Introduction to Nuclear Fusion as An Energy Source

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Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 5

2026 spring semester

Tuesday 9:00-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://reurl.cc/MMnkOL

2026/4/2 updated

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \dfrac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**

- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \dfrac{1}{q}\dfrac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**

- **Grad-B drift:** $\vec{v}_{\nabla} = \dfrac{m{v_{\perp}}^2}{2q}\dfrac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**

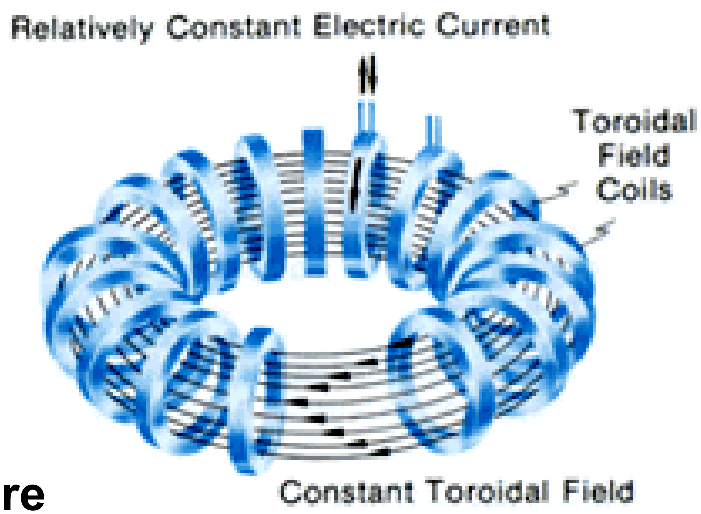
- **Curvature drift:** $\vec{v}_{R} = \dfrac{m{v_{||}}^2}{q}\dfrac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_c}^2 B^2}$ **Depended on charge**

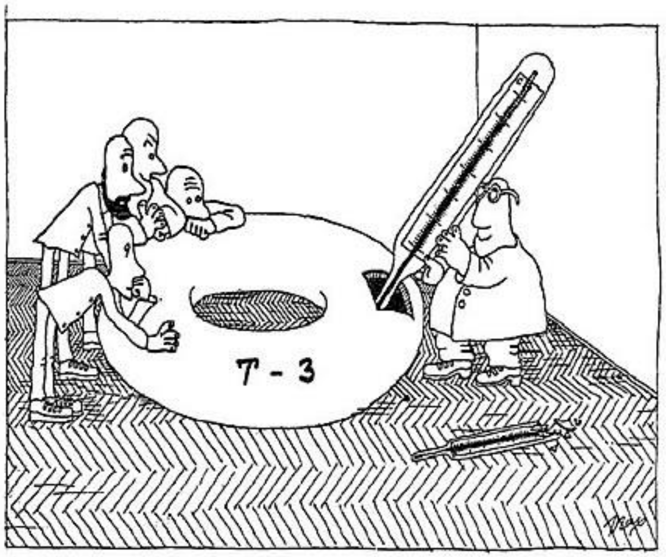
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}} B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right) = \frac{m}{q}\frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right)$$

Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field

- **Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)**





Nature

Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by
N. J. PEACOCK
D. C. ROBINSON
M. J. FORREST
P. D. WILCOCK
UKAEA Research Group, Culham Laboratory, Abingdon, Berkshire
V. V. SANNIKOV
I. V. Kurchatov Institute, Moscow

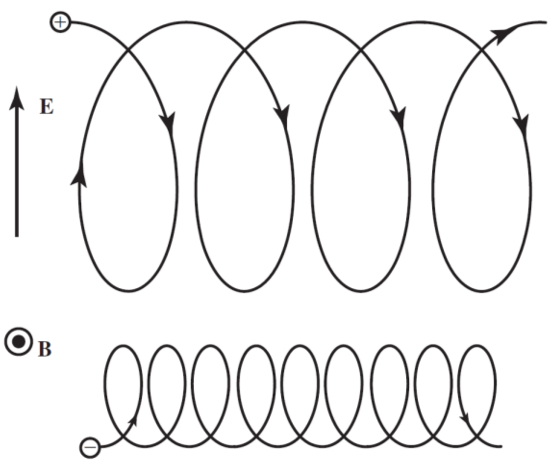
Electron temperatures of 100 eV up to 1 keV and densities in the range $1–3 \times 10^{13}$ cm^{-3} have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

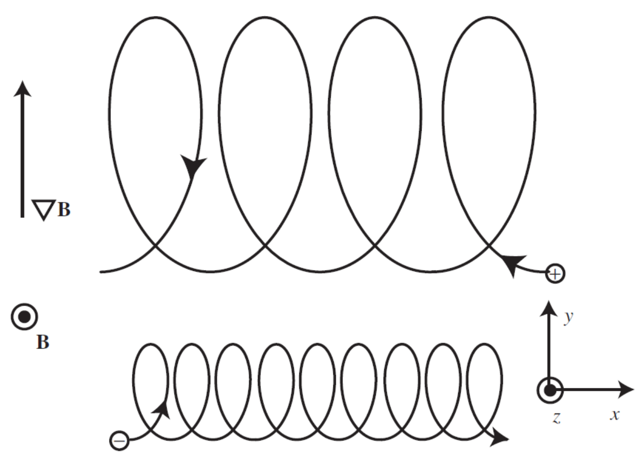
T_e = 100 ~ 1 keV
n_e = 1-3 x 10^{13} cm^{-3}

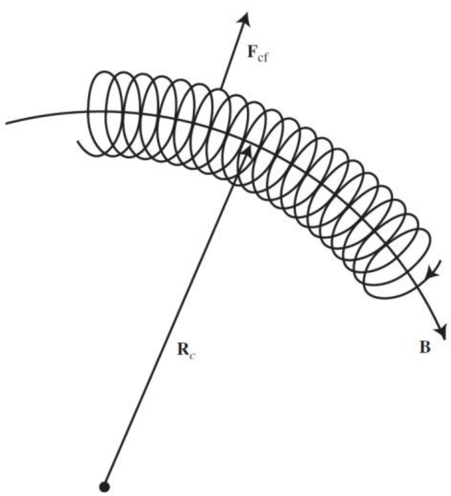
https://www.iter.org/mach/tokamak
https://en.wikipedia.org/wiki/Tokamak#cite_ref-4
Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.
N. J. Peacock,et al., Nature **224**, 488 (1969)

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \dfrac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \dfrac{m v_{\perp}{}^2}{2q} \dfrac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \dfrac{m v_{||}{}^2}{q} \dfrac{\vec{R}_{\mathrm{c}} \times \vec{B}}{R_c{}^2 B^2}$ **Depended on charge**

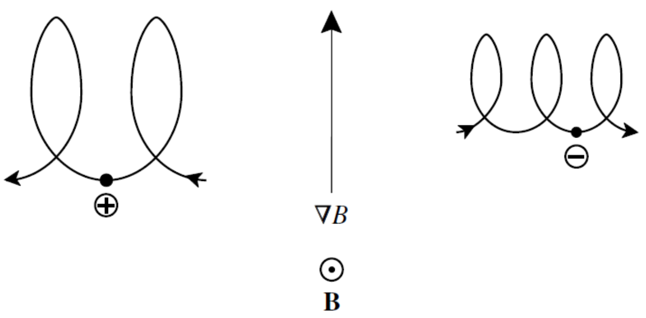




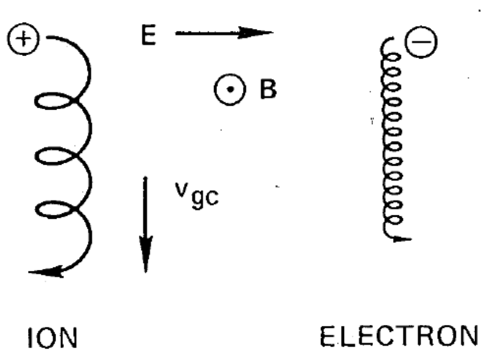


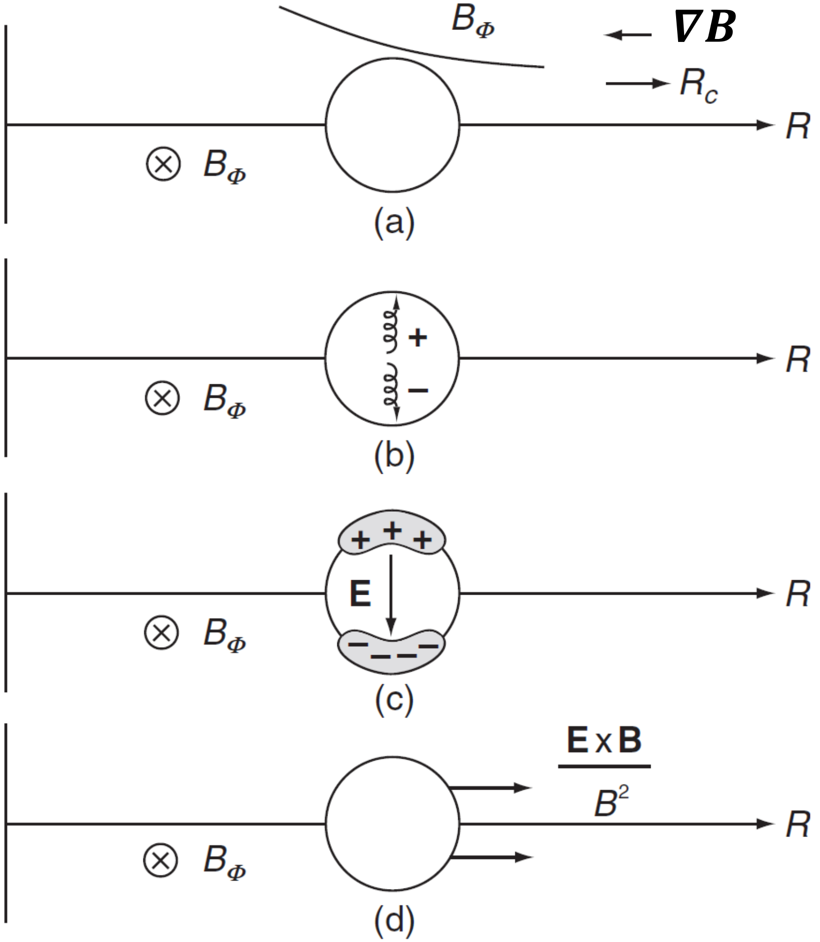
Charged particles drift across field lines

- **Grad-B drift**



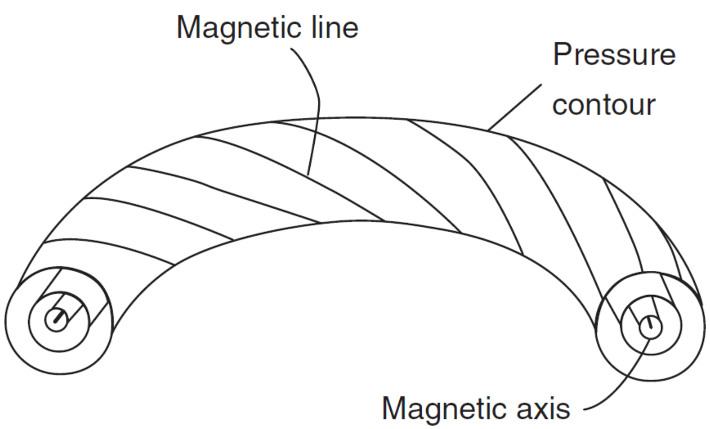
- **ExB drift**

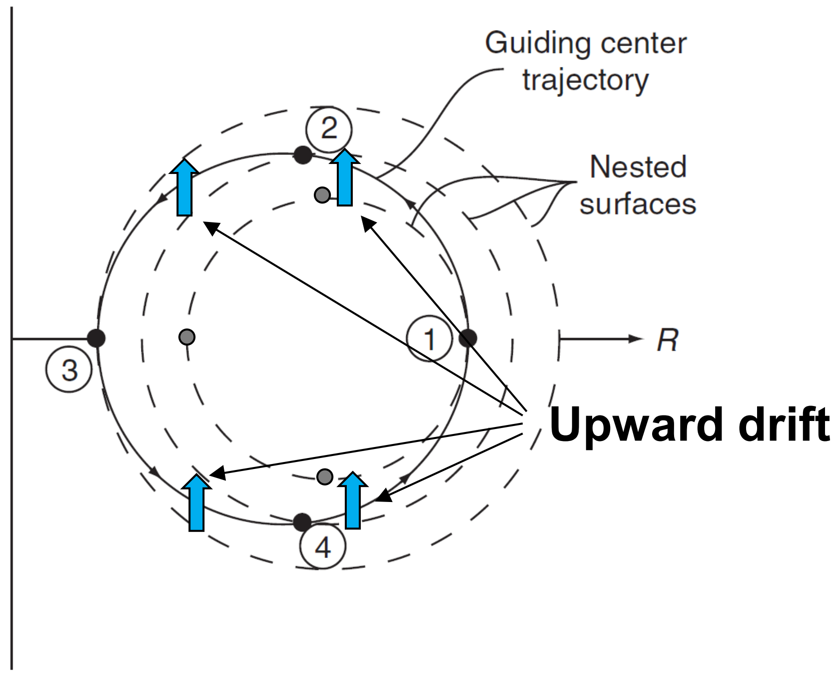




J. P. Freidberg, Ideal Magnetohydrodynamics

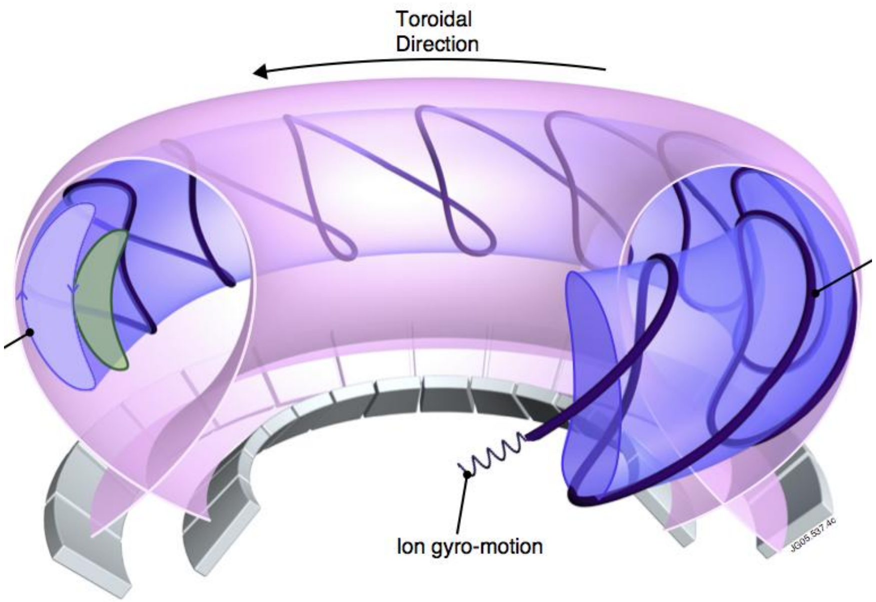
The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

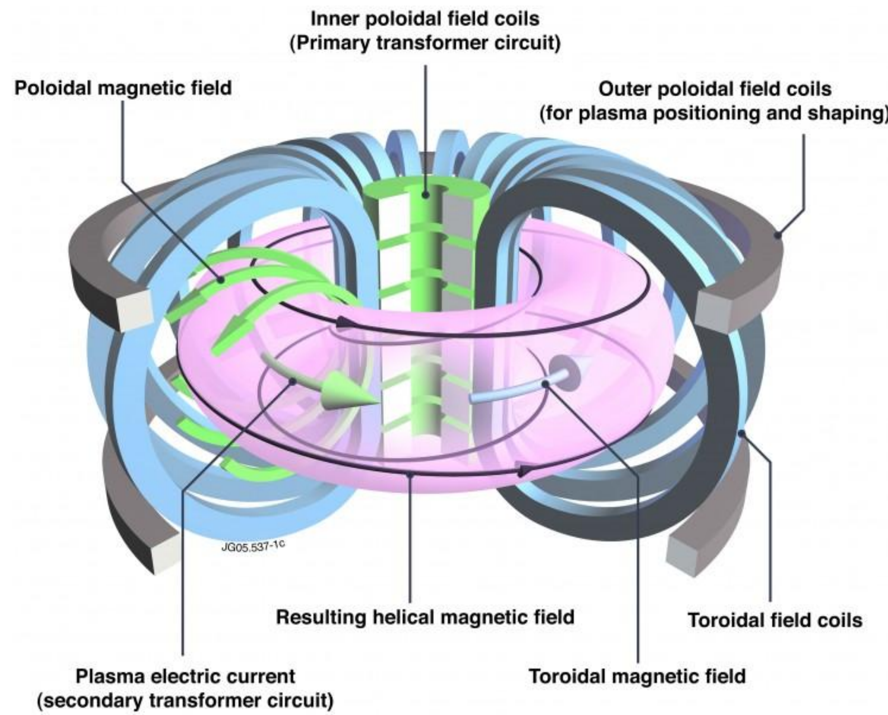




Points with no drift

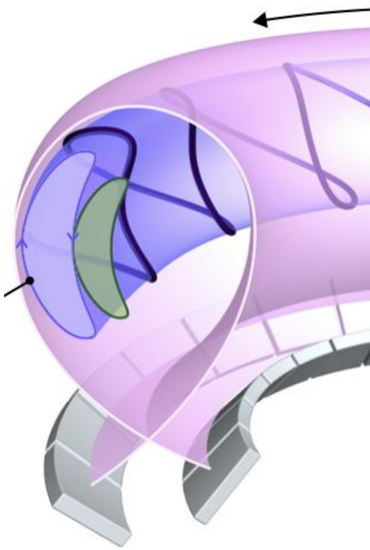
A poloidal magnetic field is required to reduce the drift across field lines

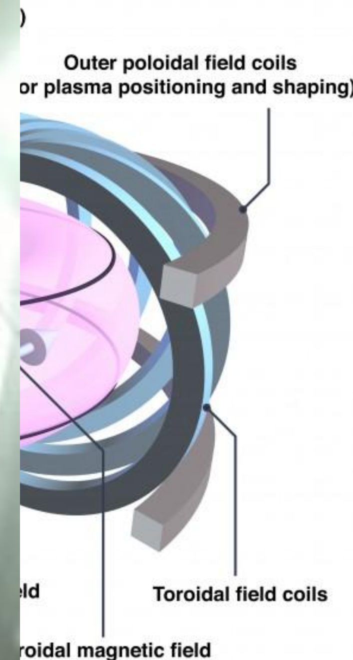




https://www.davidpace.com/keeping-fusion-plasmas-hot/
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

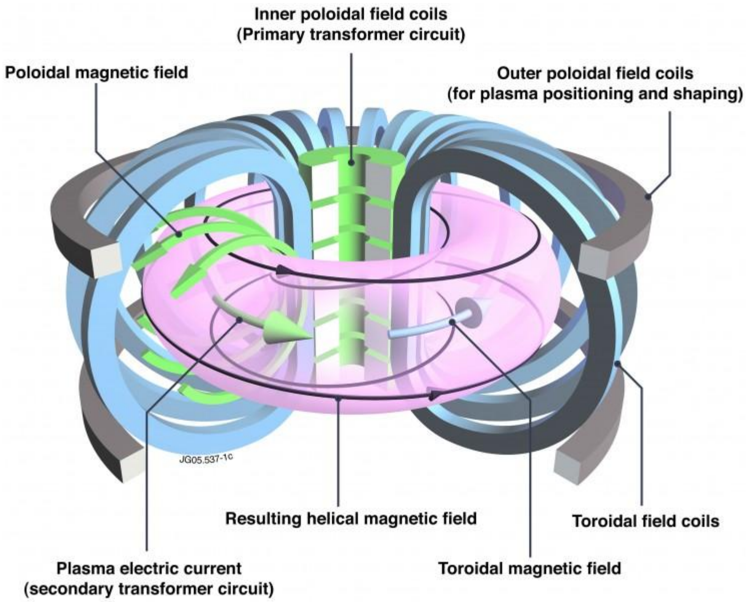
A poloidal magnetic field is required to reduce the drift across field lines



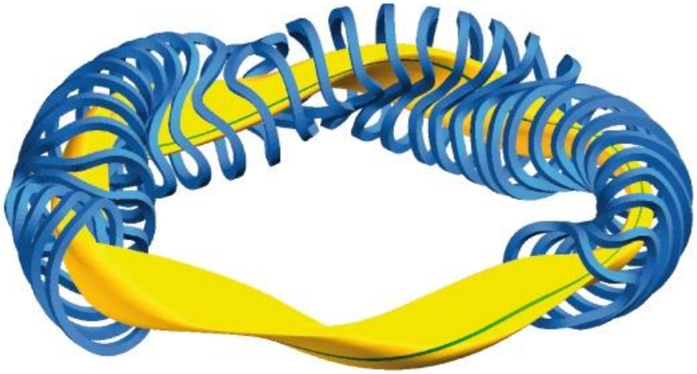


Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Magnetohydrodynamics description of plasma

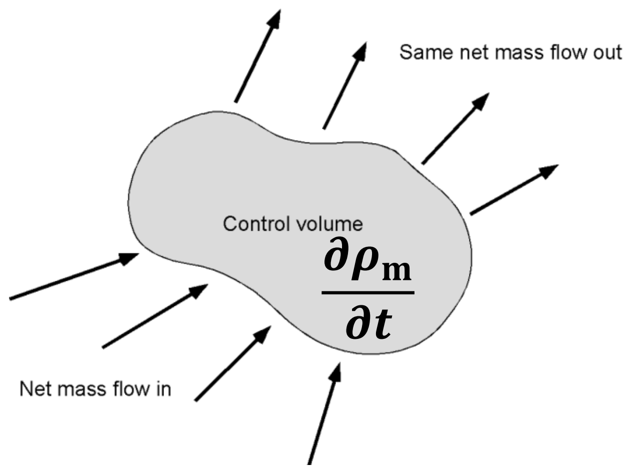
- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \rho_q \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

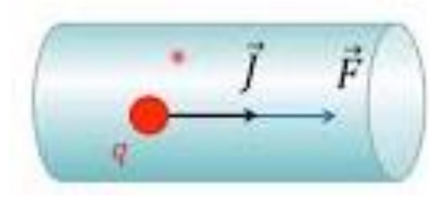
$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$





https://eaglepubs.erau.edu/introductiontoaerospaceflightvehicles/chapter/conservation-of-mass-continuity-equation/
https://www.youtube.com/watch?v=Iu0Ep8_Gp8U

Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation

- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ **w/ long wavelength ($\lambda >> \lambda_d$)**
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \cancel{\rho_q \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$
- **Ohm's law:** $\vec{j} = \sigma (\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt} \left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$
- **Maxwell's eqs:**

$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o} \approx 0$ **w/ long wavelength ($\lambda >> \lambda_d$) => quasi neutral**

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \cancel{\epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega << \omega_{pe}$)

Ideal MHD

- Continuity eq: $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- Momentum eq: $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- Ohm's law: $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- Equation of state: $$\frac{d}{dt}\left(\frac{P}{\rho_m{}^{\gamma}}\right) = 0$$
- Maxwell's eqs:

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- Requirement:
  - High collisionality – fluid model
  - Small gyro radius – low frequency
  - Small resistivity – a perfect conductor

Conflict!

