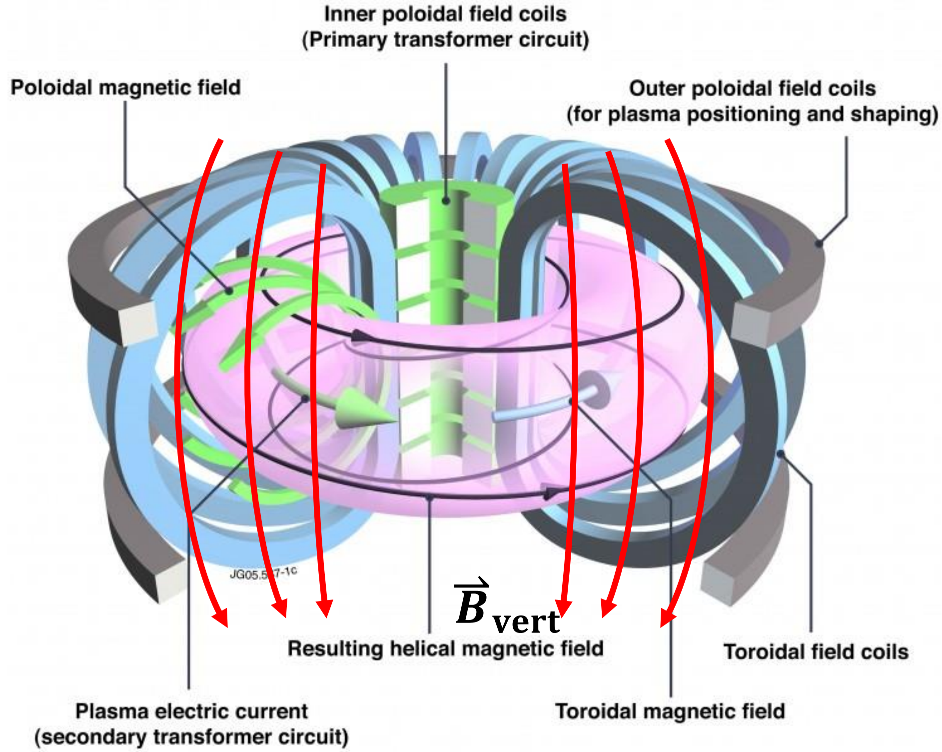




$\vec{j} \times \vec{B}_{vert}$

is not pointed along $\hat{e}_R$

Coils in a tokamak



- **Toroidal field coils (in poloidal direction) – generate toroidal field for confinement.**
- **Poloidal field coils – generate vertical field for plasma positioning and shaping.**
- **Central solenoid – for breakdown and generating plasma current (in toroidal direction) and thus generating poloidal field for confinement.**

https://www.euro-fusion.org/2011/09/tokamak-principle-2/

Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right) \qquad F = RB_\phi = -\frac{I_P}{2\pi}$$

- **The usual strategy to solve the Grad-Shafranov equation:**
  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

     $$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Application of solving Grad-Shafranov equation for designing a tokamak

- **Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, perfect conductor as the chamber.**
- **Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, insulator chamber.**
- **Given I_{plasma}, $p(\psi)$, $I(\psi)$, I_{coils}, free boundary of plasma, chamber with eddy current.**
- **Given I_{plasma}, $p(\psi)$, $I(\psi)$, fixed boundary of plasma. Then, use I_{coils}, free boundary of plasma and match the plasma shape calculated in the fixed boundary condition.**

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right)$$

$$I_P = -2\pi F(\psi)$$

$$\mu_o \vec{j} = \left(\frac{\nabla F}{R}\right) \times \hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi} \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Application of solving Grad-Shafranov equation for reconstructing a tokamak equilibrium state

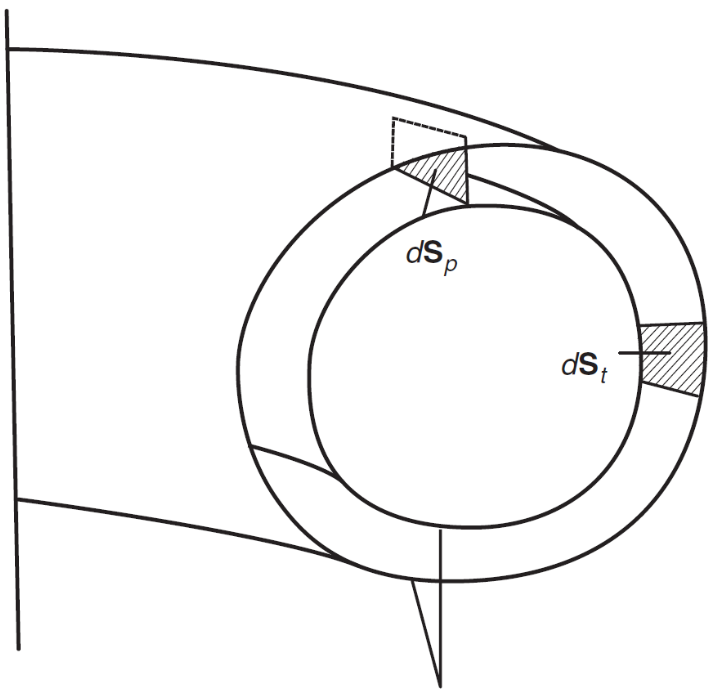
- Measure
  - boundary conditions, including ***ψ***, ***B***, etc., on the wall (using flux loop and B-dot probe).
  - Pressure.
  - Plasma current (using Rogowski coil).
- Reconstruct ***ψ***(***r***,***z***), ***j***, ***p***(***ψ***), ***I***(***ψ***), etc.

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right)$$

$$I_P = -2\pi F(\psi)$$

$$\mu_o \vec{j} = \left(\frac{\nabla F}{R}\right) \times \hat{\phi} + \left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi} \qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Fluxes and currents



- **Poloidal flux:** $$\psi_\mathrm{p} = \int \vec{B} \cdot d\,\vec{S}_\mathrm{p}$$
  $$\psi_\mathrm{p} = \psi_\mathrm{p}(p)$$
- **Toroidal flux:** $$\psi_\mathrm{t} = \int \vec{B} \cdot d\,\vec{S}_\mathrm{t}$$
- **Poloidal current:** $$I_\mathrm{p} = \int \vec{j} \cdot d\,\vec{S}_\mathrm{p}$$
- **Toroidal current:** $$I_\mathrm{t} = \int \vec{j} \cdot d\,\vec{S}_\mathrm{t}$$

Normalized plasma pressure, $\boldsymbol{\beta}$

$$\beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{2\mu_o\langle p\rangle}{B^2}$$

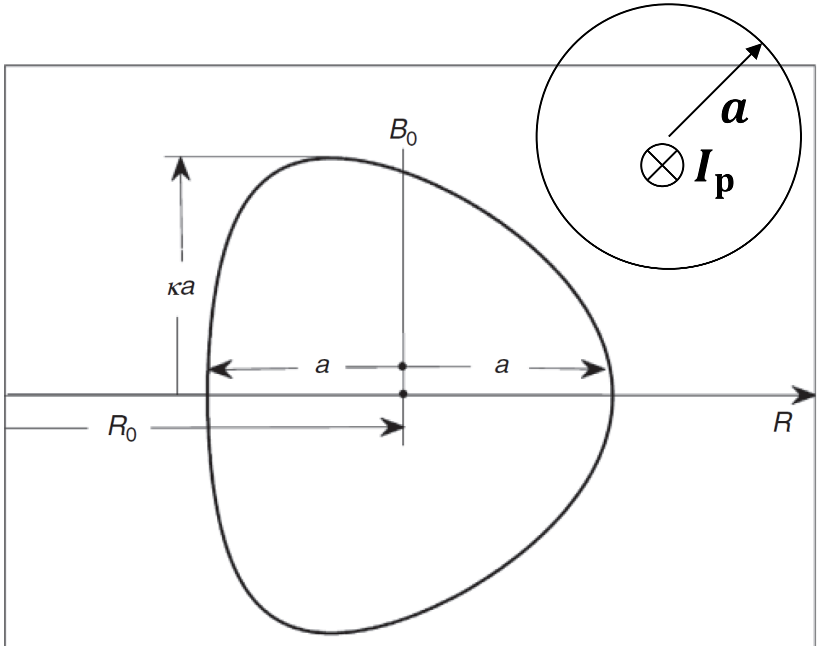
- **Plasma pressure:** $\langle p\rangle = \dfrac{1}{V_p}\displaystyle\int p\, d\,\vec{r}$
- **Magnetic pressure:** $P_B = \dfrac{B^2}{2\mu_o}$

$$B^2 = B_t^{\,2} + B_p^{\,2} = B_o^{\,2} + \left(\frac{\mu_o I_p}{2\pi a}\right)^2 \frac{2}{1+\kappa^2}$$

$$B_t^{\,2} = B_o^{\,2} \qquad B_o = B\ @R = R_o$$

$$B_p^{\,2} = \left(\frac{\mu_o I_p}{2\pi a}\right)^2 = \left(\frac{\mu_o I_p}{C_p}\right)^2$$