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a}\frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

Scale length of nonuniformity.

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
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$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

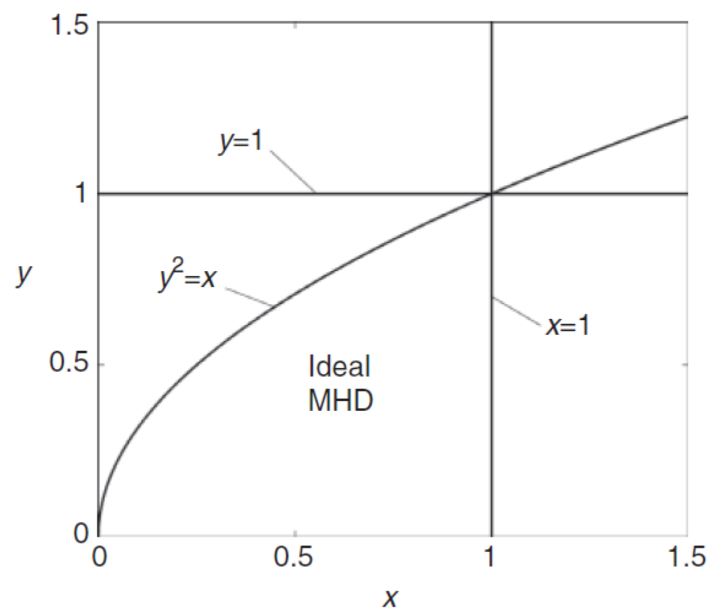
- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Conflict! (between high collisionality and small resistivity)

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a}\frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

a: **Scale length of nonuniformity.**

Region of validity for ideal MHD



Mean free path: $\lambda_\mathrm{i} \sim v_\mathrm{Ti}\tau_\mathrm{ii}$

$$x = \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \frac{v_\mathrm{Ti}\tau_\mathrm{ii}}{a}$$

$$y = \frac{r_\mathrm{Li}}{a}$$

- **Requirement:**
  - **High collisionality** $x \ll 1$
  - **Small gyro radius** $y \ll 1$
  - **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement (small η)

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad \eta\,\vec{j} = \vec{E} + \vec{v} \times \vec{B} \qquad \frac{|\eta j|}{|v \times B|} \sim ?$$

$$|j \times B| \sim |\nabla p| \qquad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a}\frac{nT}{B} \sim \frac{1}{a}\frac{n m_{\mathrm{i}} {v_{\mathrm{Ti}}}^2}{B} \qquad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_{\mathrm{Ti}}}{a} \qquad \omega_{\mathrm{ci}} = \frac{v_{\mathrm{Ti}}}{r_{\mathrm{Li}}}$$

$$\eta \sim \frac{m_{\mathrm{e}}}{ne^2\tau_{\mathrm{ei}}} \qquad \tau_{\mathrm{ei}} \sim \tau_{\mathrm{ee}} \sim \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \tau_{\mathrm{ii}} \qquad k \sim \nabla \sim \frac{1}{a} \qquad \omega_{\mathrm{ci}} = \frac{eB}{m_{\mathrm{i}}}$$

$$x = \left(\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\right)^{1/2} \frac{v_{\mathrm{Ti}}\tau_{\mathrm{ii}}}{a} \qquad y = \frac{r_{\mathrm{Li}}}{a}$$

$$\frac{|\eta j|}{|v \times B|} \sim \frac{\eta j}{v_{\mathrm{Ti}} B} \sim \frac{m_{\mathrm{e}}}{ne^2\tau_{\mathrm{ei}}}\frac{1}{a}\frac{n m_{\mathrm{i}} {v_{\mathrm{Ti}}}^2}{B}\frac{1}{v_{\mathrm{Ti}} B} = \frac{m_{\mathrm{e}}}{\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a}\frac{m_{\mathrm{i}}}{e^2B^2} = \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a}\frac{{m_{\mathrm{i}}}^2}{e^2B^2} = \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a{\omega_{\mathrm{ci}}}^2}$$

$$\sim \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ii}}}\left(\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\right)^{1/2}\frac{v_{\mathrm{Ti}}}{a{\omega_{ci}}^2} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{v_{\mathrm{Ti}}}{\tau_{\mathrm{ii}}a}\frac{{r_{\mathrm{Li}}}^2}{{v_{\mathrm{Ti}}}^2} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{1}{\tau_{\mathrm{ii}}a}\frac{{r_{\mathrm{Li}}}^2}{v_{\mathrm{Ti}}} \sim \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{1}{\omega\tau_{\mathrm{ii}}}\left(\frac{r_{\mathrm{Li}}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

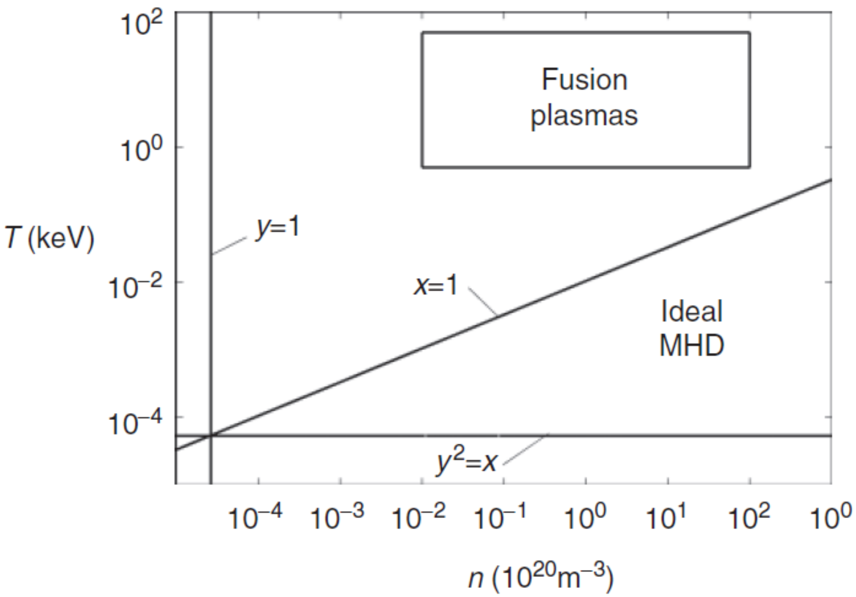
Fusion plasma is not in the ideal MHD region!

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad y = \frac{r_{Li}}{a}$$

$$10^{18}\ \mathrm{m^{-3}} < n < 10^{22}\ \mathrm{m^{-3}}$$

$$0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$$

$$\beta \equiv \frac{2\mu_o n T}{B^2}$$



- Requirement:
  - High collisionality $x = 3 \times 10^3 \frac{T^2}{an} << 1$
  - Small gyro radius $y = 2.3 \times 10^{-2} \left(\frac{\beta}{na^2}\right)^{1/2} << 1$
  - Small resistivity $\frac{y^2}{x} = 1.8 \times 10^{-7} \frac{\beta}{aT^2} << 1$

- **With strong B, the gyromotion mimic the collisional characteristics.**

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt} \left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Additional simplification of the momentum equation

- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}\right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$

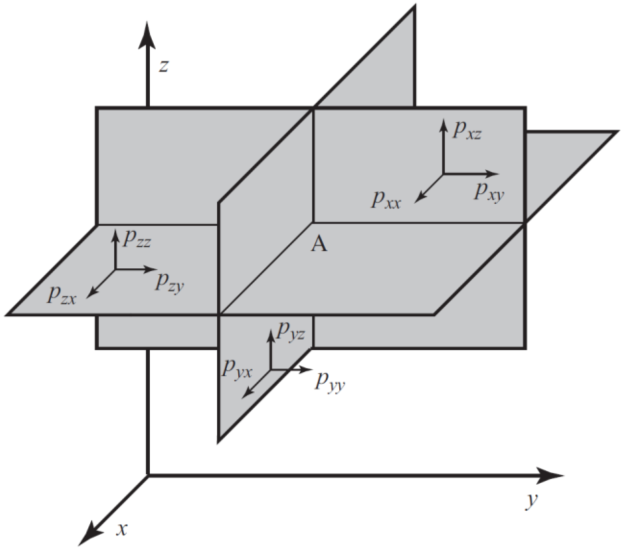
$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial p_{xx}}{\partial x} \\ \frac{\partial p_{yy}}{\partial y} \\ \frac{\partial p_{zz}}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{pmatrix}$$

Additional simplification of the momentum equation

$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$



$$= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{pmatrix}}_{\text{Viscosity } \nabla \cdot \overleftrightarrow{\Pi}}$$

- **Isotropic plasma:** $p_{xx} = p_{yy} = p_{zz} \equiv p$

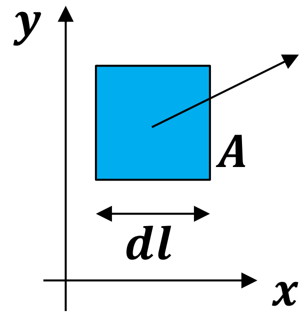
$$\begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} \equiv p\overleftrightarrow{1}$$

$$\nabla \cdot \overleftrightarrow{P} = \nabla p + \nabla \cdot \overleftrightarrow{\Pi}$$

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

Viscosity is negligible in a collision-dominated plasma

- Y component of momentum transfer through the surface A.



$$\pi_{\mathrm{xy}}^{\mathrm{ii}} \sim (m v_{\mathrm{y}} n) \frac{d v_{\mathrm{x}}}{dx} dl \sim \mu \frac{d v_x}{dx}$$

$$\mu \sim mnvdl \sim mnv(v\tau_{\mathrm{ii}}) \sim mn\left(\sqrt{\frac{T_{\mathrm{i}}}{m}}\right)^2 \tau_{\mathrm{ii}} \sim nT_{\mathrm{i}}\tau_{\mathrm{ii}}$$

$$\overleftrightarrow{\Pi} \sim \mu\left(2\nabla_{\parallel} \cdot \vec{v}_{\parallel} - \frac{2}{3}\nabla \cdot \vec{v}\right) \sim \mu \frac{v_{\mathrm{Ti}}}{a}$$

$$\left|\frac{\nabla \cdot \overleftrightarrow{\Pi}}{\nabla p}\right| \sim \frac{\overleftrightarrow{\Pi} a}{ap} \sim \frac{nT_{\mathrm{i}}\tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{ap} \sim \frac{\tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{a} \sim \frac{\lambda_i}{a} \ll 1 \qquad \lambda_i \sim v_{\mathrm{Ti}}\tau_{\mathrm{ii}}$$

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

⇓

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla p$$

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

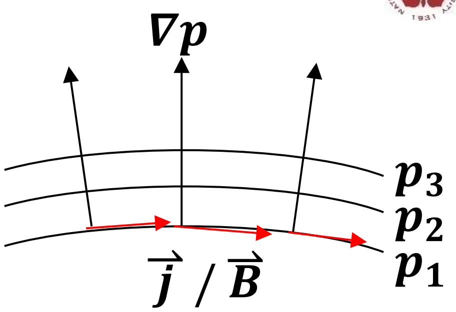
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"



- **Equilibrium state:**

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] = \vec{j}\times\vec{B} - \nabla p \equiv 0$$

$$\vec{j}\times\vec{B} = \nabla p$$

$$\nabla\times\vec{B} = \mu_o\,\vec{j}$$

$$\vec{j}\times\vec{B} = \frac{1}{\mu_o}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = \frac{1}{\mu_o}\left[\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} - \frac{1}{2}\,\vec{\nabla}\,B^2\right] = \nabla p$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$$

Magnetic pressure | **Magnetic tension** ← **Forces caused by curvature of the field lines**

$\vec{j}\perp\nabla p$ $\vec{B}\perp\nabla p$ ⇒ $\vec{j}\cdot\nabla p = 0$ $\vec{B}\cdot\nabla p = 0$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

Pressure can be written as a function of flux for an equilibrium state with axisymmetric

$$\vec{j} \times \vec{B} = \nabla p \qquad \nabla \times \vec{B} = \mu_o \vec{j}$$

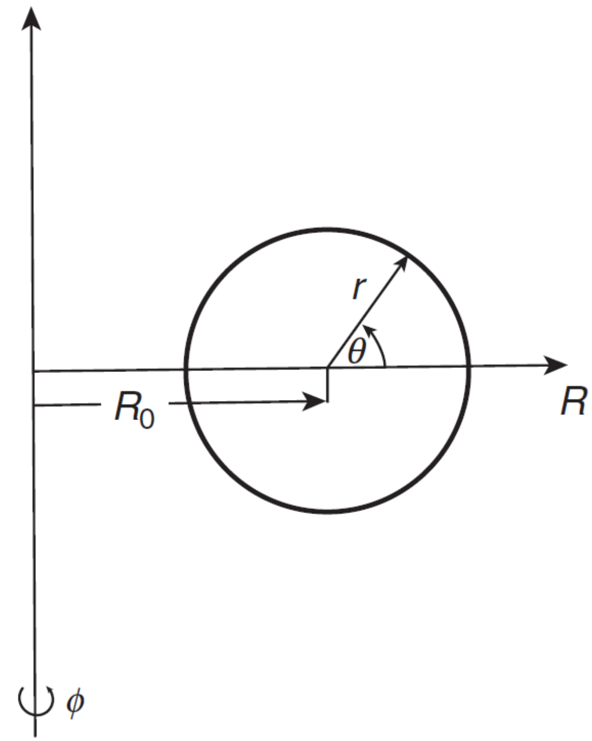
$$\nabla \cdot \vec{j} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = (B_R, B_\phi, B_z) \qquad \text{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \cancel{\frac{1}{R}\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$



- Represent the magnetic field using a vector potential *A*:

$$\vec{B} = \nabla \times \vec{A} = \hat{R}\left(\cancel{\frac{1}{R}\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \cancel{\frac{1}{R}\frac{\partial A_R}{\partial \phi}}\right)$$

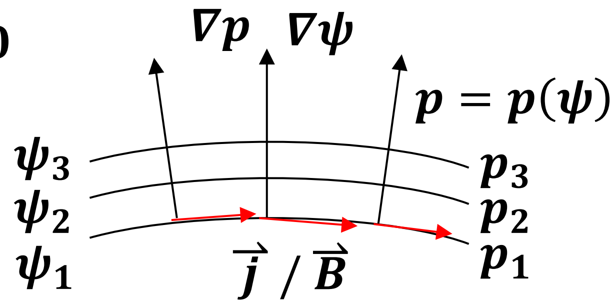
$$= \hat{R}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\right)$$

$$\equiv \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z$$

$$B_R = -\frac{\partial A_\phi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

Pressure can be written as a function of flux

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R)+\frac{\partial B_z}{\partial z}=0$$

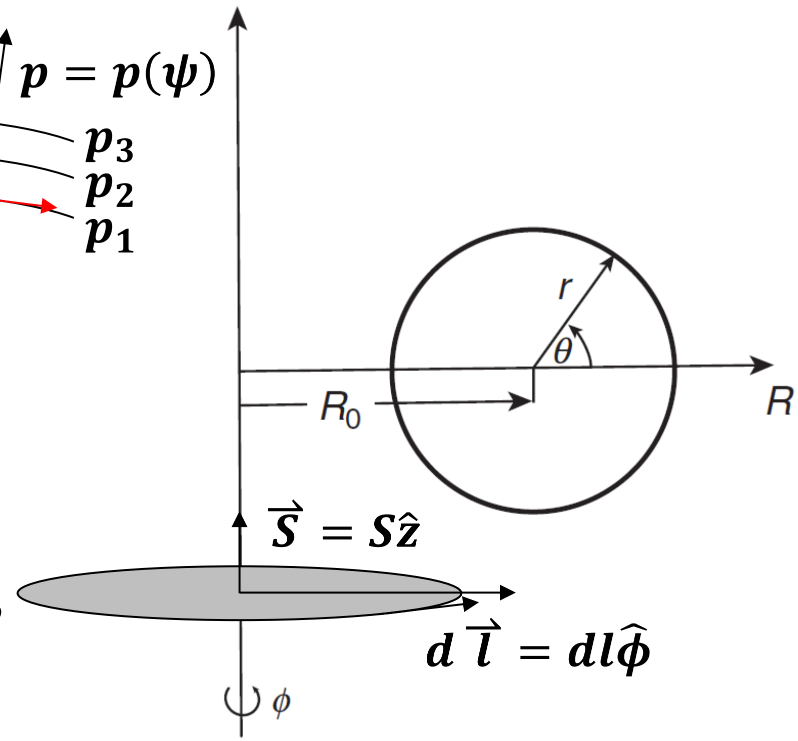


$$B_R=-\frac{\partial A_\phi}{\partial z}$$

$$B_z=\frac{1}{R}\frac{\partial}{\partial R}\left(RA_\phi\right)$$

$$\psi\equiv\frac{1}{2\pi}\int\vec{B}\cdot d\,\vec{S}=\frac{1}{2\pi}\int\left(\nabla\times\vec{A}\right)\cdot d\,\vec{S}$$

$$=\frac{1}{2\pi}\int\vec{A}\,2\pi R\cdot d\,\vec{l}=\int\vec{A}\cdot\hat{\phi}R\,dl=RA_\phi$$



$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial z}\qquad B_z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}\cdot\nabla\psi=B_R\frac{\partial\psi}{\partial R}+B_\phi\frac{1}{R}\frac{\partial\psi}{\partial\phi}+B_z\frac{\partial\psi}{\partial z}=B_R\frac{\partial\psi}{\partial R}+B_z\frac{\partial\psi}{\partial z}$$

$$=\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)\frac{\partial\psi}{\partial R}+\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\frac{\partial\psi}{\partial z}=0$$

$$\vec{B}\cdot\nabla\psi=0$$

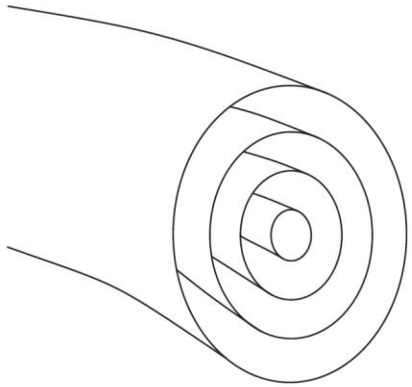
$$\vec{B}\cdot\nabla p=0$$

for $\nabla p\neq 0$:

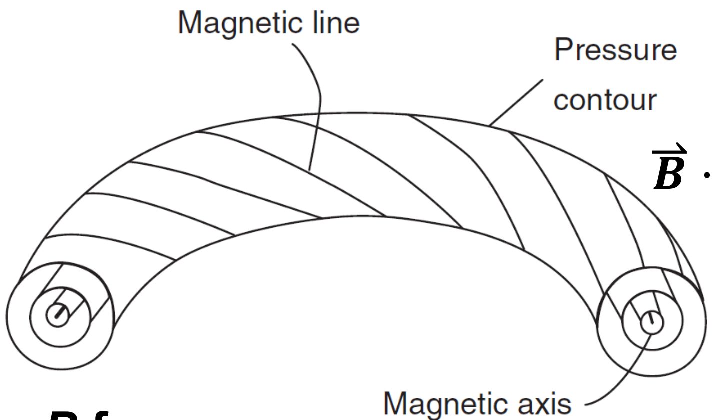
$$p=p(\psi)$$

Magnetic lines lying on pressure contour

- **Contours of constant pressure**



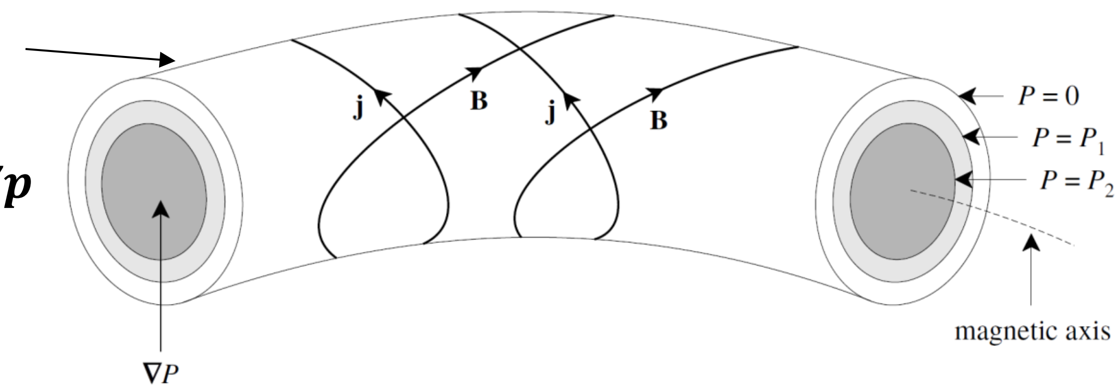
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