$$C_p \approx 2\pi a\sqrt{\frac{1+\kappa^2}{2}}$$

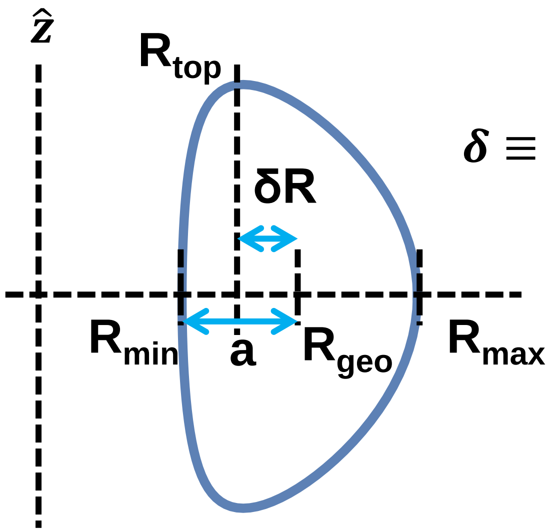


- **Elongation:** κ

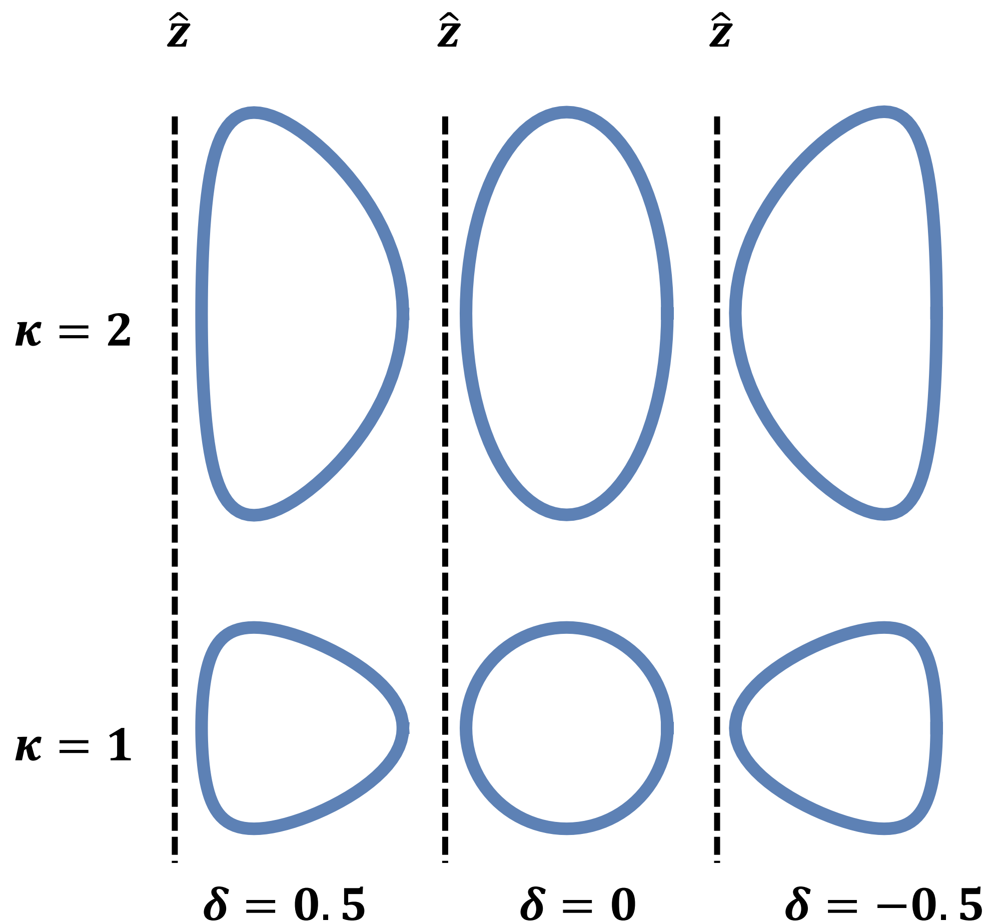
$$\beta_t = \frac{2\mu_o\langle p\rangle}{B_o^{\,2}} \qquad \beta_p = \frac{4\pi^2 a^2(1+\kappa^2)p}{\mu_o I_p^{\,2}}$$