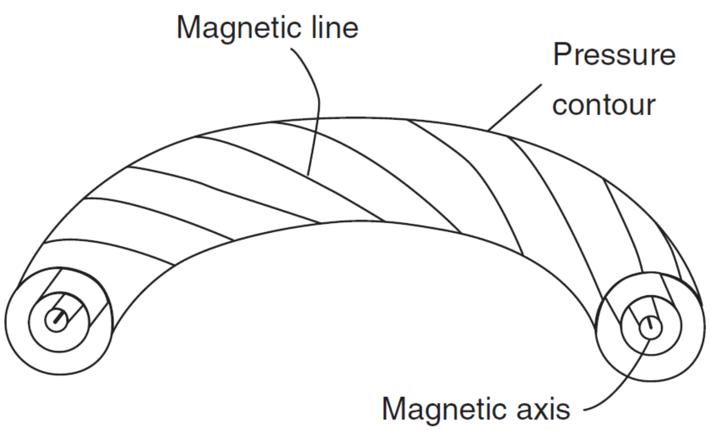
$$\vec{j} \cdot \nabla p = 0$$

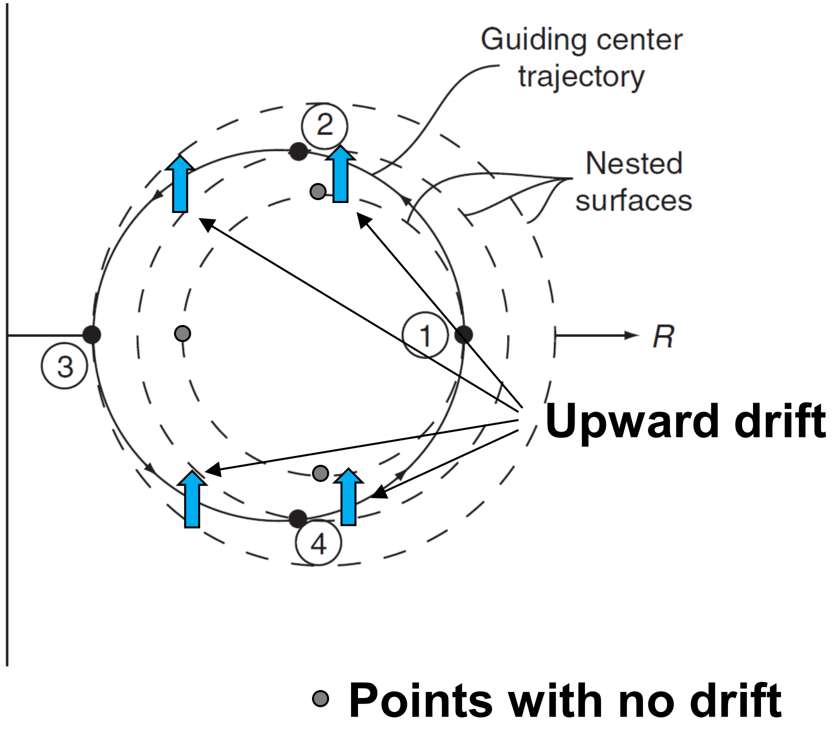
$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \Psi = 0$$

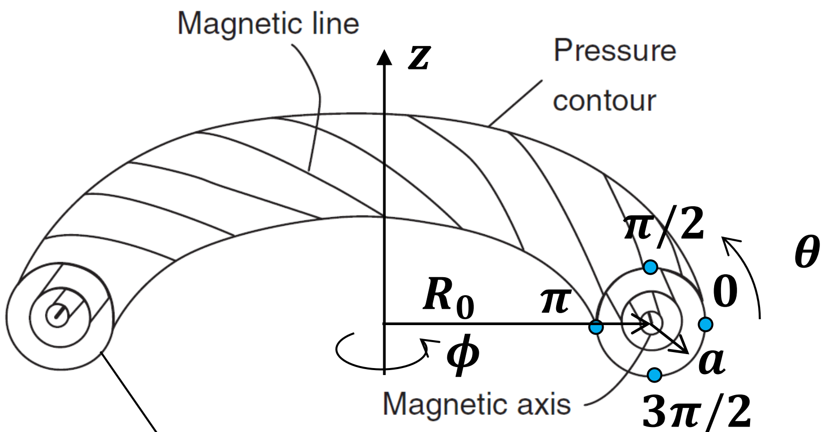
- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

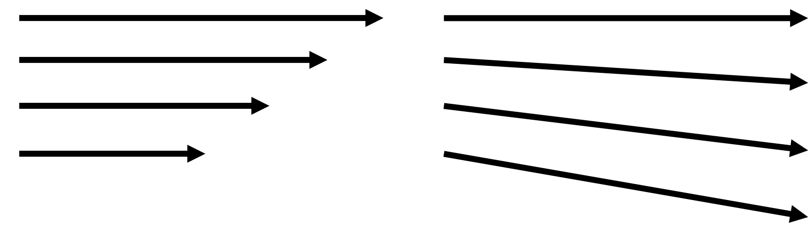




Local magnetic shear

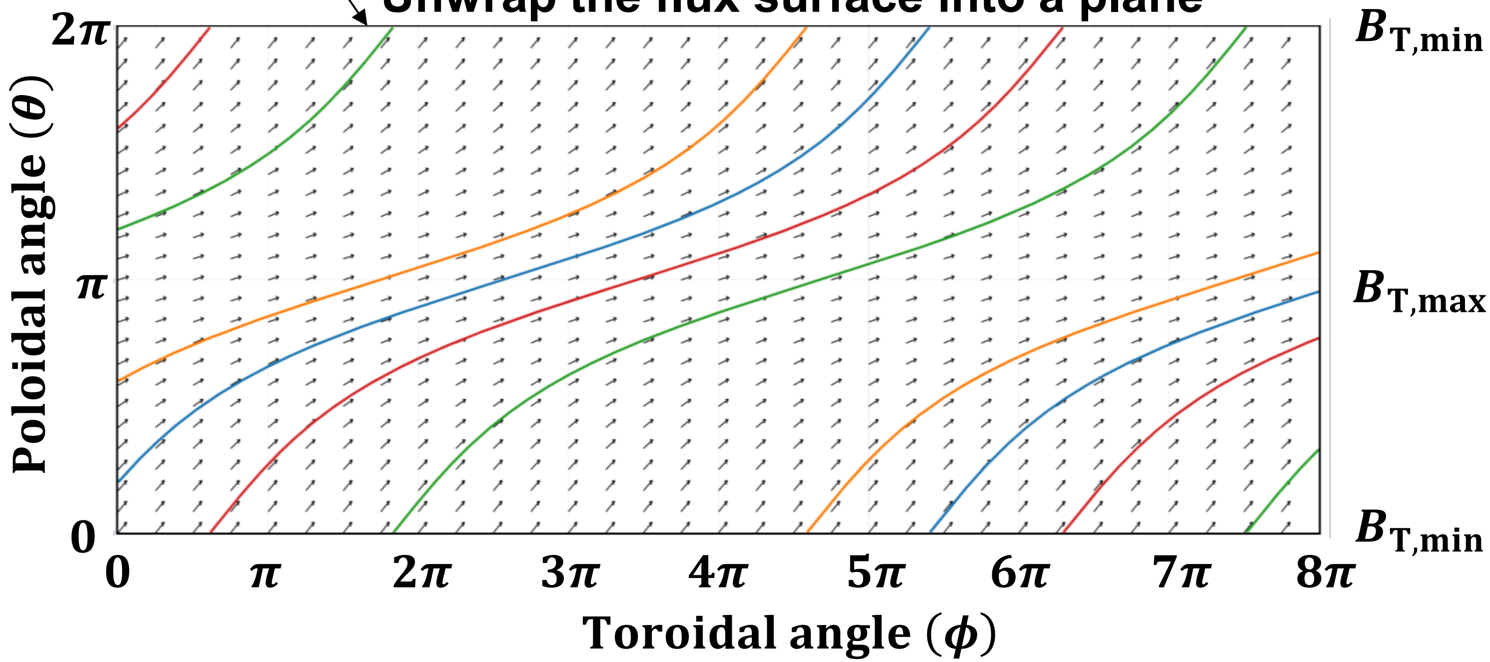


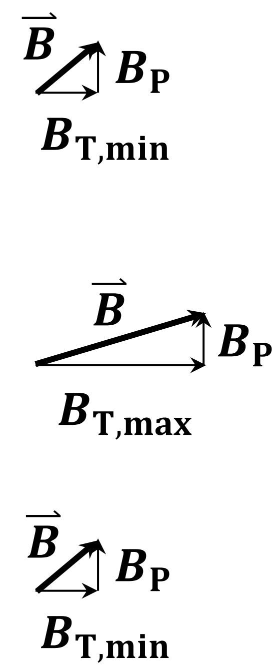
- **Shear**



$$B_\mathrm{T} \propto \frac{1}{R(\theta)} \qquad R(\theta) = R_0 + a\cos(\theta)$$

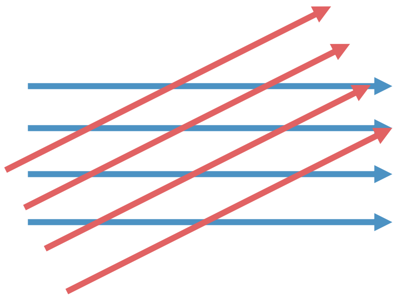
Unwrap the flux surface into a plane

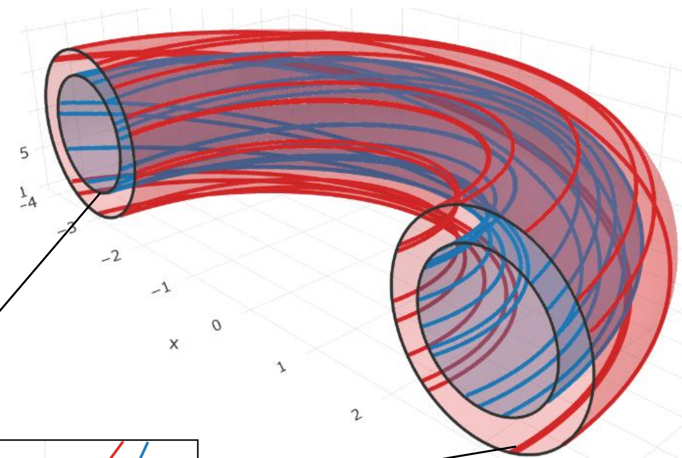


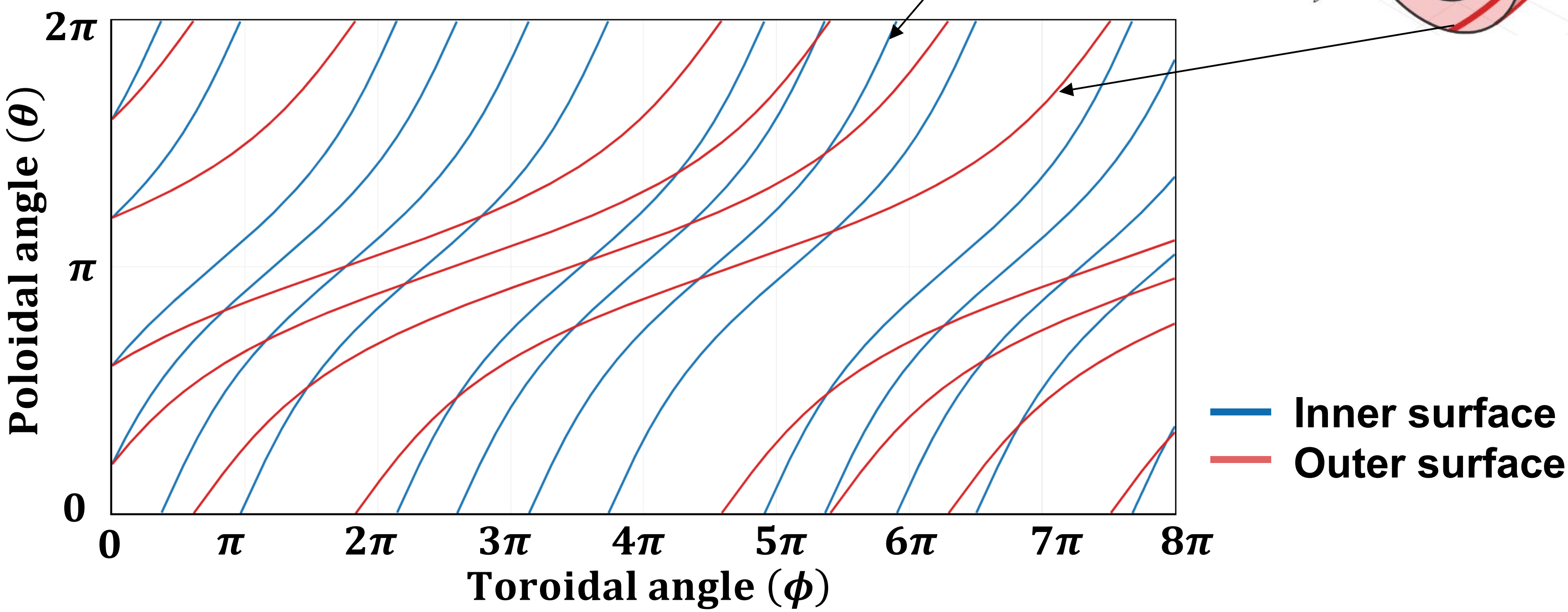


Global magnetic shear

- **Shear**



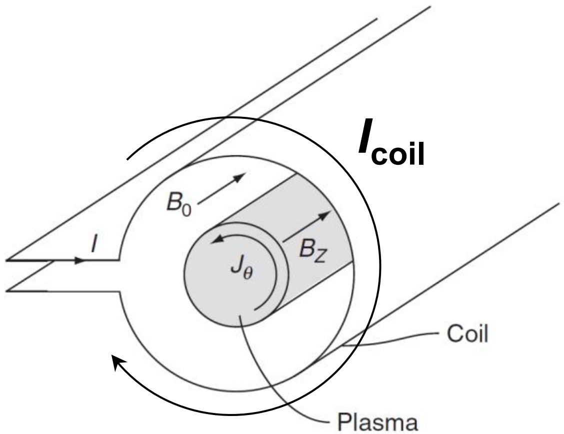




Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z\hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}(\vec{B}\cdot\vec{\nabla})\vec{B} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

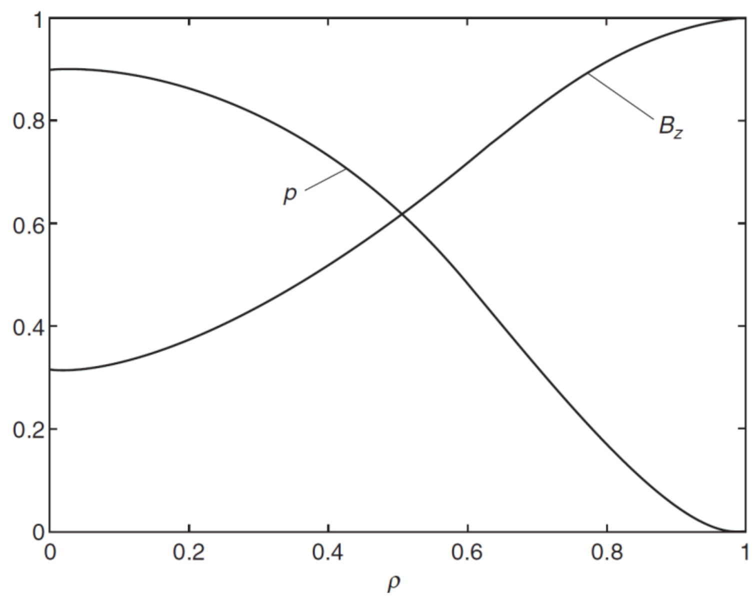
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma

- **Example:**

$$\frac{2\mu_o p(r)}{B_o^{\,2}} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$



$$j_\theta B_z = \frac{dp}{dr} \quad \Rightarrow \quad \frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{B_o^{\,2}} \qquad \rho = \frac{r}{a}$$

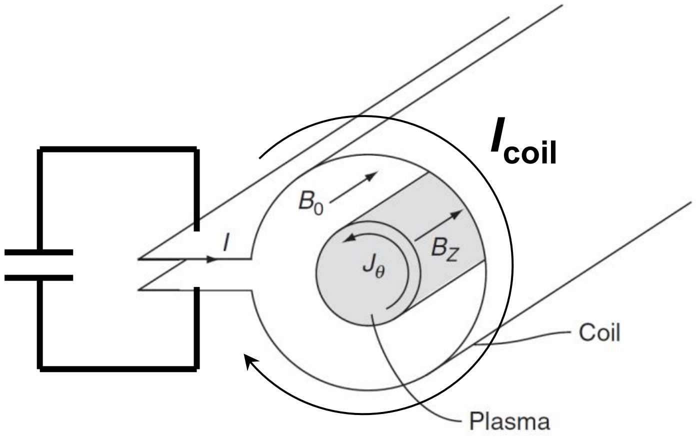
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{B_o^{\,2}} = \frac{4\mu_o}{a^2 B_o^{\,2}}\int_0^a p r dr = 2\int_0^1\left(1-\frac{B_z^{\,2}}{B_o^{\,2}}\right)\rho d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_o}{2}\ , \beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad \Rightarrow \quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$$0 < \beta < 1$$

Theta pinches provide good radial confinement but NOT axially

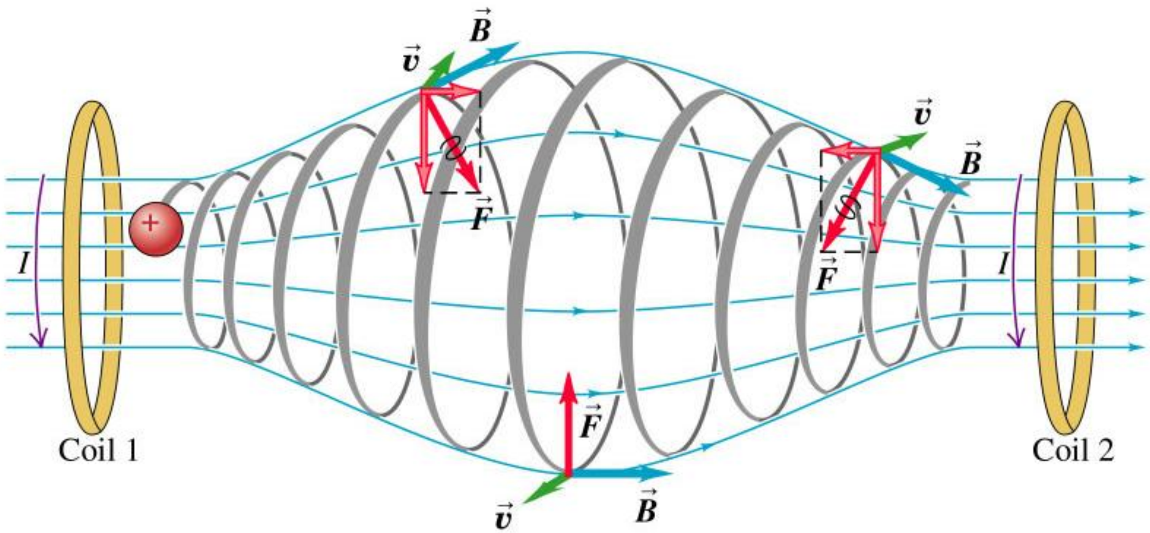


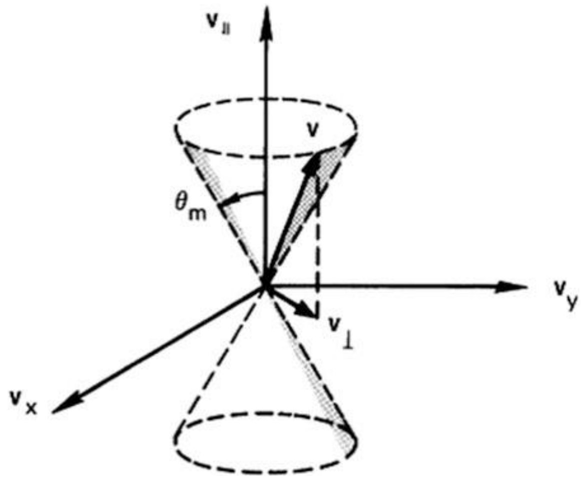
- The gas is initially preionized.
- The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.
- The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.
- This energy is ultimately converted to heat after repeated reflections off the converging piston.
- T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.
- The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{\mathrm{Ti}} \sim 10 \mu s$ for L = 5 m.

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.





$$v'_\perp{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_\perp{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv sin^2\theta$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_\perp{}^2 \qquad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_\perp{}^2}{B}$$

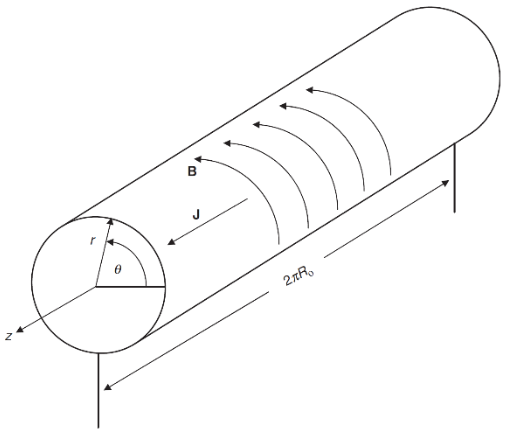
$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = sin^2\theta_m$$

- Large $v_{||}$ may occur from collisions between particles.

- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

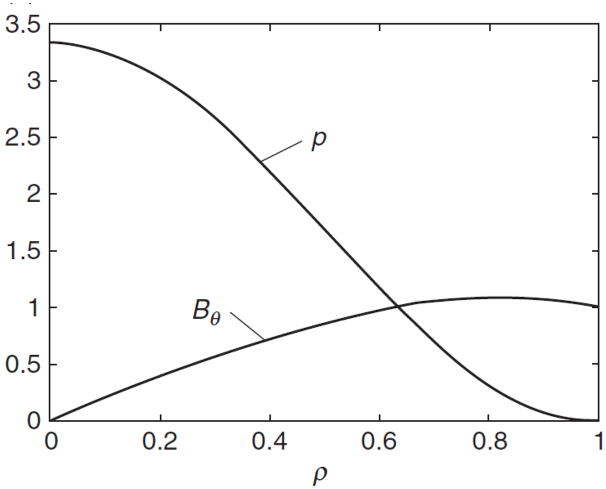
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

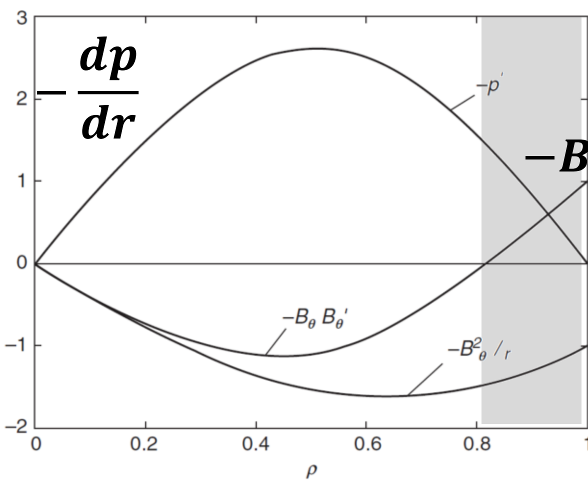
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β





$$-\frac{dp}{dr}$$

$$-B_\theta\frac{dB_\theta}{dr}$$

$$-\frac{B_\theta^2}{\mu_o r}$$

$$\frac{d}{dr}\left(p+\frac{B_\theta^2}{2\mu_o}\right)+\frac{B_\theta^2}{\mu_o r}=0$$

$$\frac{2\mu_o p(r)}{{B_{\theta a}}^2}=\frac{2}{3}\left(5-2\rho^2\right)\left(1-\rho^2\right)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}}=2\rho\left(1-\frac{\rho^2}{2}\right)$$

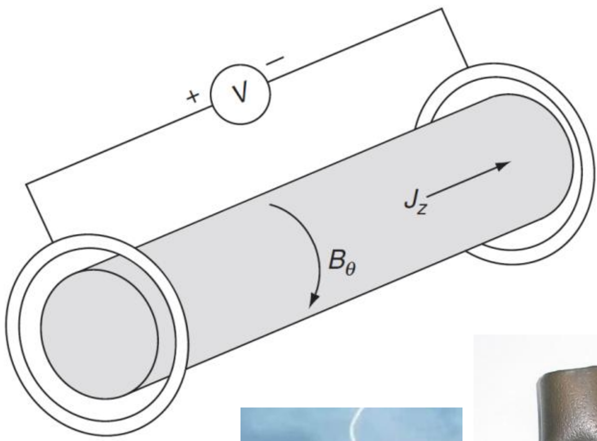
$$\frac{a\mu_o j_z(r)}{B_{\theta a}}=4\left(1-\rho^2\right)$$

$$B_{\theta a}\equiv B_\theta(a)=\frac{\mu_o I}{2\pi a}$$

$$\beta\equiv\beta_p=\frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2}=\frac{4\mu_o}{a^2{B_{\theta a}}^2}\int_0^a prdr=1$$

Bennett pinch relation: $\beta=1$

Huge instabilities occur in a z pinch

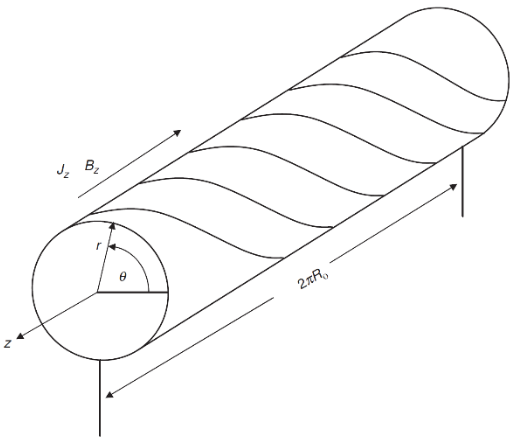


- **A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.**
- **The gas is ionized by the high voltage and produces a z current flowing along the plasma.**
- **Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.**

Main issue: unstable.

Facebook@鄭明典 (https://www.facebook.com/share/p/18N8x8kFmu/)
https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

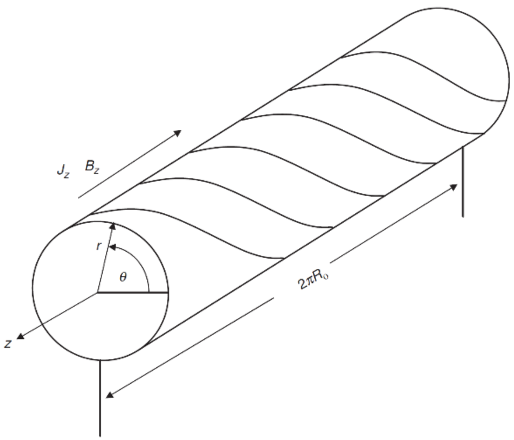
$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

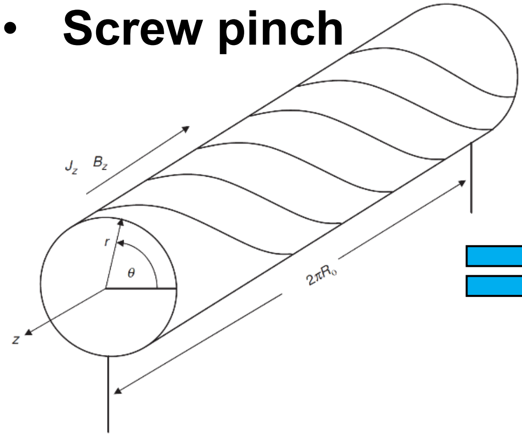
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch is flexible with varies range of β

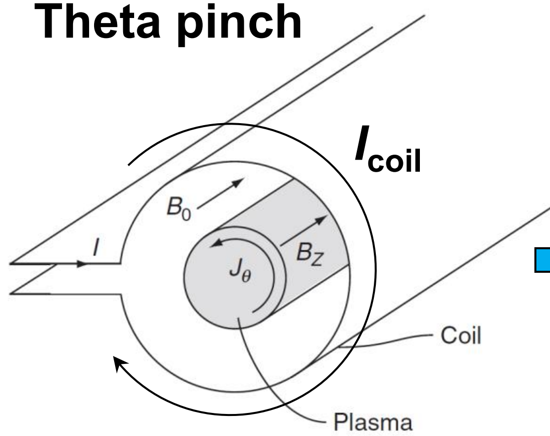
- **Screw pinch**



$$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$$
$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$
$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

=

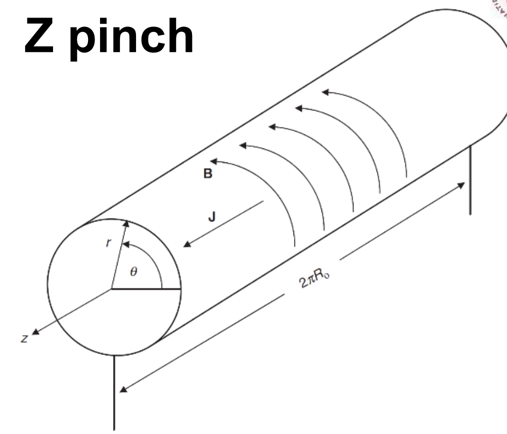
- **Theta pinch**



$$\vec{B} = B_z\hat{z}$$
$$\vec{j} = j_\theta\hat{\theta}$$
$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

\+

- **Z pinch**



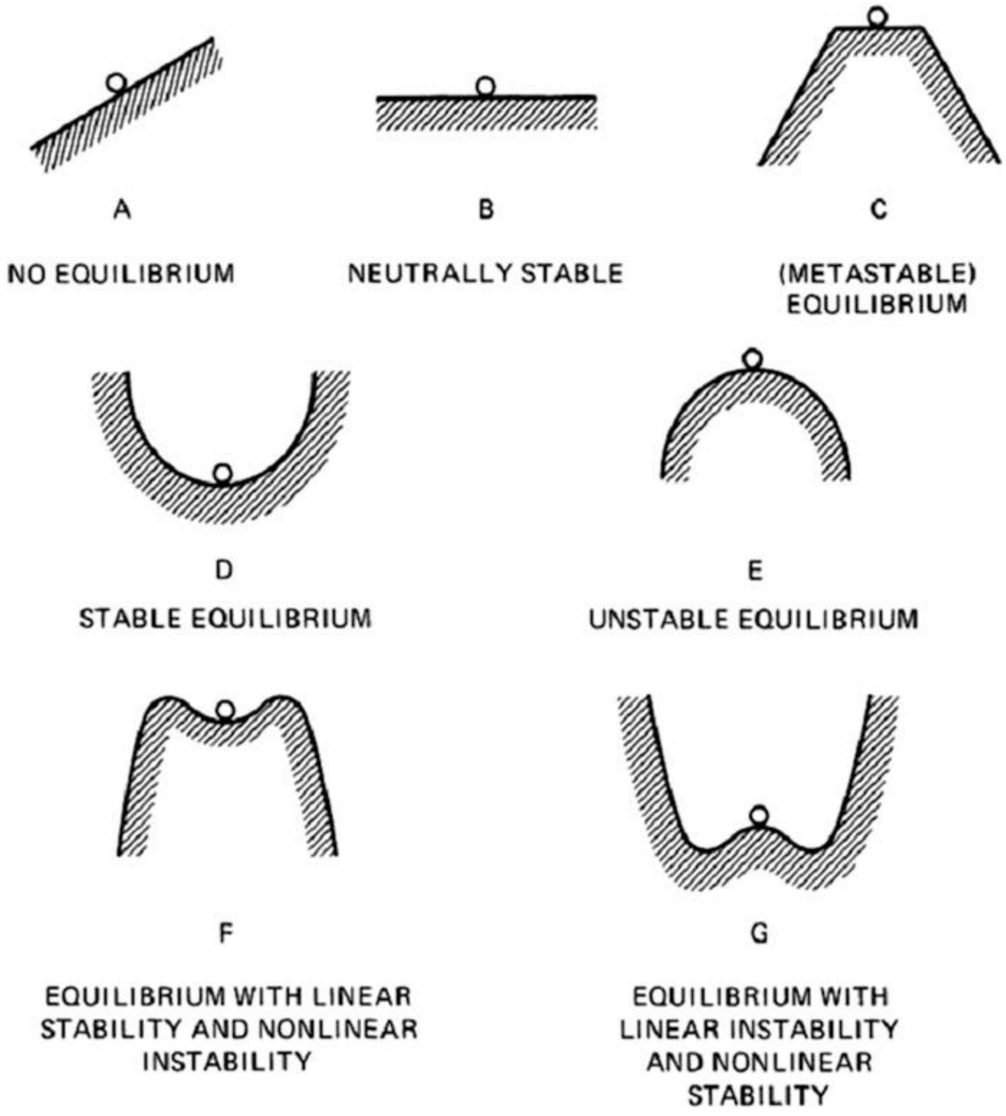
$$\vec{B} = B_\theta\hat{\theta}$$
$$\vec{j} = j_z\hat{z}$$
$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$\int_0^a \pi r^2 dr \left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0 \qquad \langle p\rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2\rangle\right)$$

$$\beta_t = \frac{2\mu_o\langle p\rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t\beta_p}{\beta_t + \beta_p} = \frac{2\mu_o\langle p\rangle}{B_o^2 + {B_{\theta a}}^2}$$

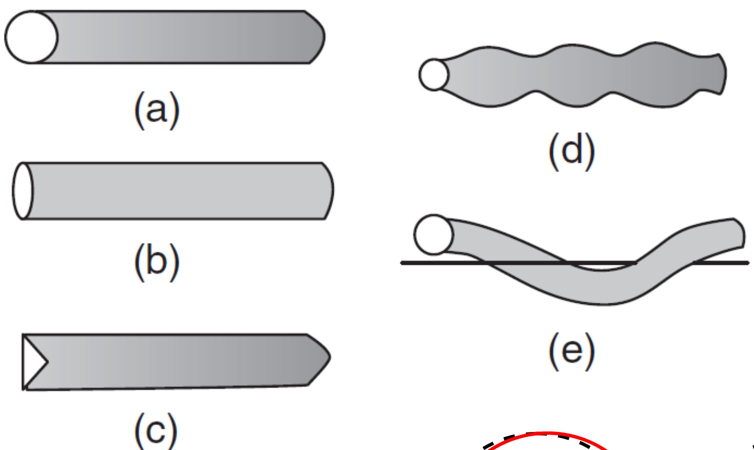
$$0 \le \langle\beta\rangle \le 1$$

An equilibrium state may not be stable

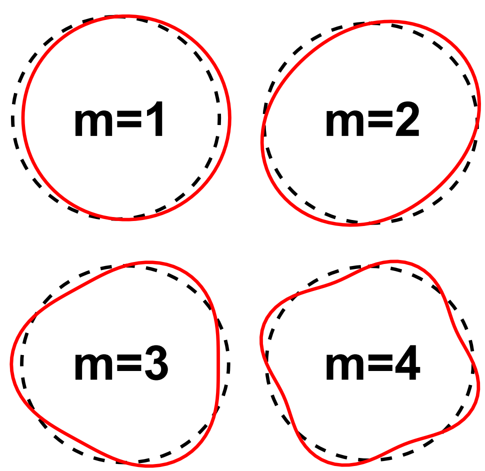


A cylindrical plasma column may not be stable

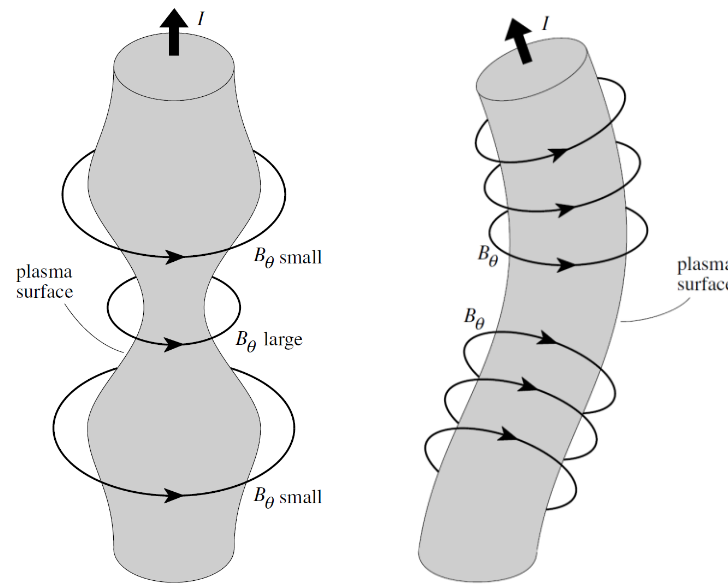
- **Instabilities of theta pinch**



(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0



- **Instabilities of z pinch**



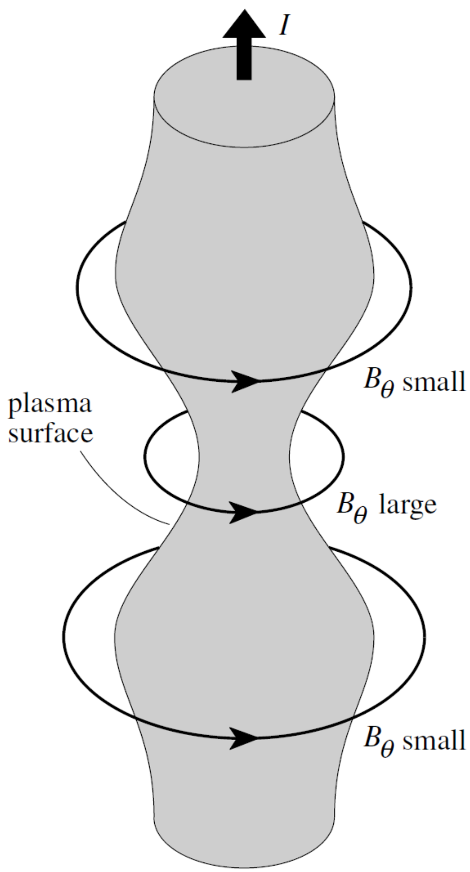
Sausage instability (m=0)

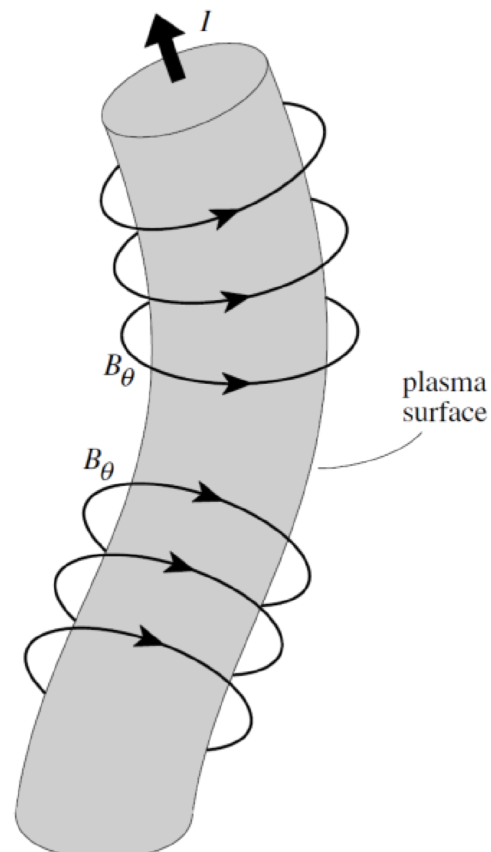
Kink instability (m=1)

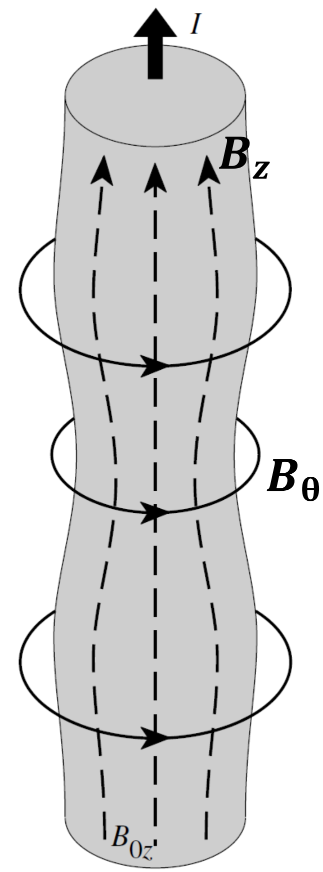
$$\zeta(\vec{r}) = \zeta(r)exp(im\theta + ikz)$$

A cylindrical plasma column is stable when the safety factor is greater than unity (Kruskal–Shafranov limit)

- **Sausage instability (m=0)**
- **Kink instability**

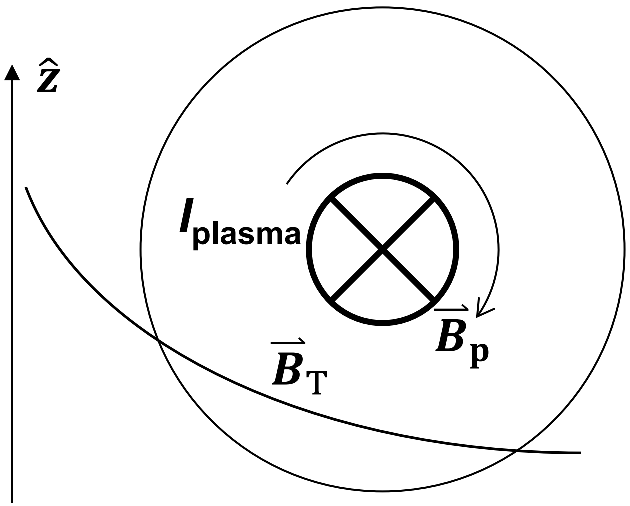






- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_o B_\theta(r)}$
- **Kruskal–Shafranov limit:** $q(r) > 0$ **: stable**
- **The tension of B_z provides the stabilizing force and suppresses the instabilities.**

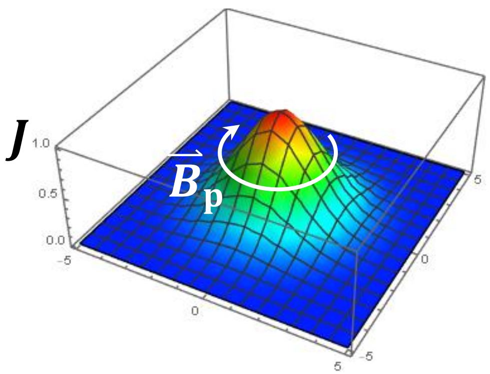
Sawtooth oscillation is initiated when the core is heated

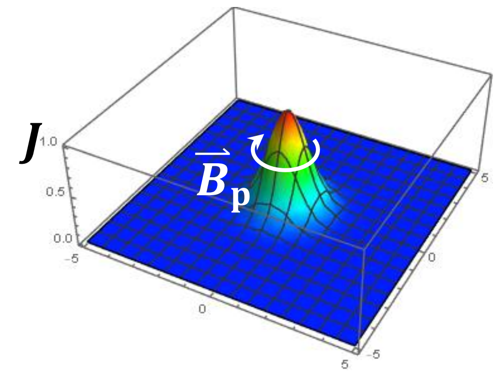


$\eta \propto T_e^{-3/2}$ $\quad E_\parallel = \eta J$

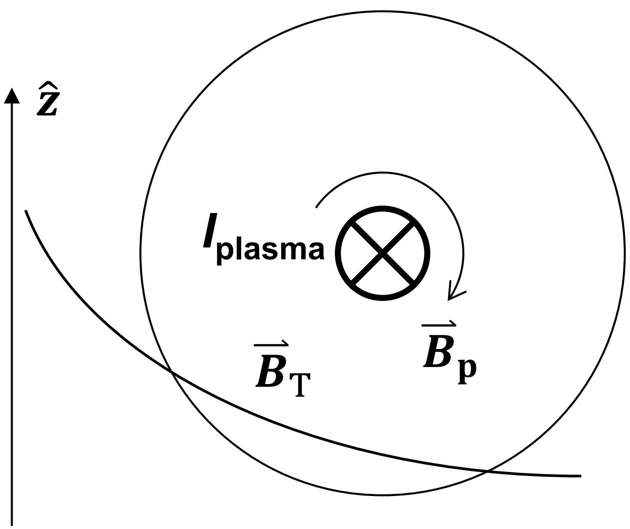
$T_e \uparrow \Rightarrow \eta \downarrow \Rightarrow J \uparrow$

- Current is less diffused.
- Current density with a higher peak is formed.





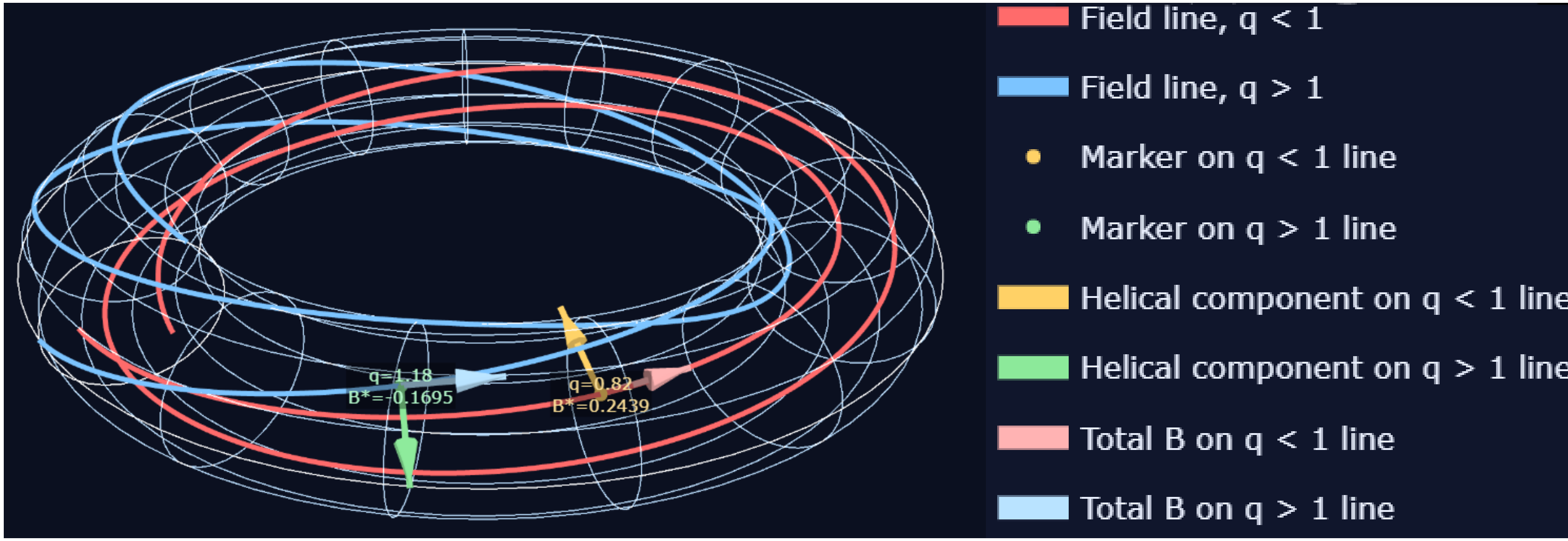
$$J \uparrow \Rightarrow \vec{B}_p \uparrow \Rightarrow q(r) = \frac{rB_T(r)}{R_o B_p(r)} \downarrow$$



- For a certain radius a,

$$q(a) = \frac{rB_T(a)}{R_o B_p(a)} < 0 \;\Rightarrow \text{kink unstable}$$

The helical field component B* is opposite across the surface of q=1



$$B_* = \vec{B} \cdot \nabla(m\theta - n\phi)$$

$$B_* = \vec{B} \cdot \nabla(\theta - \phi) = \frac{B_\mathrm{p}}{r} - \frac{B_\mathrm{T}}{R} = \frac{B_\mathrm{T}}{R}\left(\frac{1}{q} - 1\right)$$

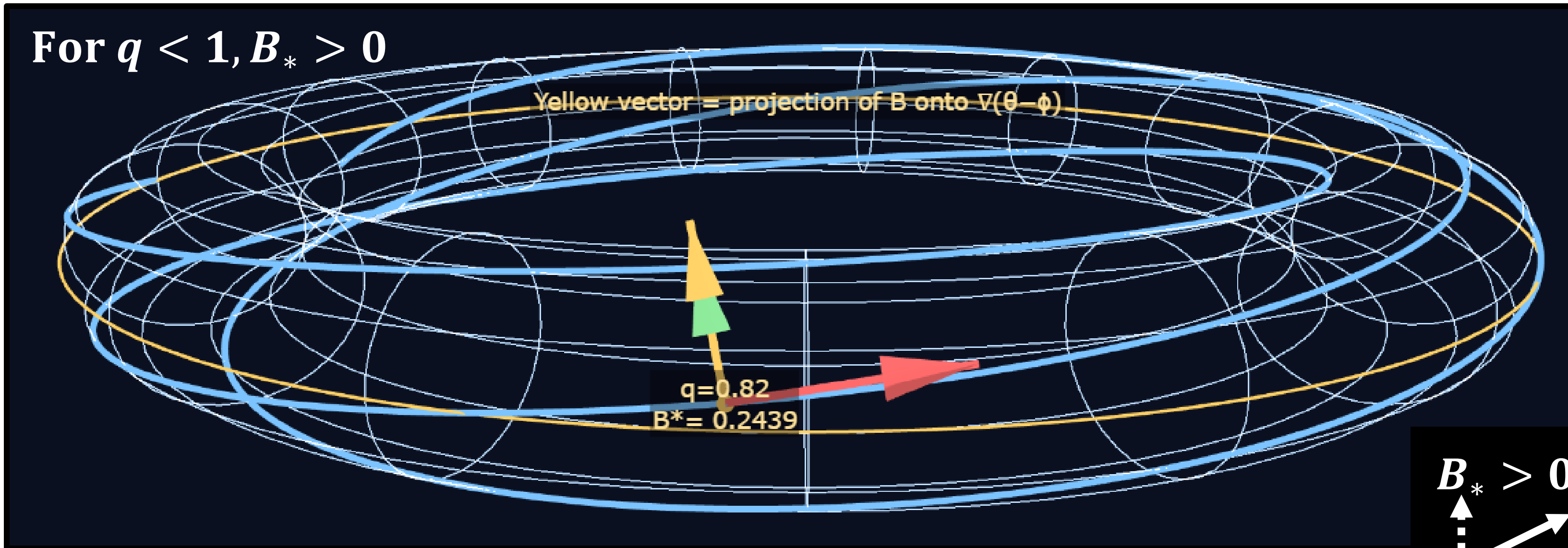
$$q(r) = \frac{rB_\mathrm{T}(a)}{R_\mathrm{o}B_\mathrm{p}(a)}$$

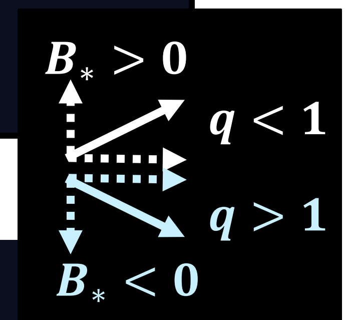
For $q < 1$, $B_* > 0$

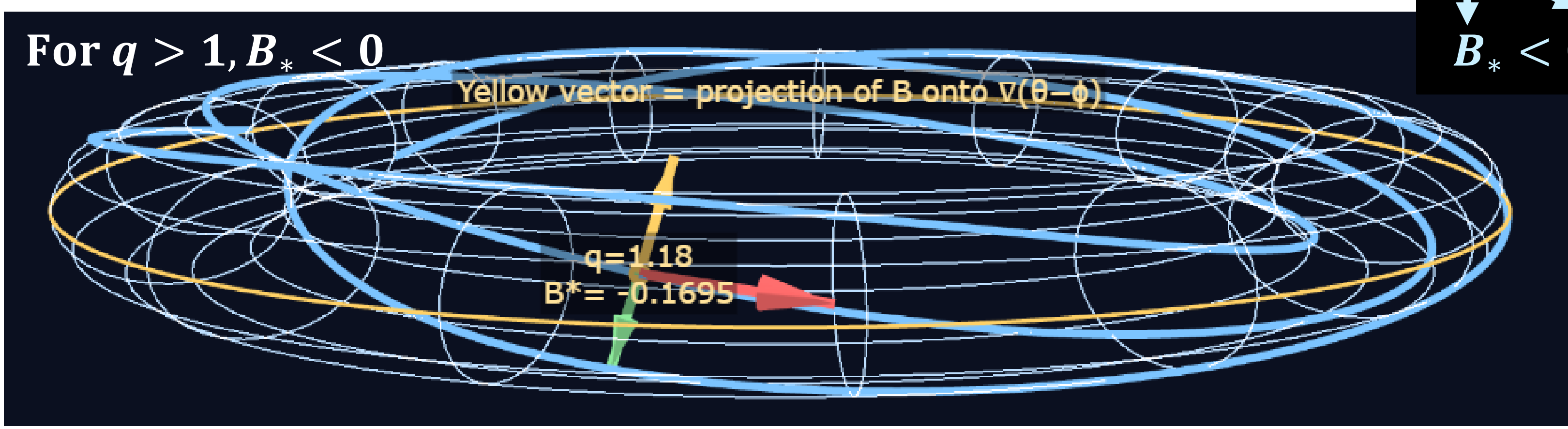
For $q = 1$, $B_* = 0$

For $q > 1$, $B_* < 0$

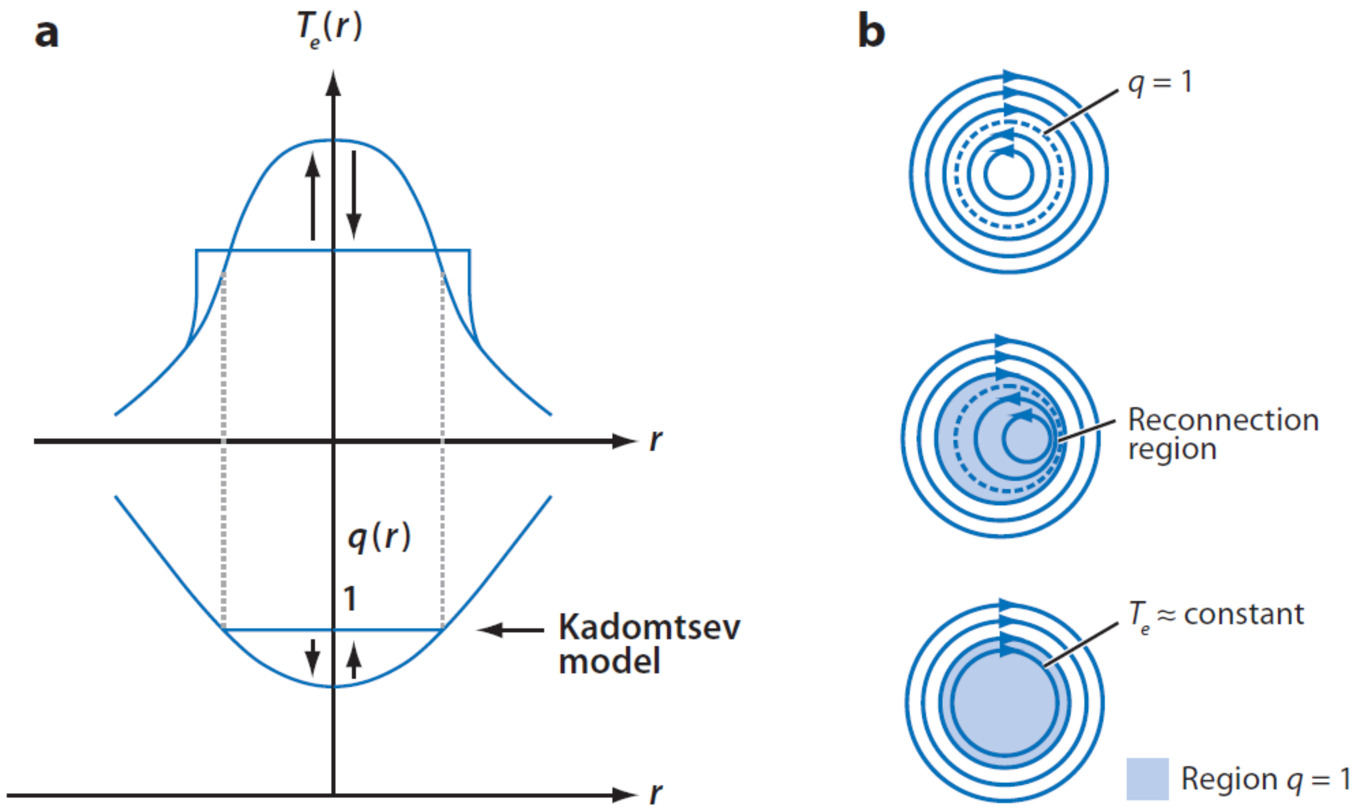
The helical field component B* is opposite across the surface of q=1





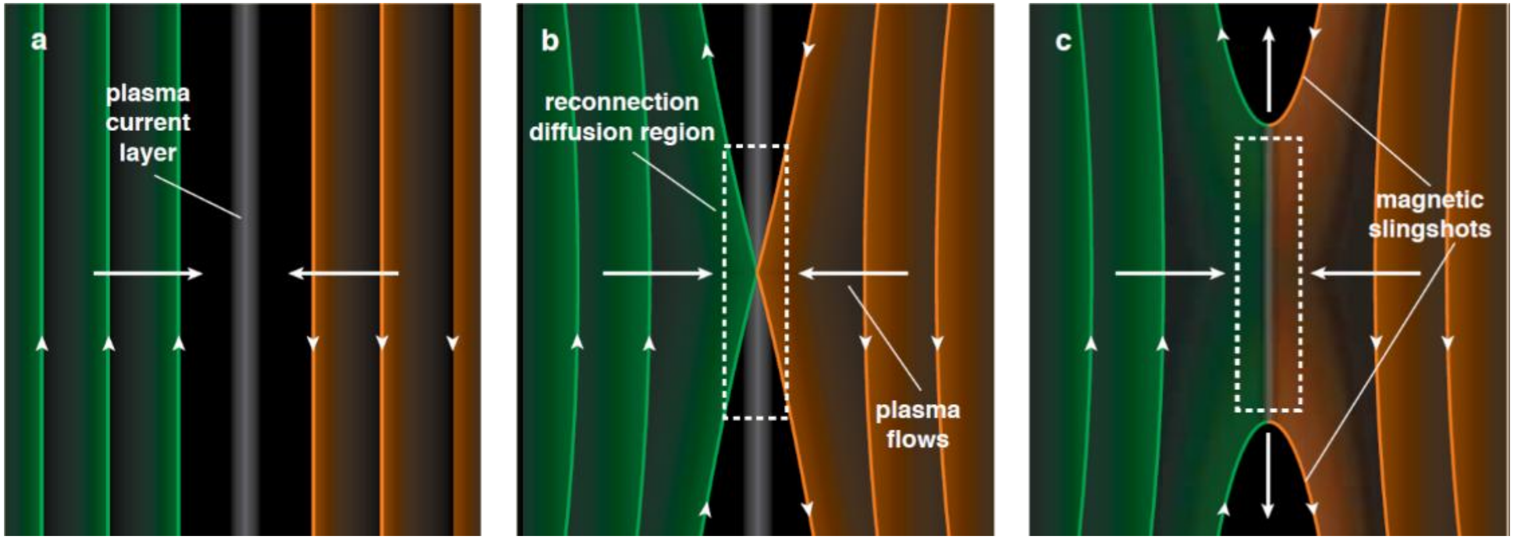


Kink instabilities occur at q <0 leading to reconnection events



E. G. Zweibel and M. Yamada, Annu. Rev. Astron. Astrophys., 47, 291 (2009)

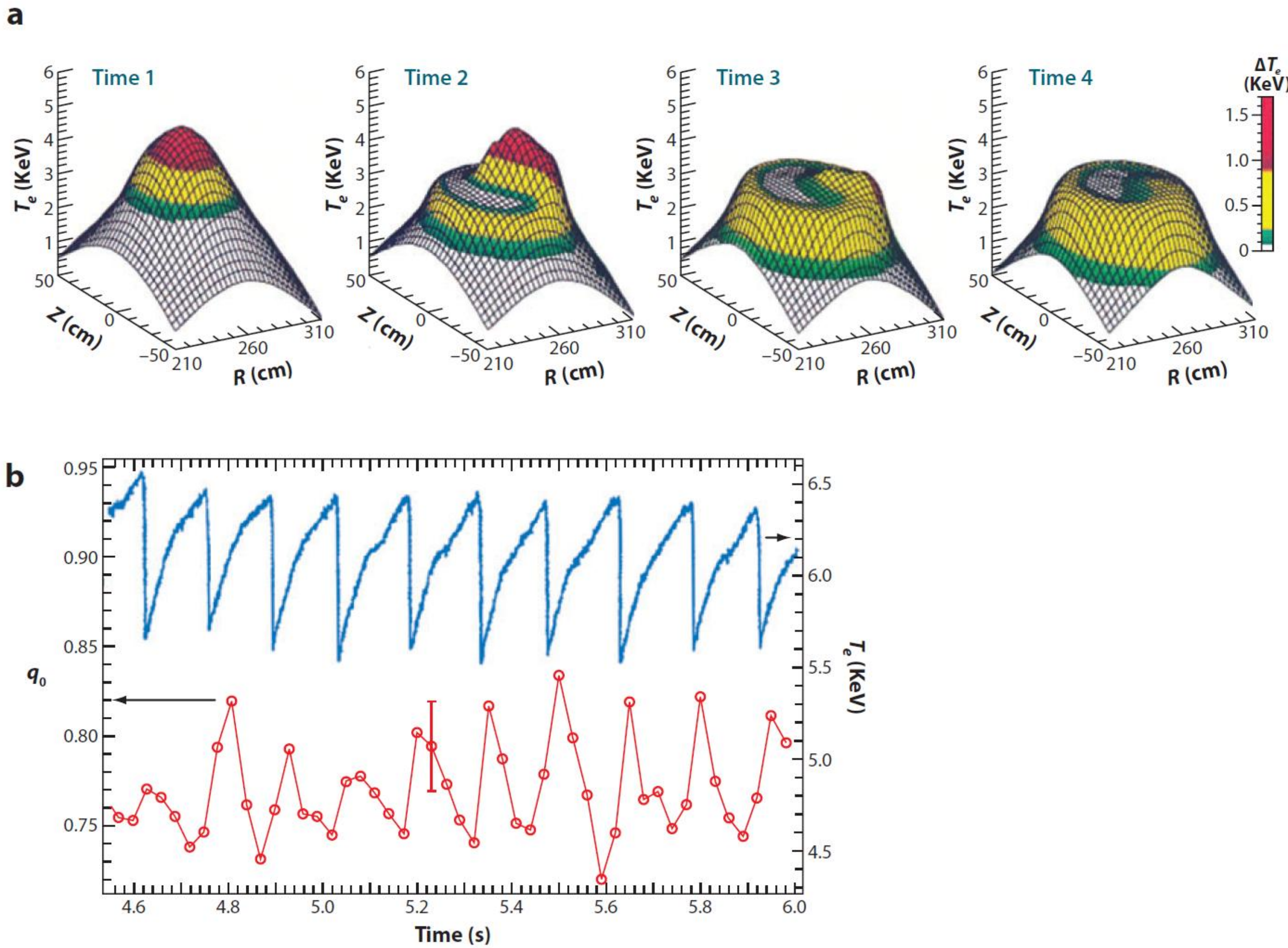
Reconnection converts the magnetic field energy to kinetic energy of particles



https://www.youtube.com/watch?v=7sS3Lpzh0Zw

- Energetic particles are thermalized when they collide with the surrounding particles.