$$\frac{1}{\beta} = \frac{1}{\beta_t} + \frac{1}{\beta_p}$$

Different poloidal shapes



$$\delta \equiv \frac{\delta R}{a}$$



$$r = R + a\mathrm{Cos}\big(\theta + \delta \mathrm{Sin}(\theta)\big)$$

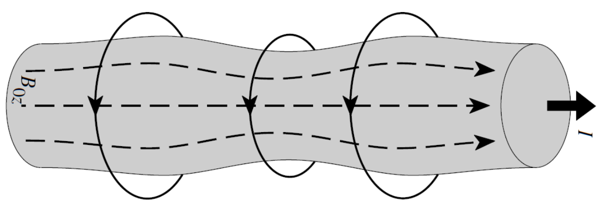
$$z = a\kappa \mathrm{Sin}(\theta)$$

- **Aspect ratio:** $\frac{R}{a}$
- **Elongation:** κ
- **Triangularity:** δ

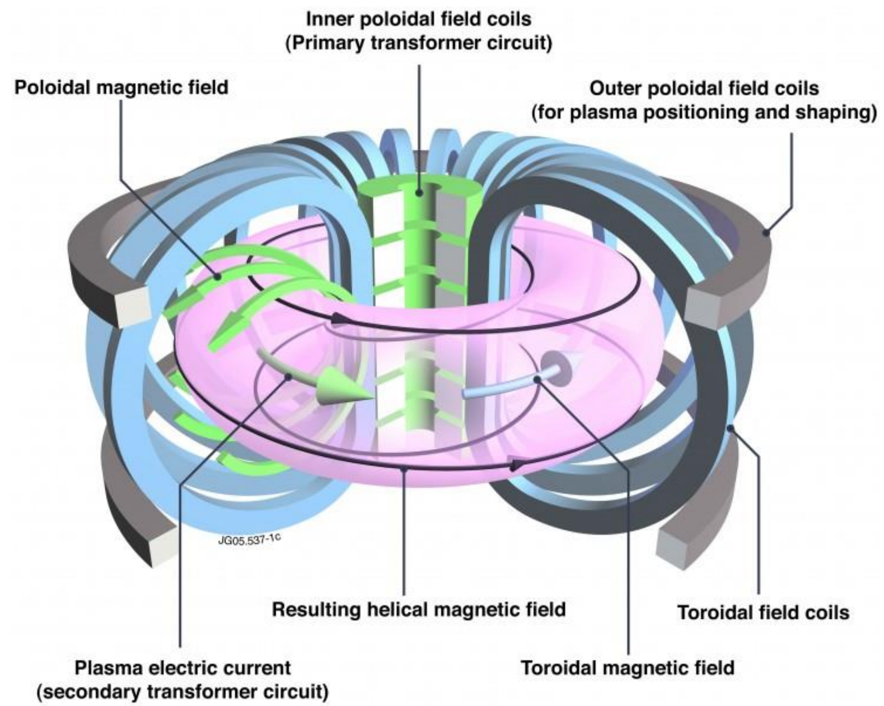
Safety factor

- **Kink Safety Factor:**

$$q^*(r) = \frac{aB_\mathrm{o}}{R_oB_\mathrm{p}} = \frac{2\pi a^2\kappa B_\mathrm{o}}{\mu_\mathrm{o}R_\mathrm{o}I_\mathrm{o}}$$

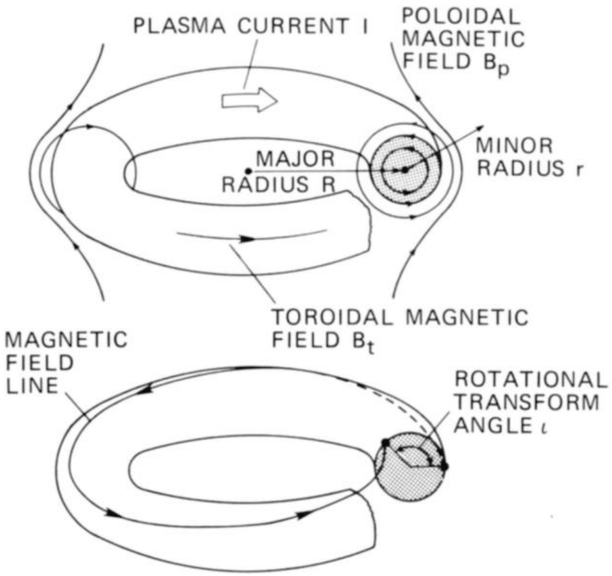


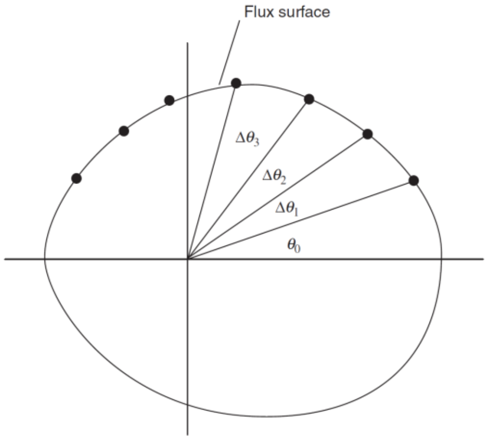
$$q(r) = \frac{rB_z(r)}{R_oB_\theta(r)}$$

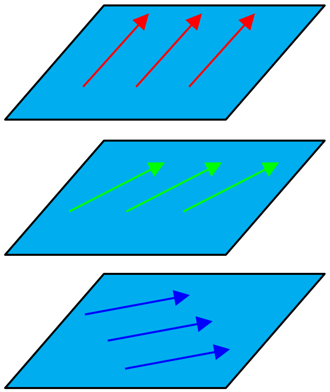


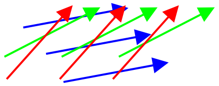
Safety factor

- **Shear:**









- **Rotational transform:** $$\iota \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \Delta\theta_{\mathrm{n}}$$

- **MHD safety factor:** $$q(V) \equiv \frac{2\pi}{\iota(V)} = \frac{d\psi_t/dV}{d\psi_p/dV}$$

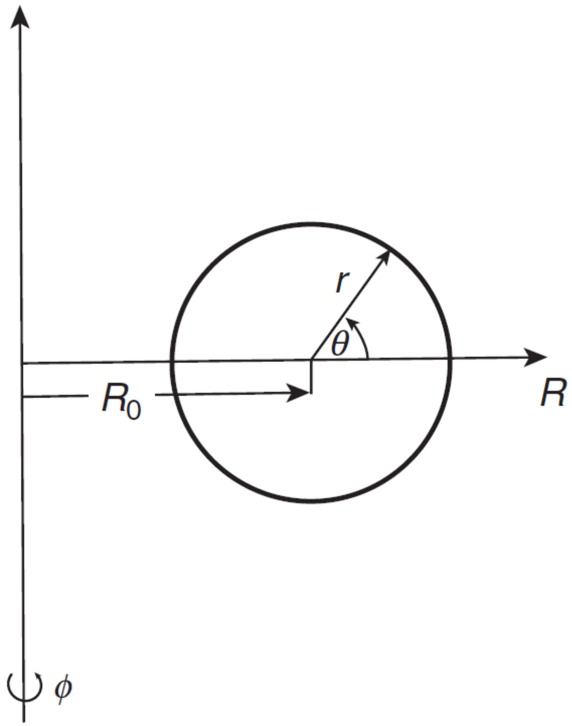
$$\psi_{\mathrm{t}} = \psi_{\mathrm{t}}(V) \qquad \psi_{\mathrm{p}} = \psi_{\mathrm{p}}(V)$$

- **Shear:** $$s(V) \equiv 2\frac{V}{q}\frac{dq}{dV}$$

$$\iota(V) \equiv 2\pi\left(\frac{d\psi_p/dV}{d\psi_t/dV}\right)$$

C. C. Baker, *et al*, Nuclear Technology/Fusion, 1, 5 (1981)

Volume of the constant flux



$$R = R_o + r\cos\theta$$

$$Z = r\sin\theta$$

$$\psi = \psi(r, \theta)$$

$$r = \hat{r}(\theta, \psi)$$

$$V(\psi) = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\hat{r}} R r\,dr\,d\theta\,d\phi$$

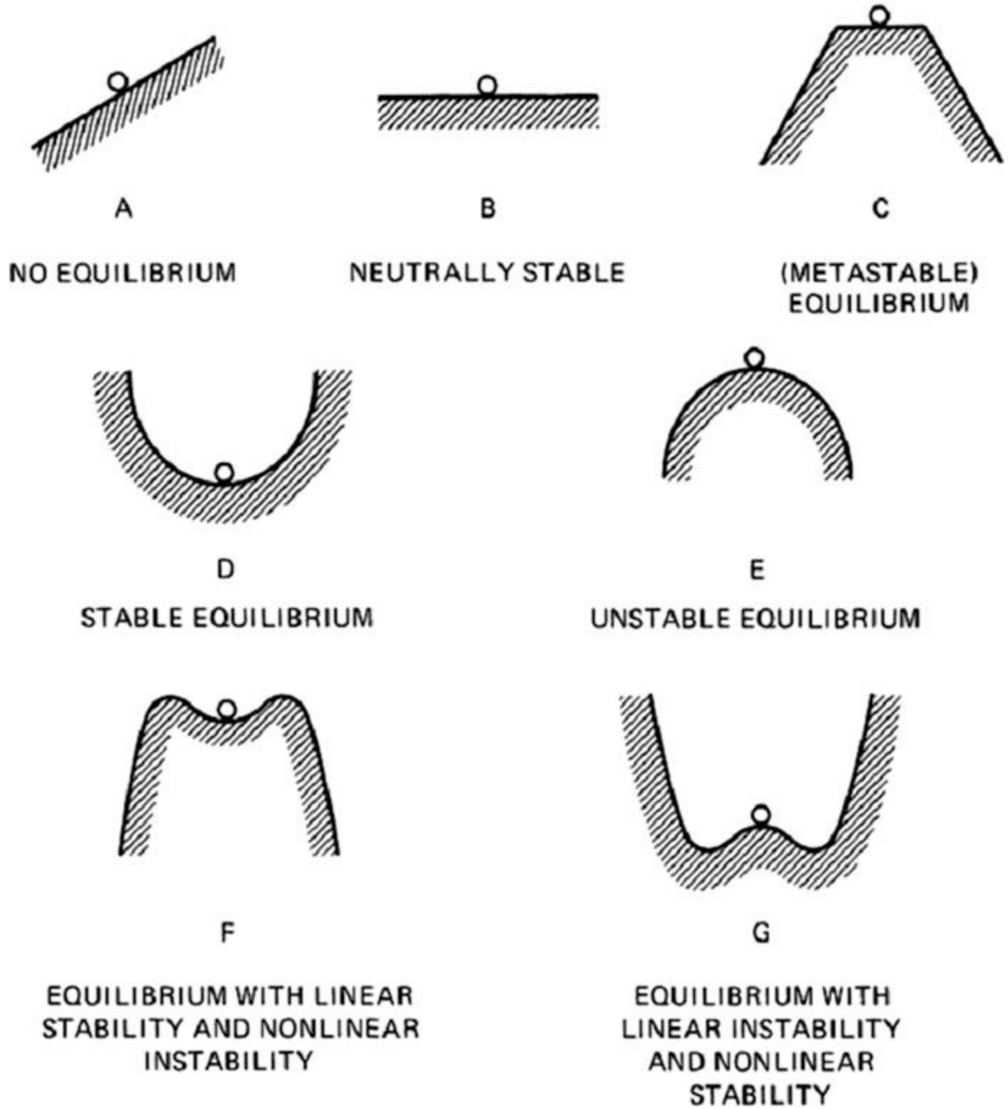
$$V(\psi) = \pi R_o \int_0^{2\pi} d\theta\,\hat{r}\left[1 + \frac{2}{3}\left(\frac{\hat{r}}{R_o}\right)cos\theta\right]$$

Magnetic well

$$\widehat{W} = 2\frac{V}{\langle B^2\rangle}\frac{d}{dV}\left\langle\frac{B^2}{2}\right\rangle$$