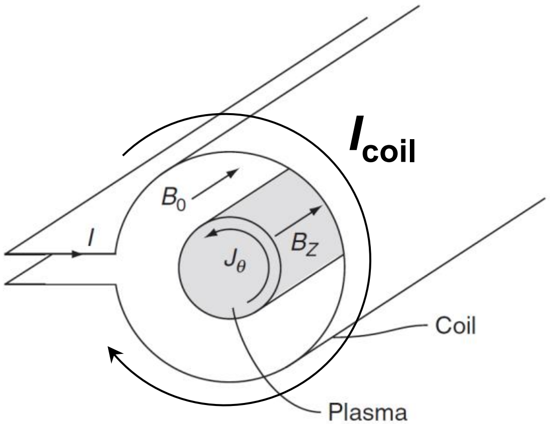
Temperature and safety factor oscillates during the sawtooth crash



E. G. Zweibel and M. Yamada, Annu. Rev. Astron. Astrophys., 47, 291 (2009)

Theta pinch is stable while z pinch is unstable

- **Theta pinch**

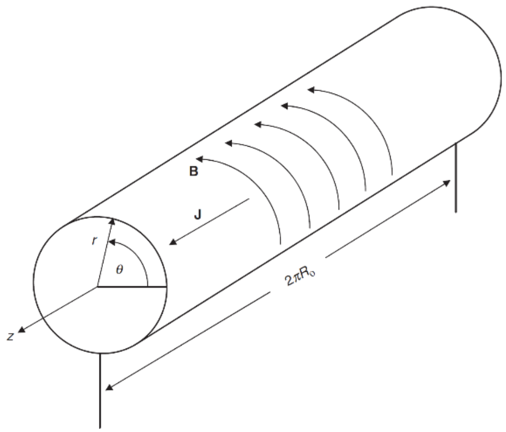


$\vec{B} = B_z\hat{z}$

$q_\theta = \infty$

Stable

- **Z pinch**

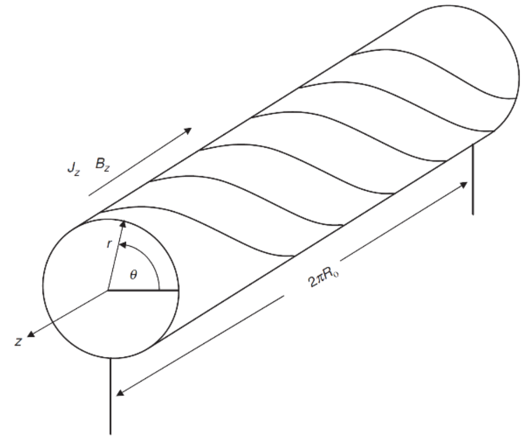


$\vec{B} = B_\theta\hat{\theta}$

$q_z = 0$

Unstable

- **Screw pinch**



$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

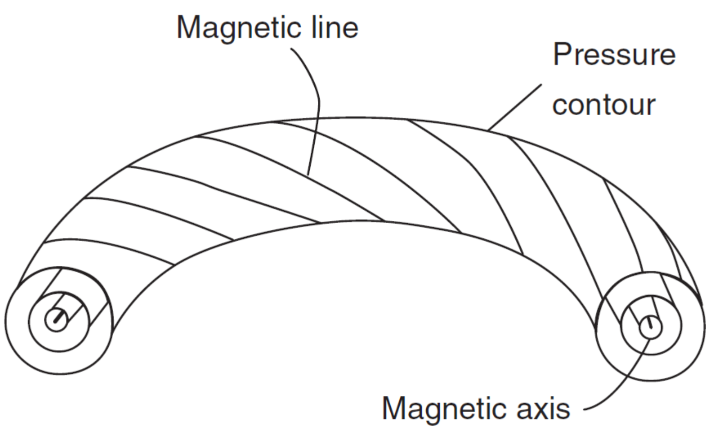
$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$

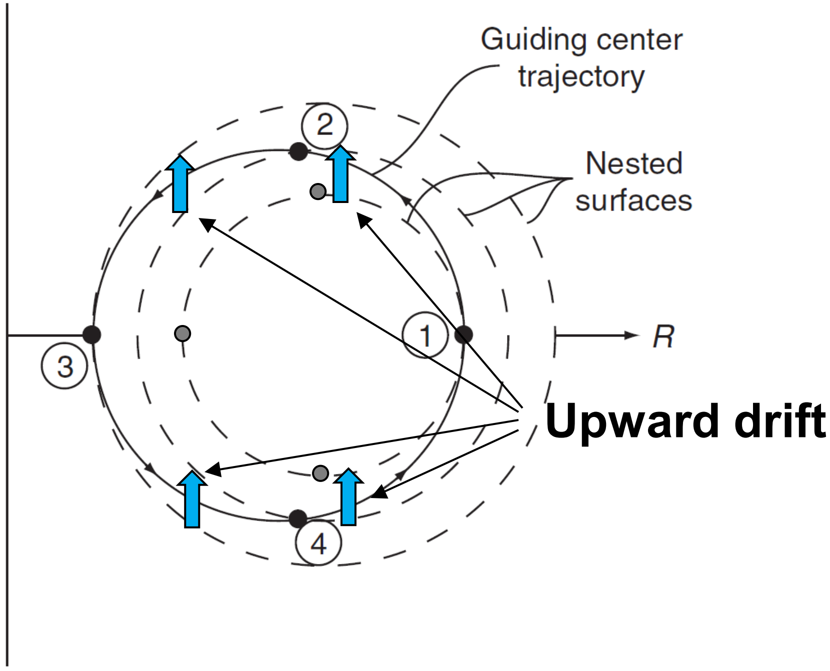
q can be controlled.

Stable/Unstable

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

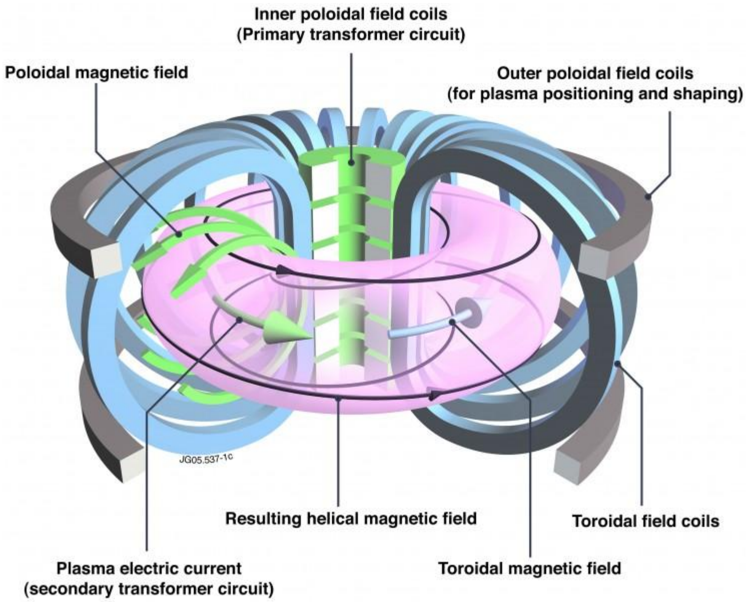




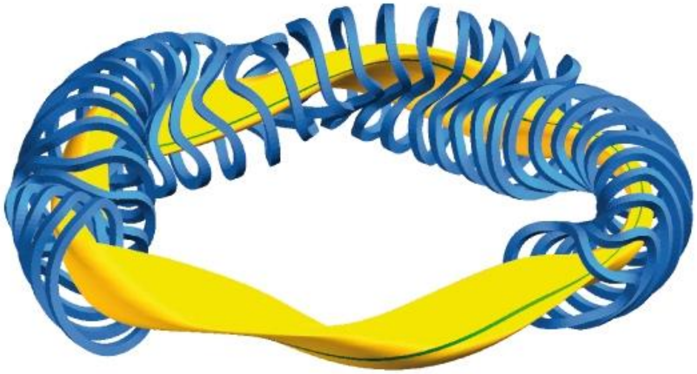
Points with no drift

Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



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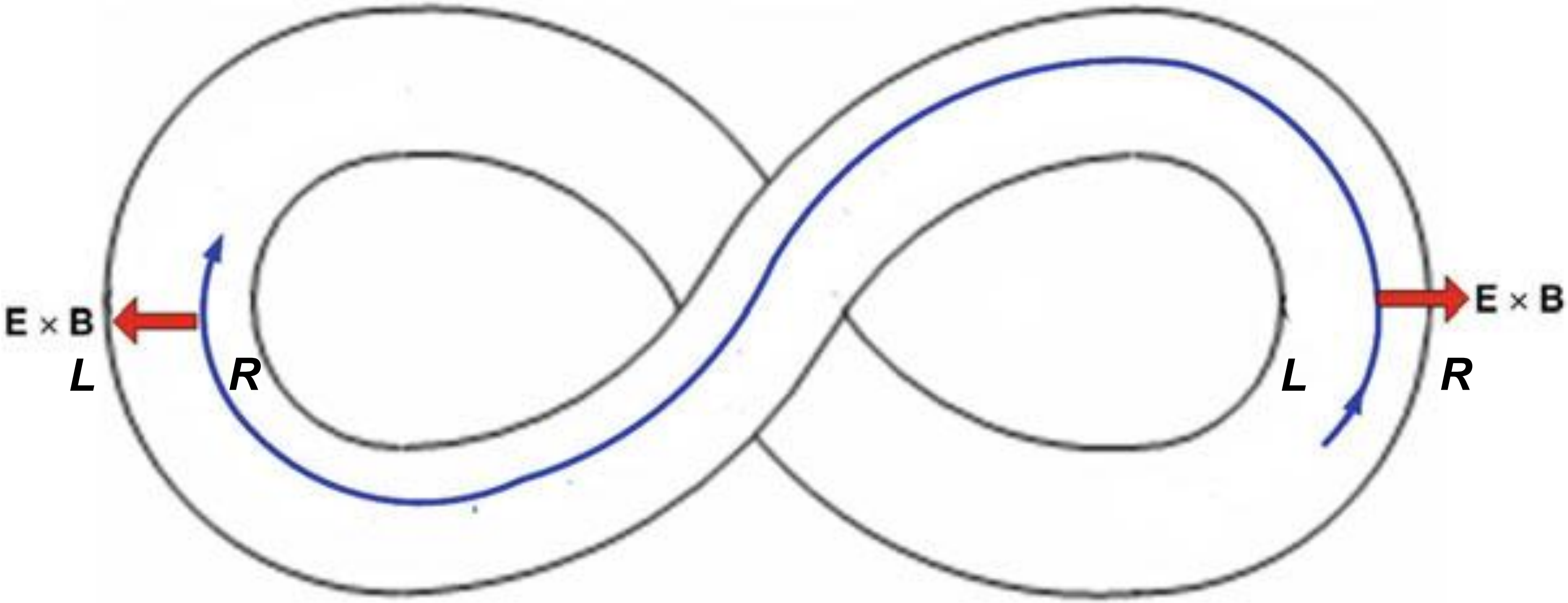
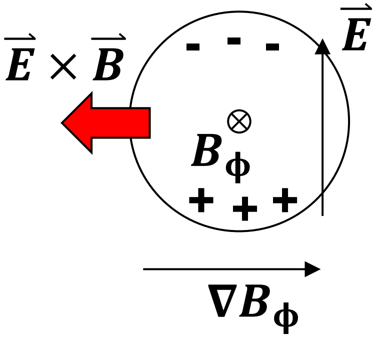
https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

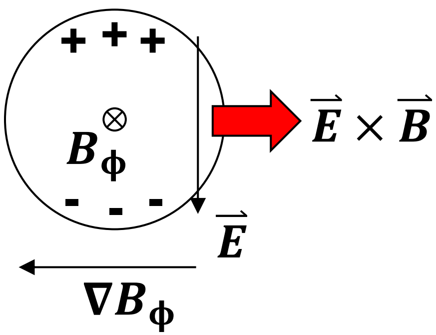
Stellerator

- Figure eight shape
- Oval shape (racetrack)
- Torsatron
- Heliotron
- Heliac (Helical Axis stellarator)
- Helias (W7-x)

A figure-8 stellarator solved the drift issues





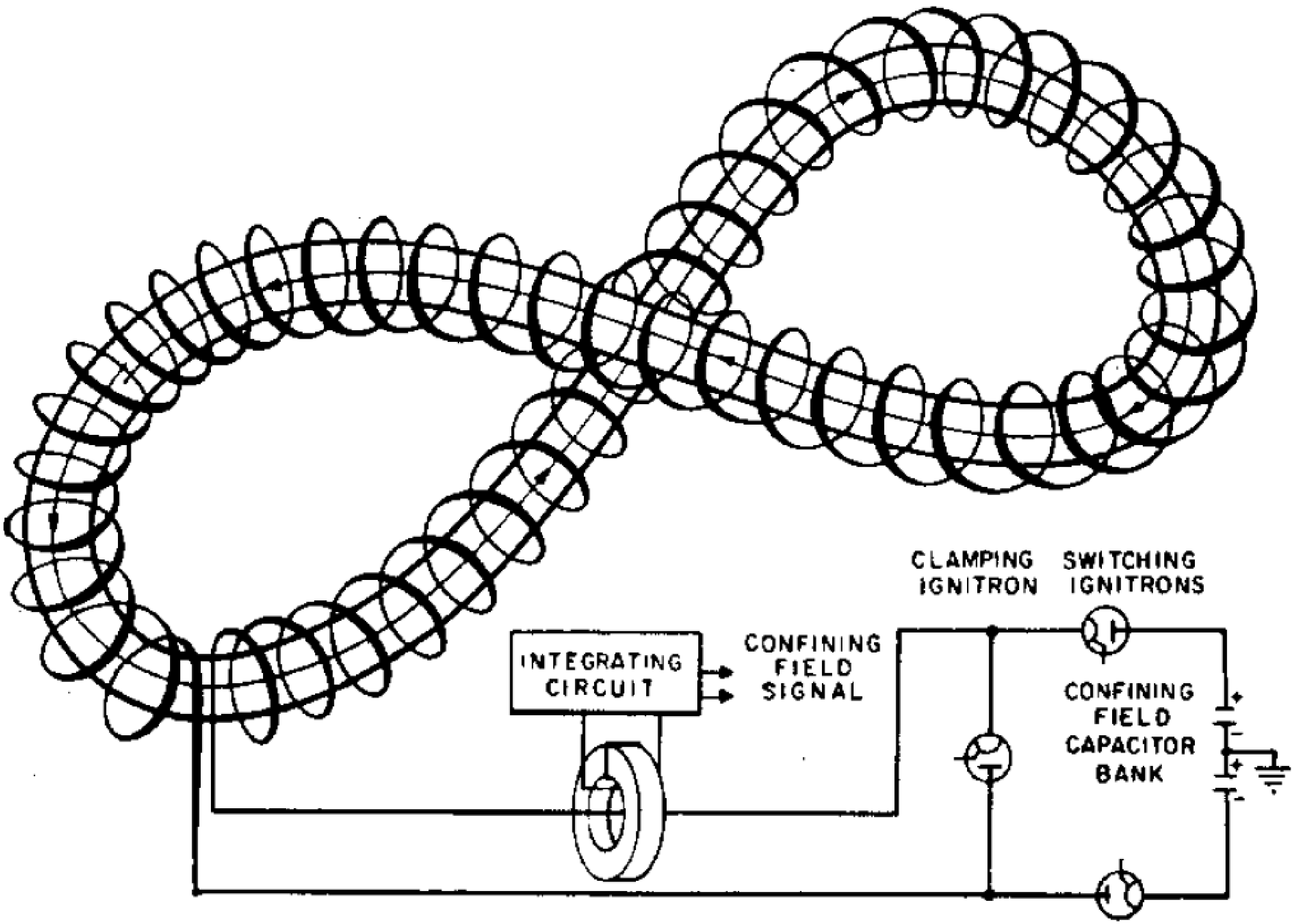


A figure-8 stellarator solved the drift issues

Lyman Spitzer, Jr. came out the idea during a long ride on a ski lift at Garmisch-Partenkirchen

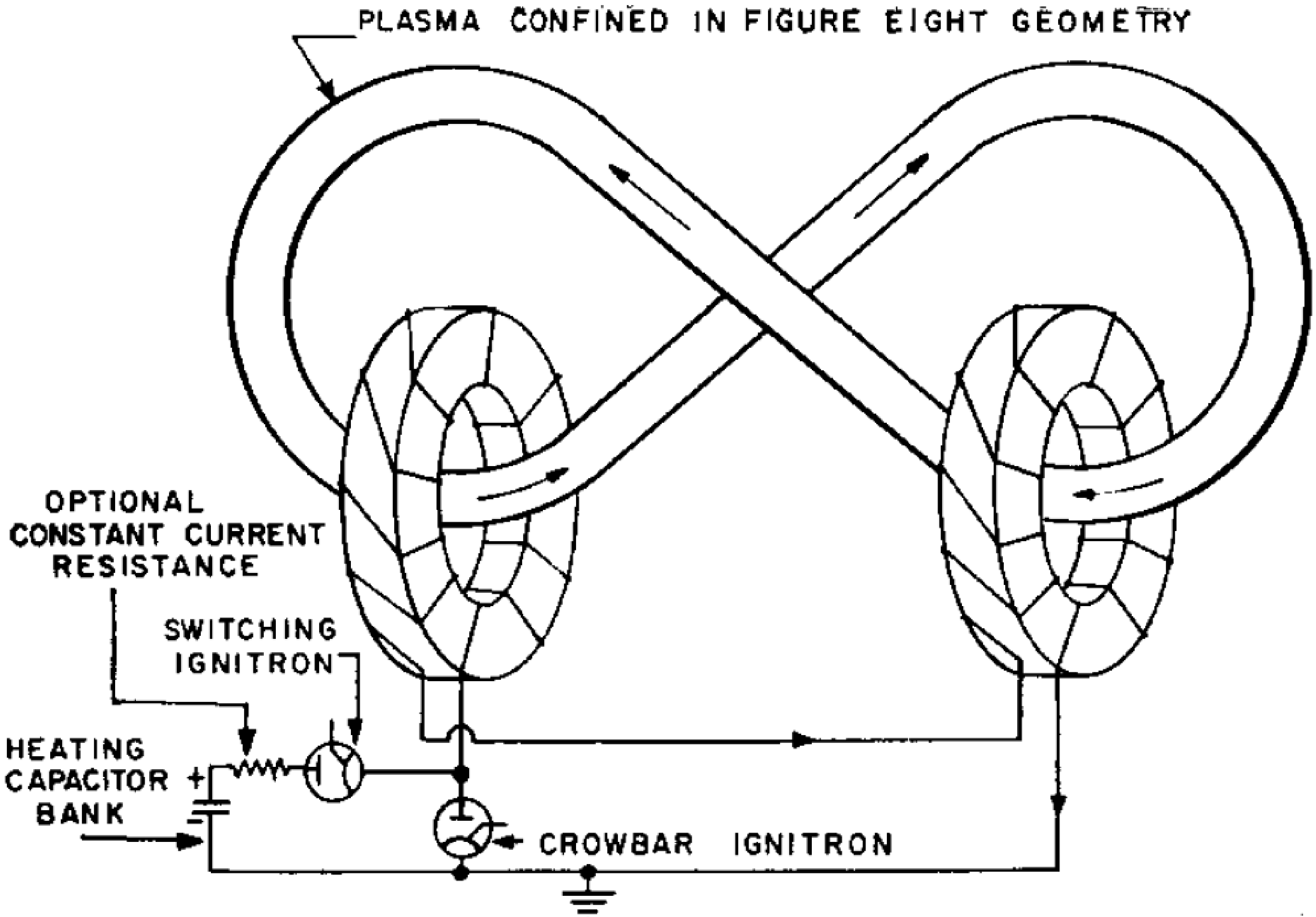
https://www.snowtrex.de/magazin/skigebiete/garmisch-classic-zugspitze/

Concept of figure-8 stellarator



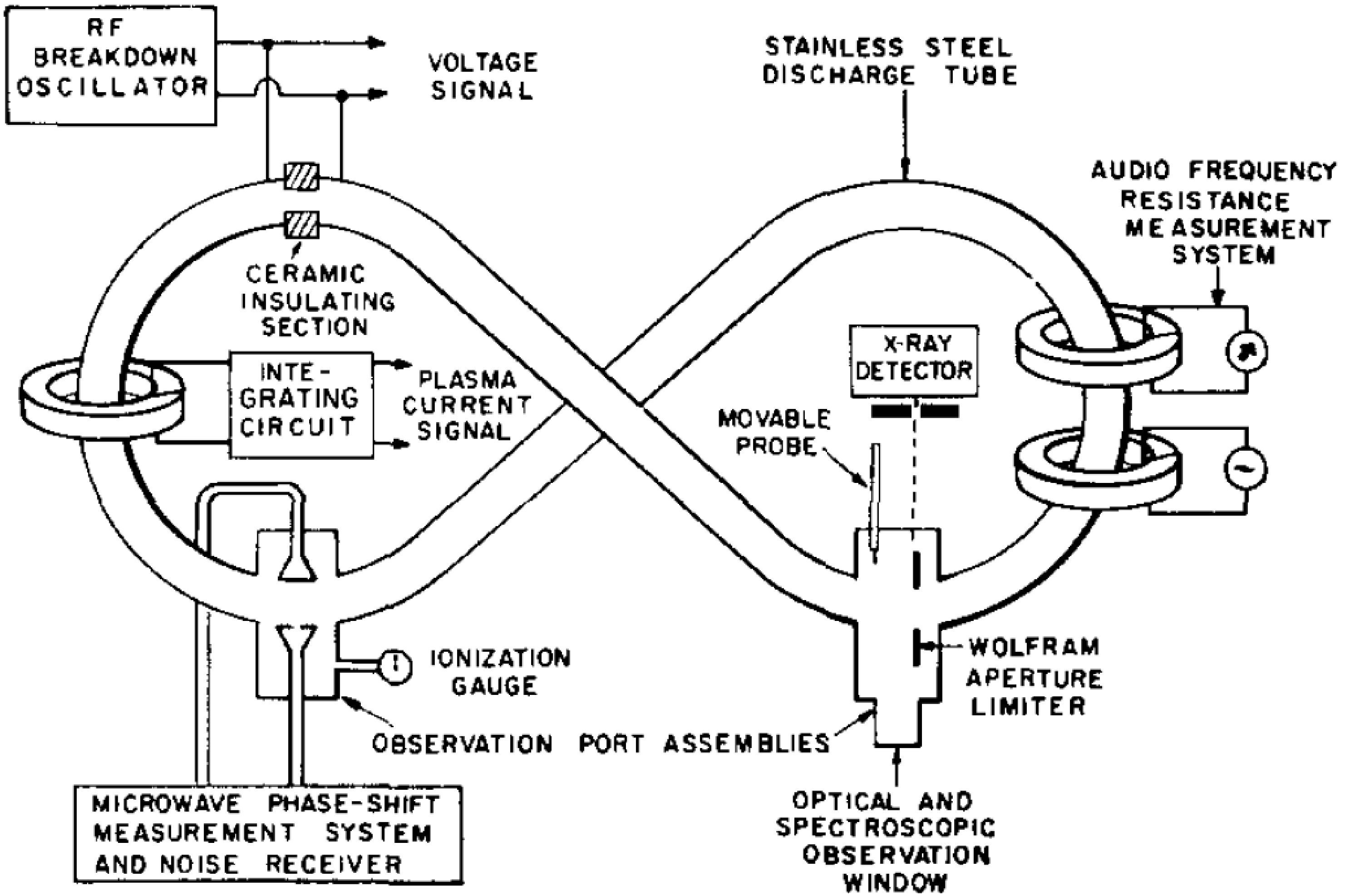
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Figure-8 stellarator with ohmic heating apparatus