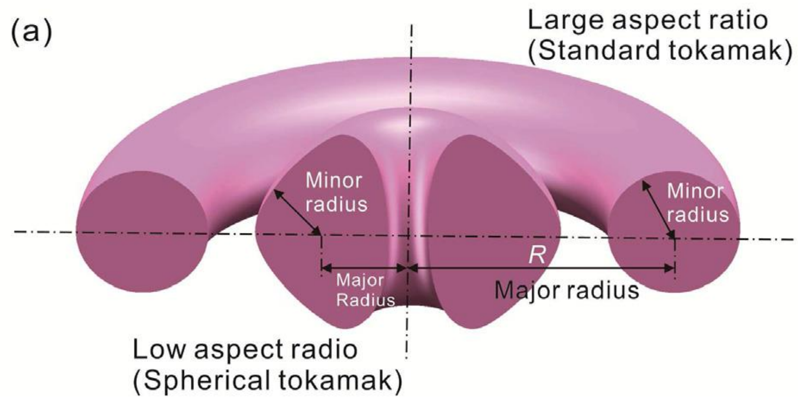
$$= 2\frac{V}{\langle B^2\rangle}\frac{d}{dV}\left\langle\mu_o p + \frac{B^2}{2}\right\rangle$$

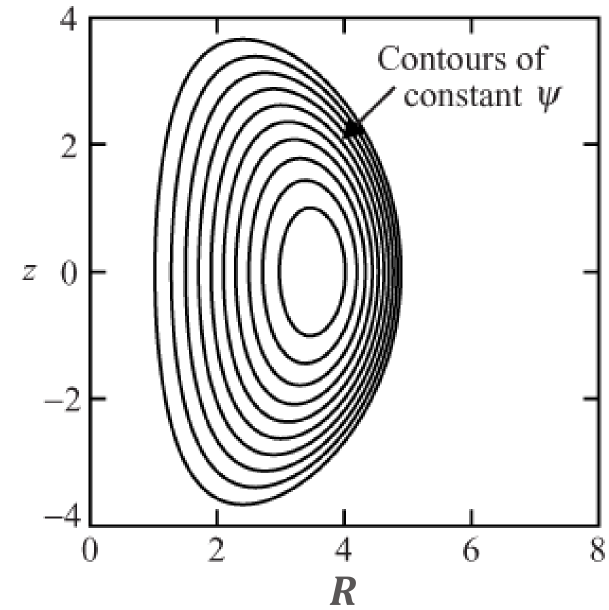
- **A magnetic well is a quantity that measures plasma stability against short perpendicular wavelength modes driven by the plasma pressure gradient.**