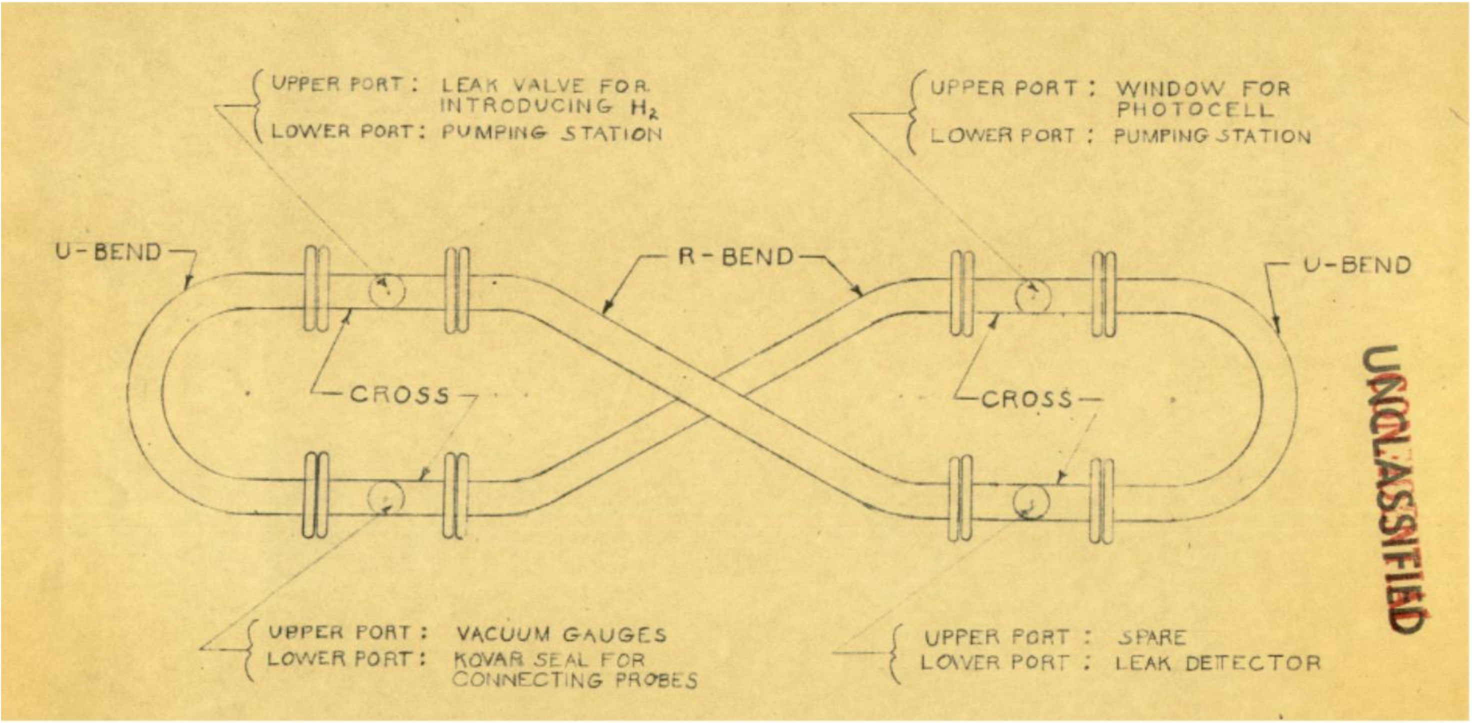
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Schematic diagram of B-1 stellarator

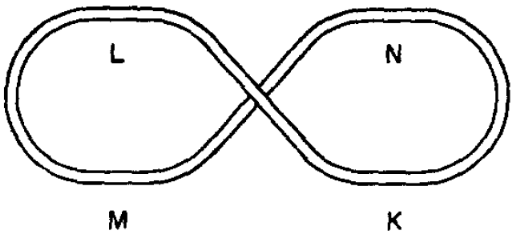


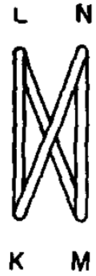
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Figure-eight (Princeton Model A) – 1953-1958



- Top view
- Side view



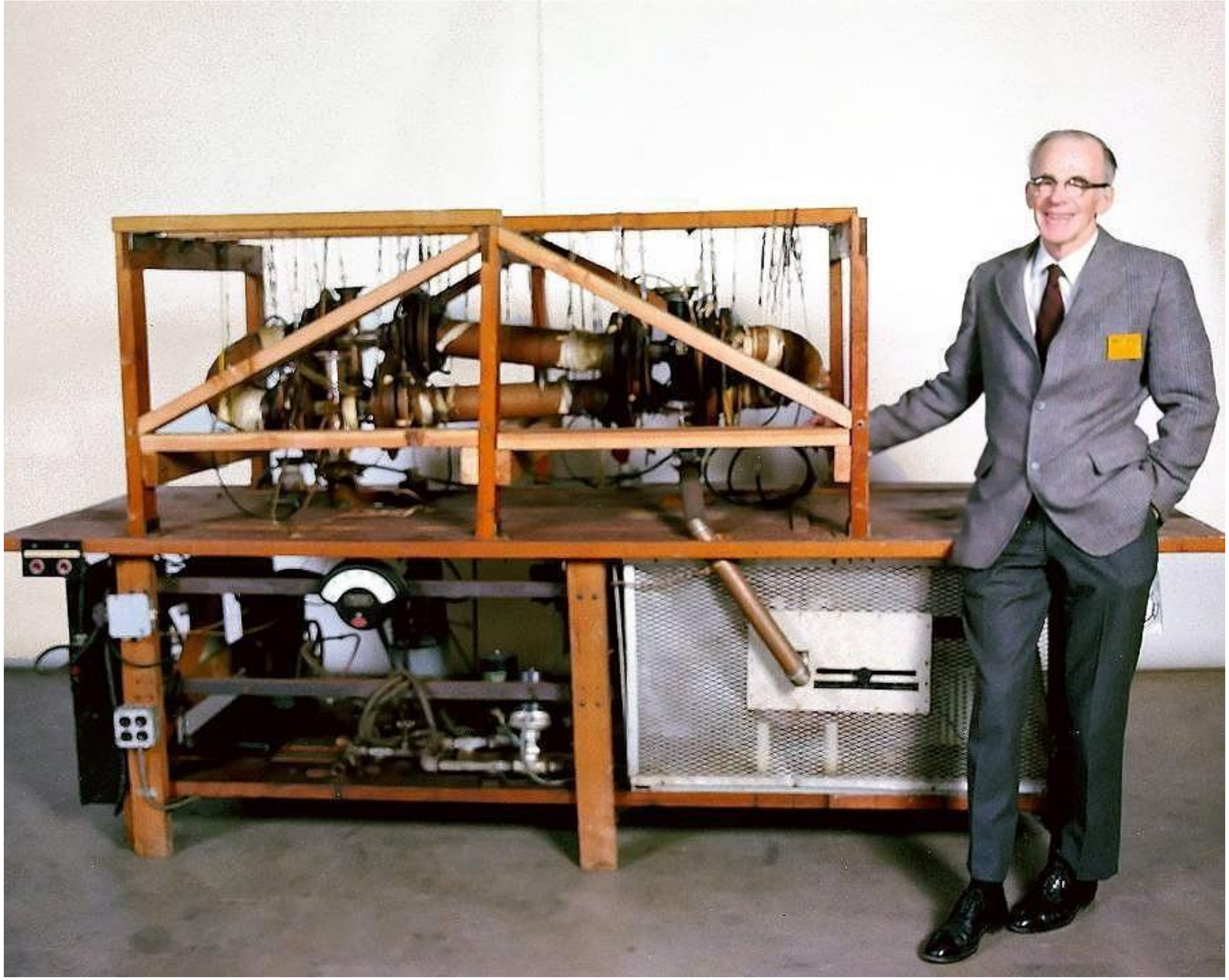


C. H. Willis, NJ Project Matterhorn (1953)
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Model A stellarator

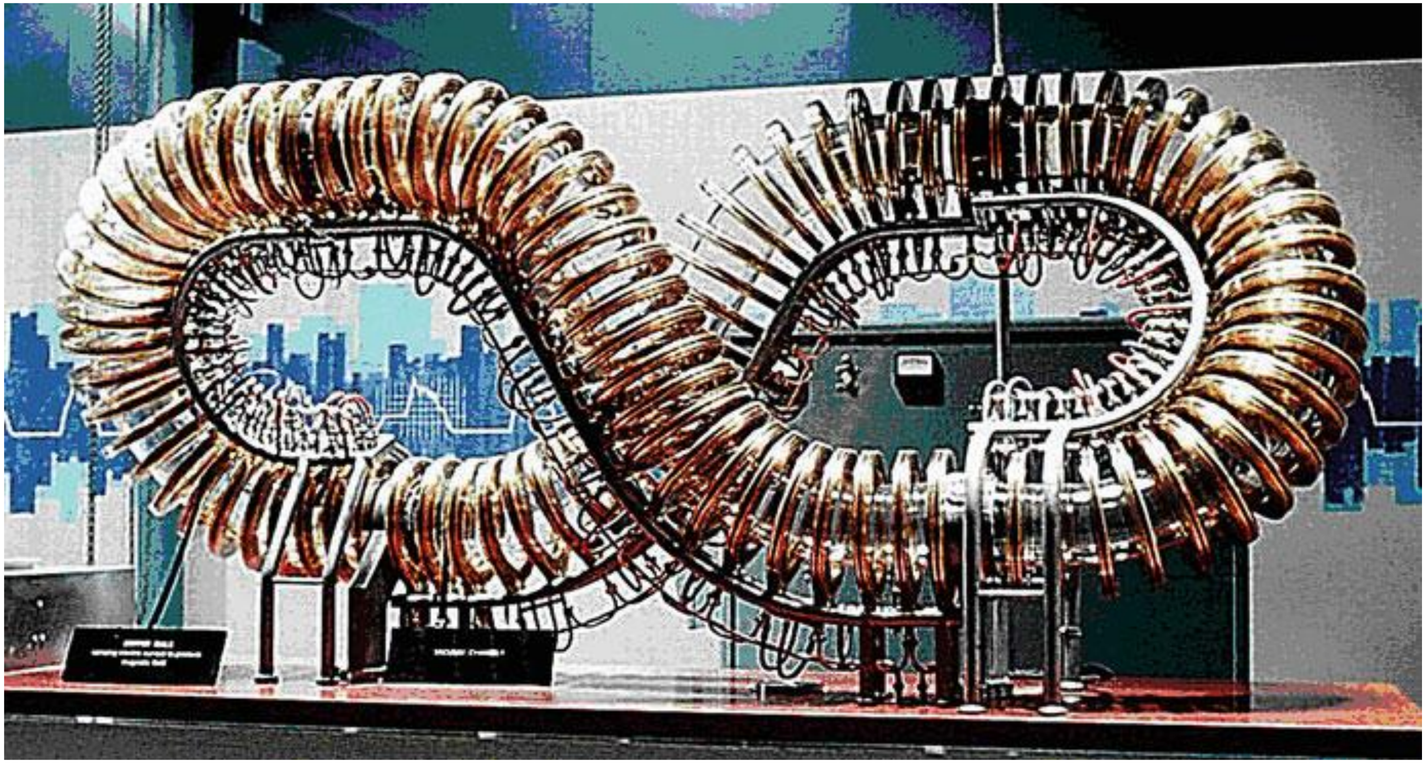
https://www.pppl.gov/timeline

Model A stellarator



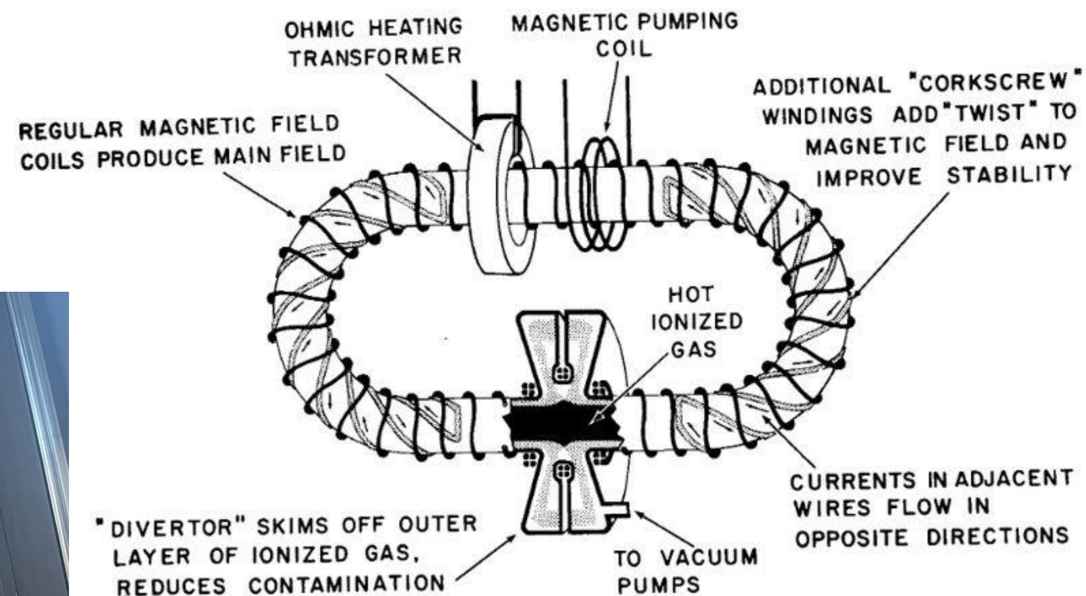
https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

Exhibit model of a figure-8 stellarator for the Atoms for Peace conference in Geneva in 1958



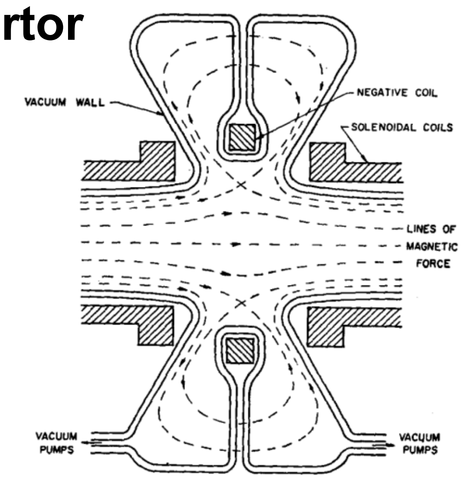
Racetrack Stellarator (Project Matterhorn)

FIG. 4: SCHEMATIC "RACETRACK" STELLARATOR



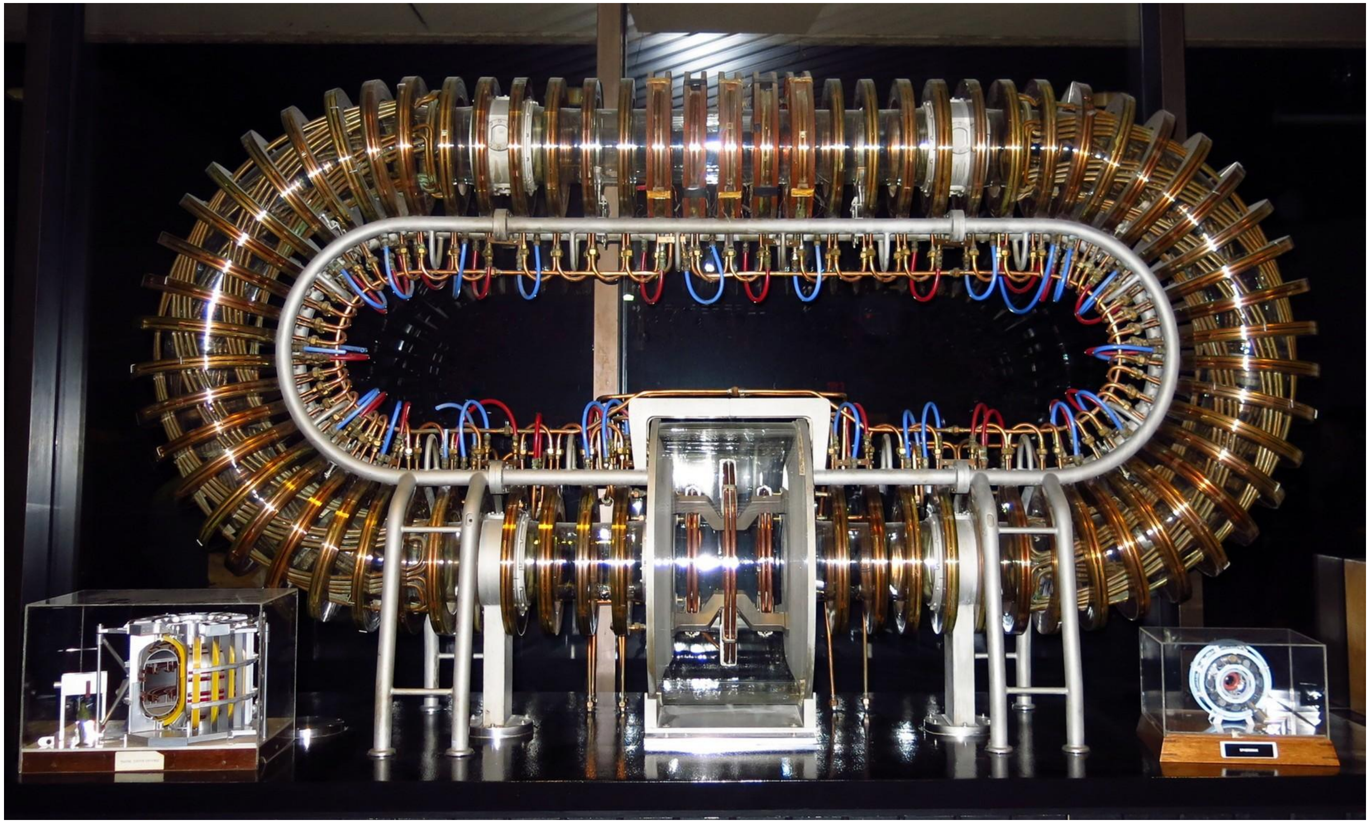
SEPTEMBER 19, 1958 ★ 9

- **Divertor**



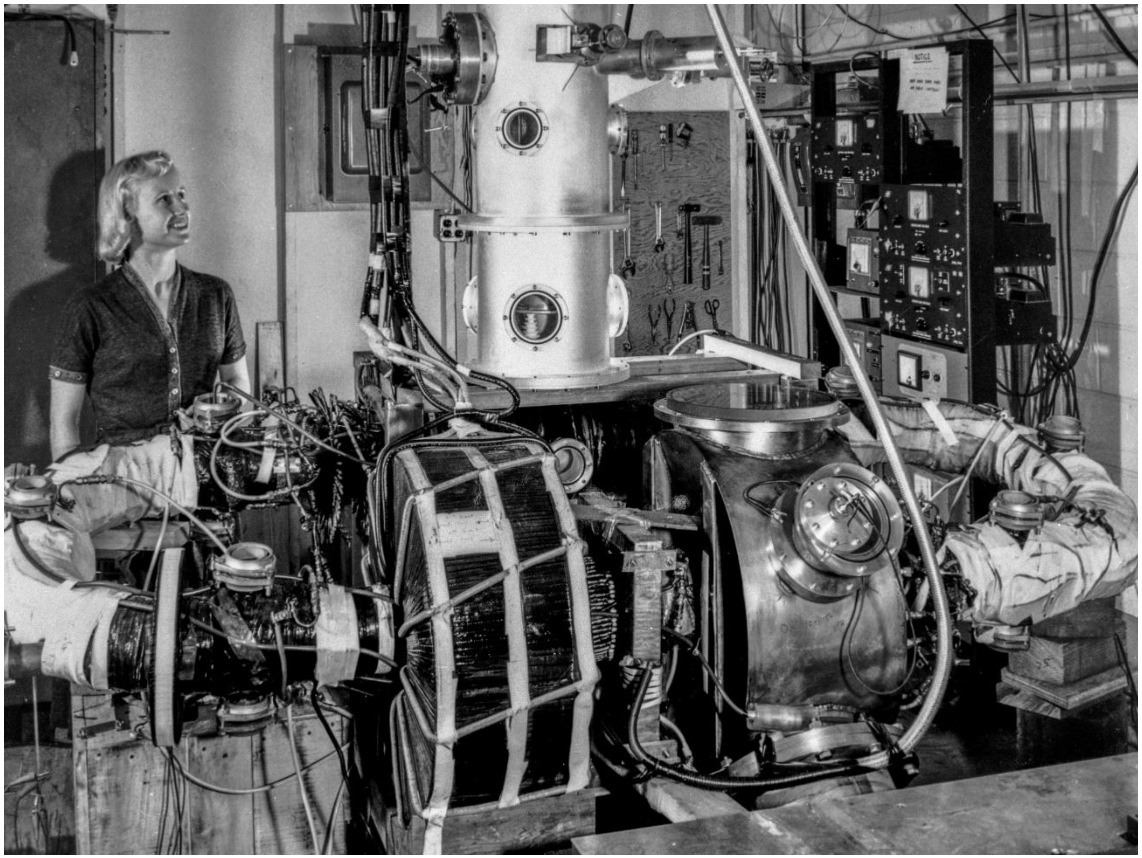
https://twitter.com/quantumwriters/status/1724959216966775270
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Racetrack Stellarator