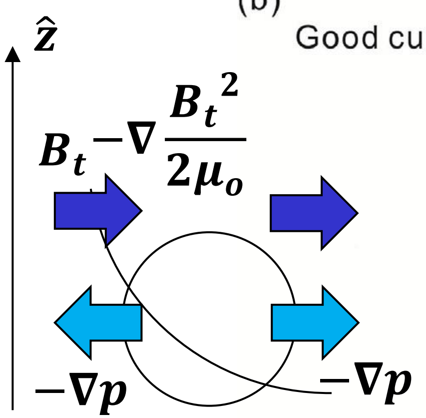


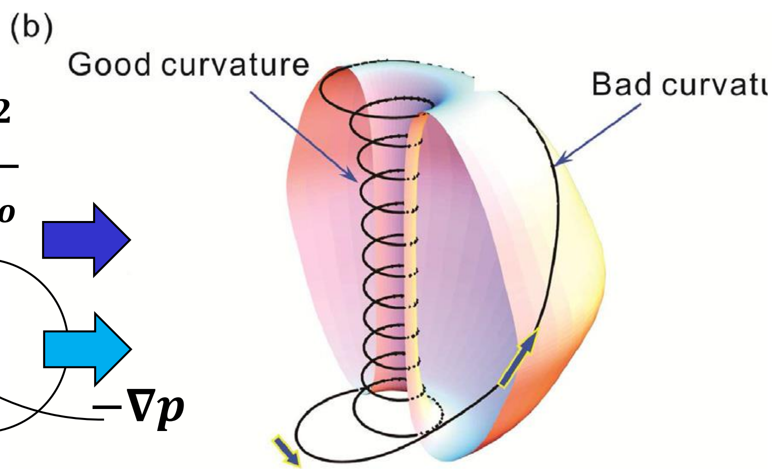
The Spherical tokamak

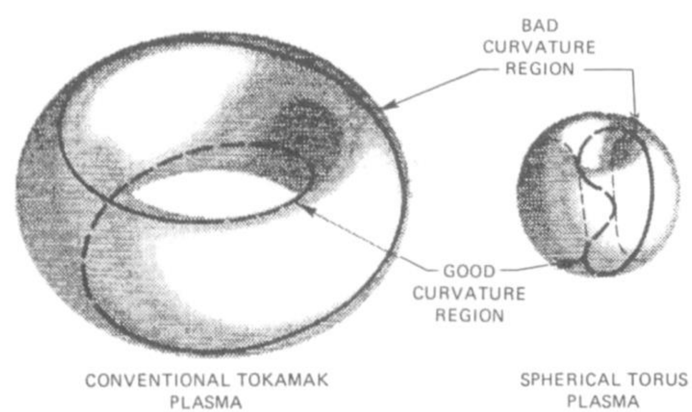
- **Aspect ratio R_o/a ~ 1.6**









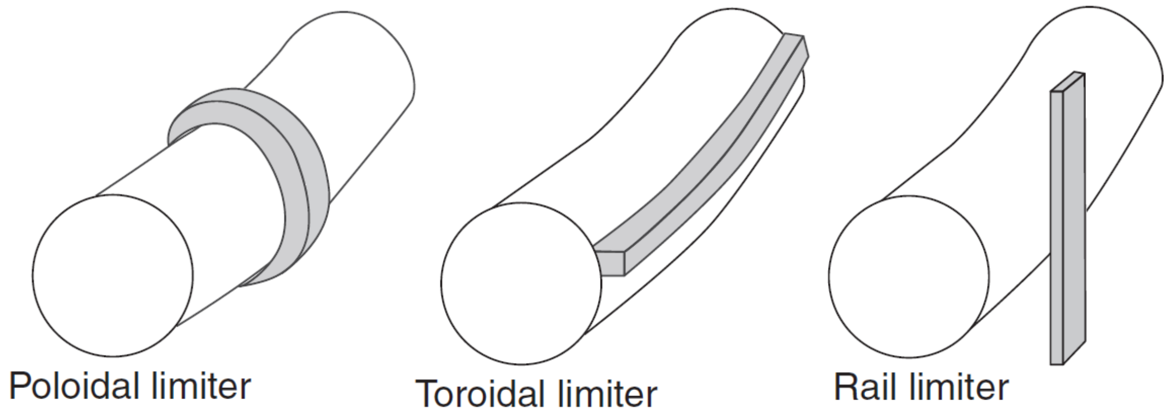


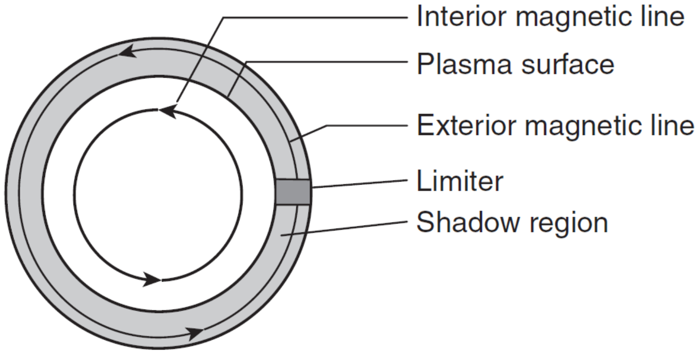
Zhe Gao, Matter and Radia. Extrem., **1**, 153 (2016)
Y-k. M.Peng and D. J. Strickler, Nucl. Fusion **26**, 769 (1986)

The Spherical tokamak

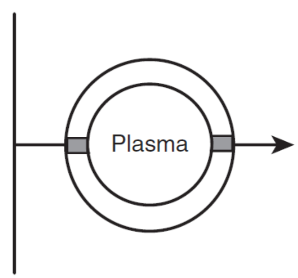
- **Aspect ratio $R_o/a \sim 1.6$**
- **Advantages:**
  - **Higher $\boldsymbol{\beta}_t$ limit.**
  - **A compact design almost spherical in appearance.**
- **Challenges:**
  - **Minimum space is given in the center of the torus to accommodate the toroidal field coils.**
  - **With a very compact design the technology associated with the construction and maintenance of the device may be more difficult than for a "normal" tokamak.**
  - **Large currents will have to be driven noninductively, a costly and physically difficult requirement.**

Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma

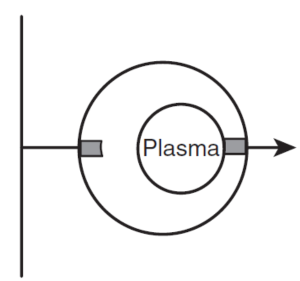




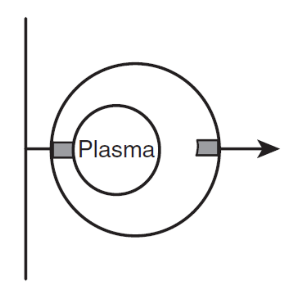
- **Vertical field is correct.**



- **Vertical field is too small.**

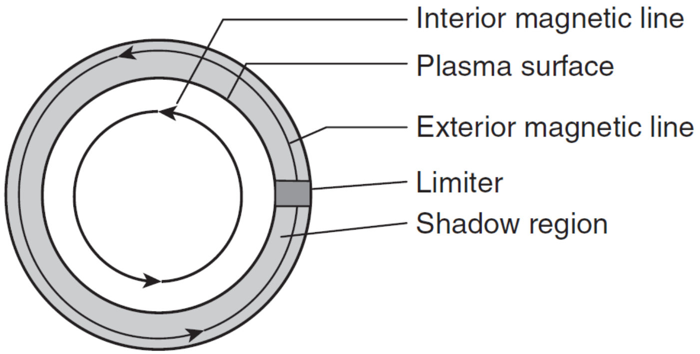


- **Vertical field is too large.**

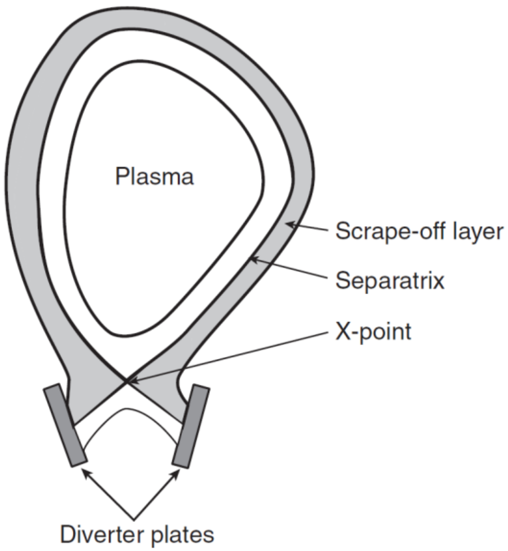


Limiter protects the vacuum chamber from plasma bombardment and defines the edge of the plasma

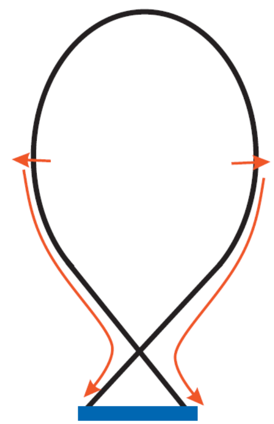
- **A mechanical limiter is a robust piece of material, often made of tungsten, molybdenum, or graphite placed inside the vacuum chamber.**
- **Some of the particles of the limiter surface may escape. Neutral particles can penetrate some distance into the plasma before being ionized.**
- **The high-z impurities can lead to significant additional energy loss in the plasma through radiation.**
- **In ignition experiments and fusion reactors, the bombardment is more intense and extends over longer periods of time. In addition, if the impurity level is too high, it may not be possible to achieve a high enough temperature to ignite.**



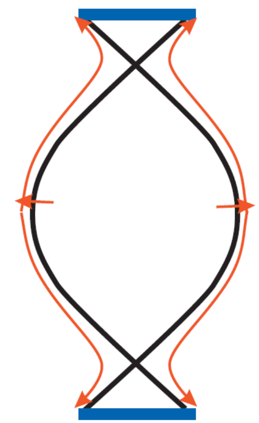
The magnetic divertor – guide a narrower layer of magnetic lines away from the edge of the plasma



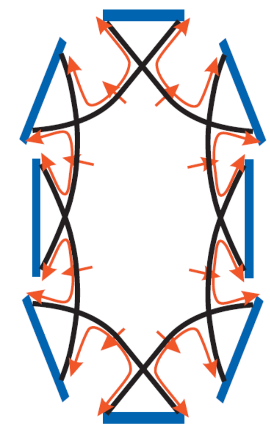
- **Single-null poloidal-field divertors for tokamak**



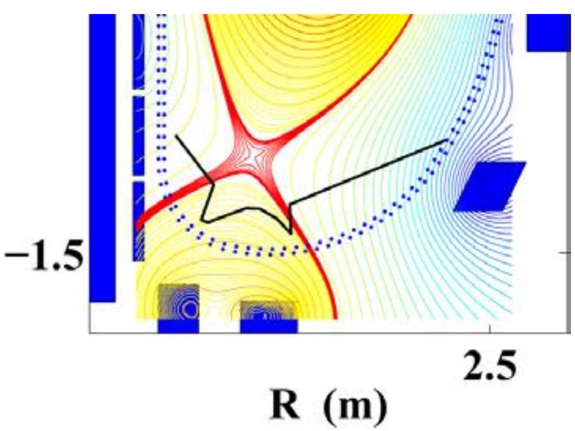
- **Double-null poloidal-field divertors for tokamak**



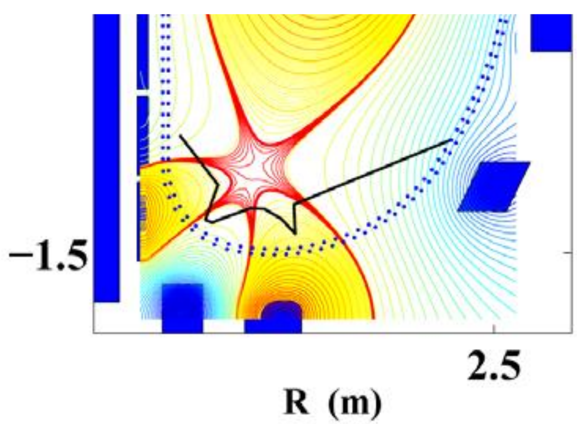
- **Island divertor for stellerators**



- **Standard**



- **Snowflake**



Y. Feng, Nucl. Fusion, **46**, 807 (2006)
L Xue *et al*, Plasma Phys. Control. Fusion **58**, 055005 (2016)

Pros and cons of a divertor

- **Advantages:**
  - **The collector plate is remote from the plasma. There is space available to spread out the magnetic lines.**
  - **A lower intensity of particles and energy bombard the collector plate leading to a longer replacement time.**
  - **It is more difficult for impurities to migrate into the plasma.**
  - **There are longer distance distances to travel and if a neutral particle becomes ionized before or during the time it crosses the divertor layer on its way toward the plasma, its parallel motion then carries it back to the collector plate.**
  - **The larger divertor chamber provides more access to pump out impurities.**
  - **The plasma edge is not in direct contact with a solid material such as a limiter.**
- **Disadvantages: larger and more complex system and more expensive.**