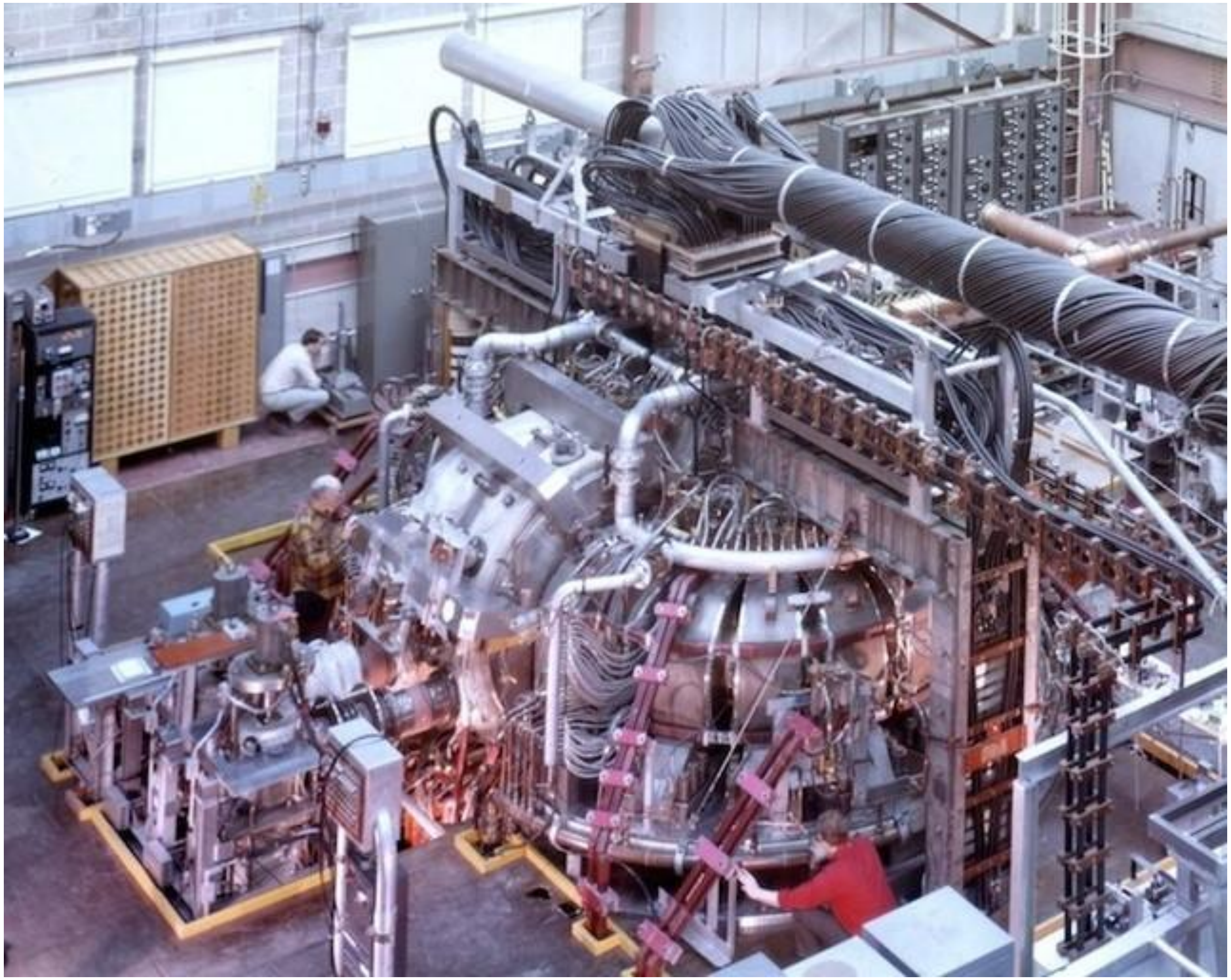
https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

B-65 stellarator



https://www.pppl.gov/timeline
Elizabeth Paul, An introduction to stellarators,
Princeton Alumni Weekly, Sep. 19, 1958

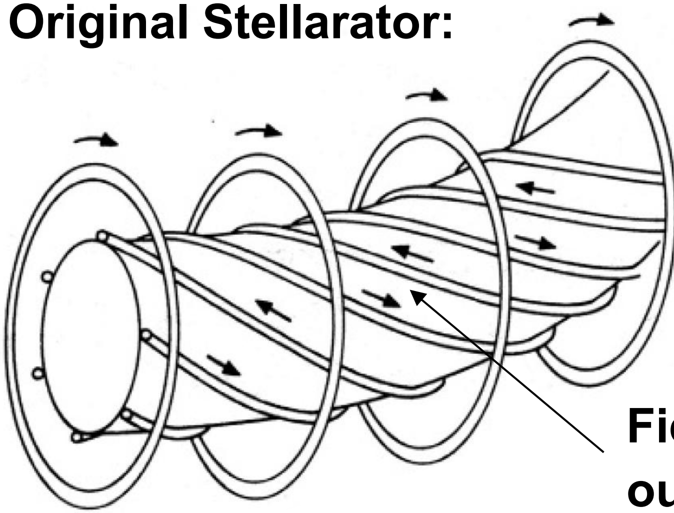
Racetrack (Princeton Model C) – 1962-1969



https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

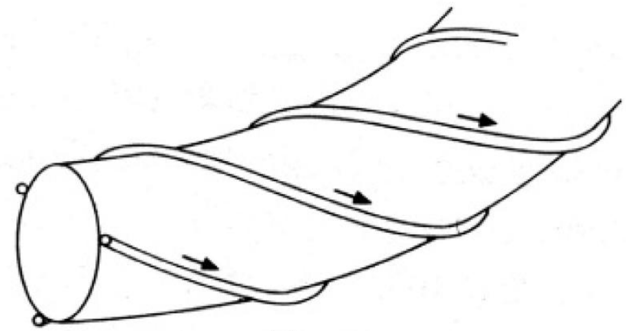
Different types of stellarators

- **Original Stellarator:**

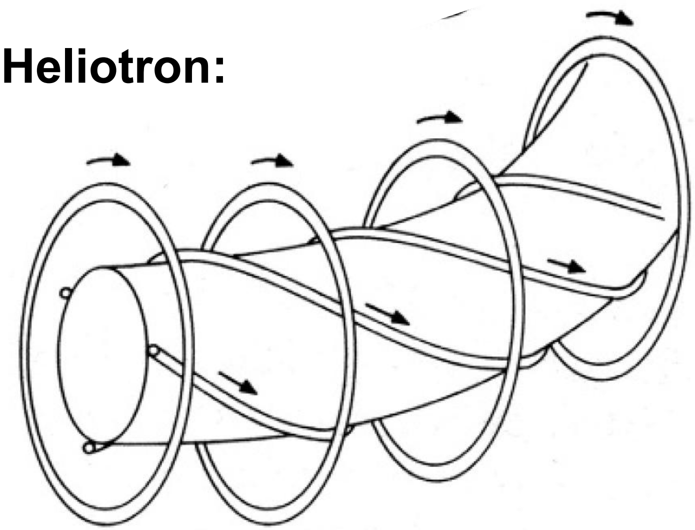


Field cancels out on axis

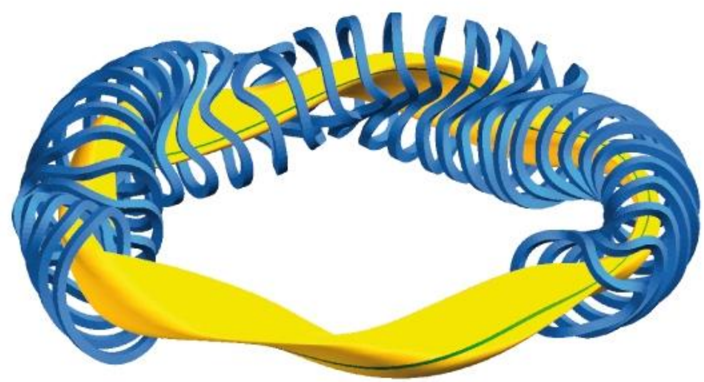
- **Torsatron:**



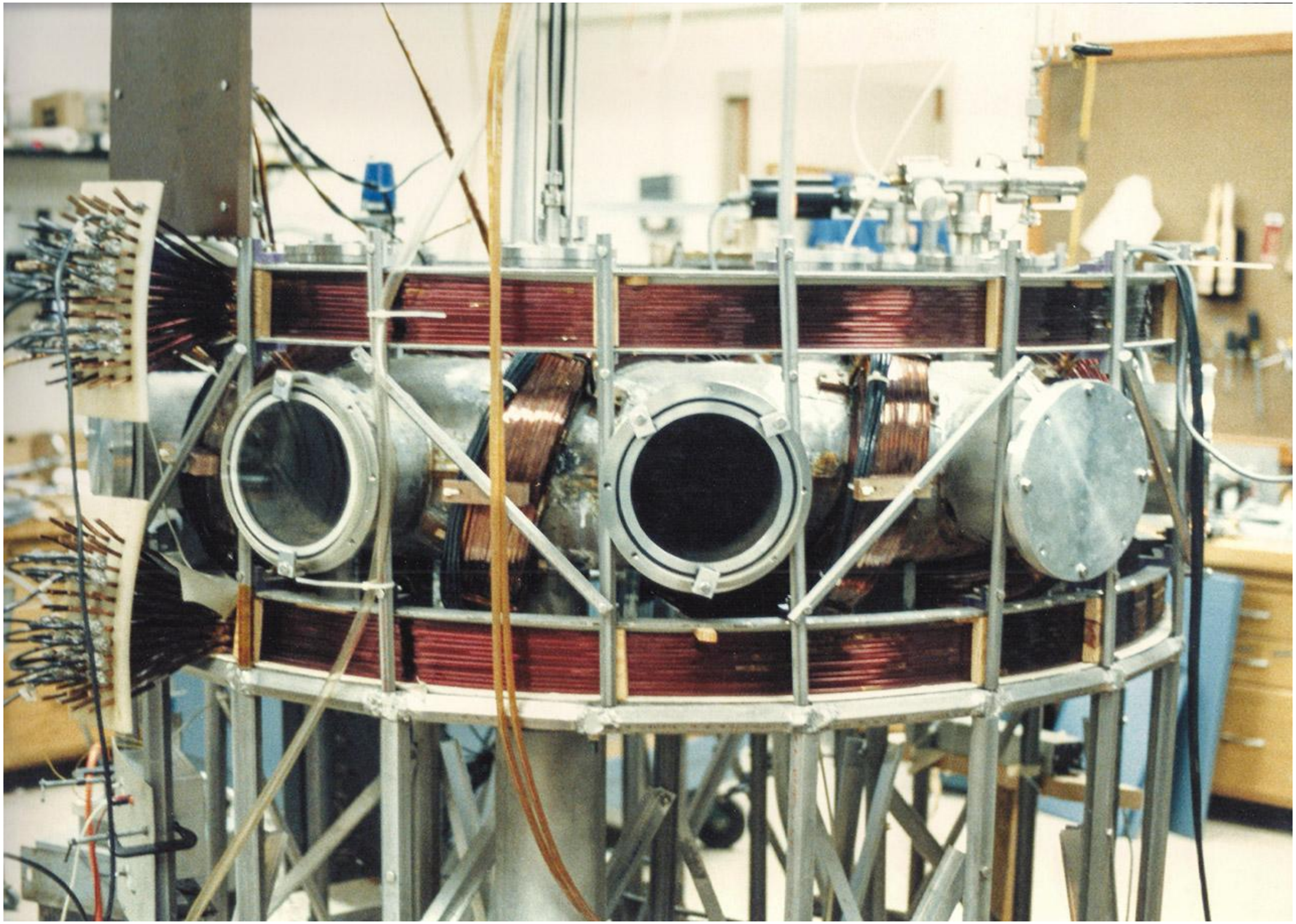
- **Heliotron:**



- **Helias:**

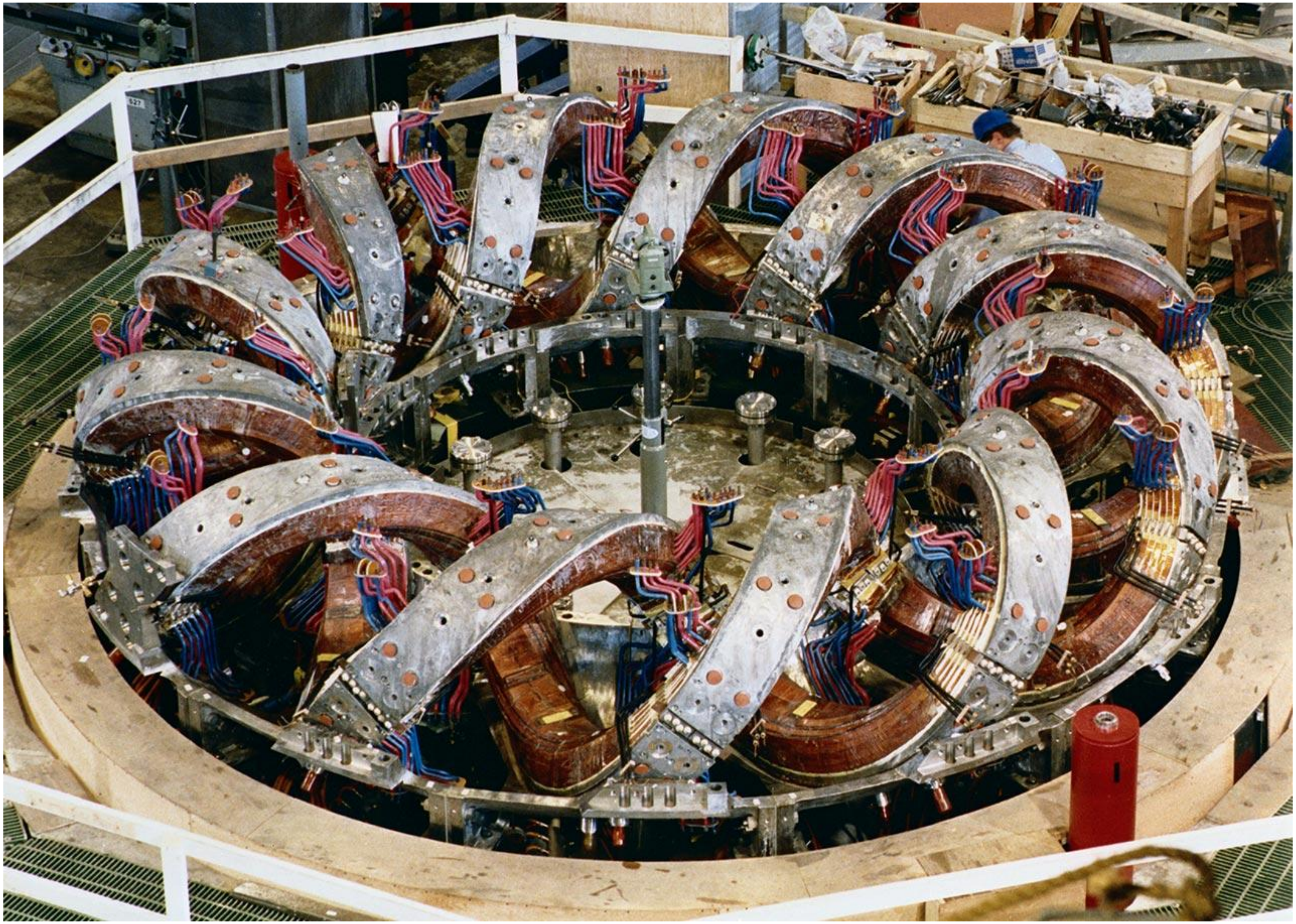


Auburn torsatron — winding of both helical and poloidal coils can be seen



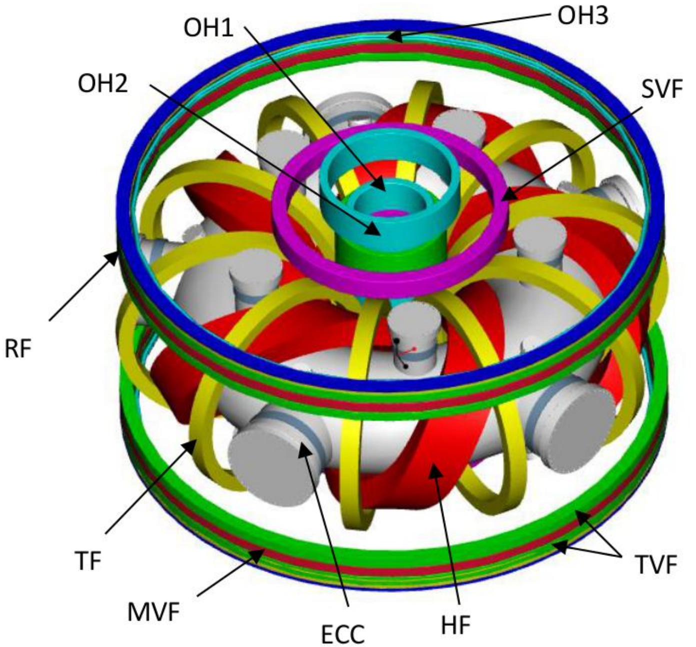
https://www.energyencyclopedia.com/en/glossary/torsatron

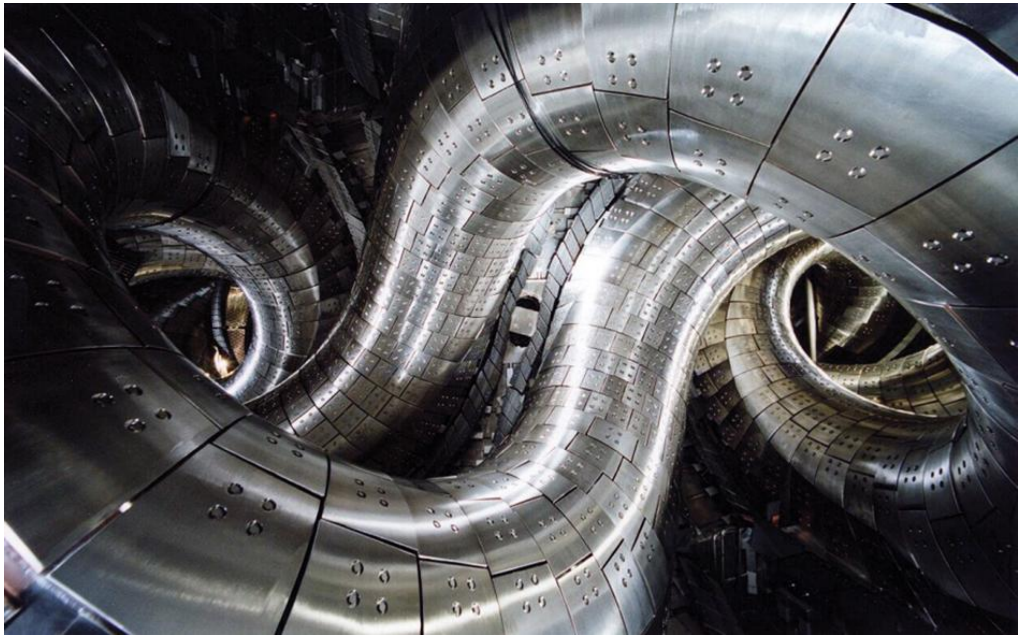
Construction of a pair of helical magnetic coils for the Advanced Toroidal Facility torsatron

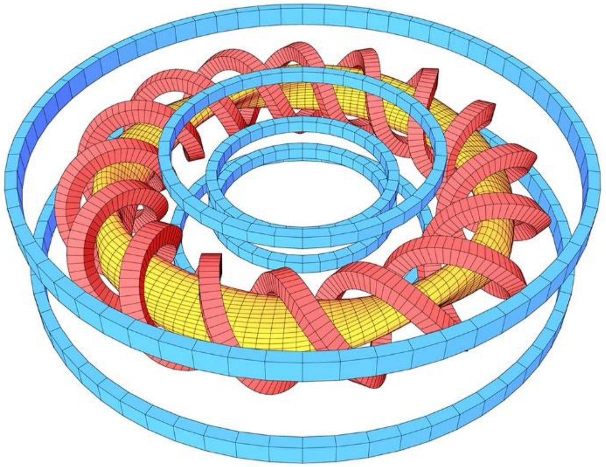


https://www.energyencyclopedia.com/en/glossary/torsatron

LHD stellarator in Japan (Heliotron)

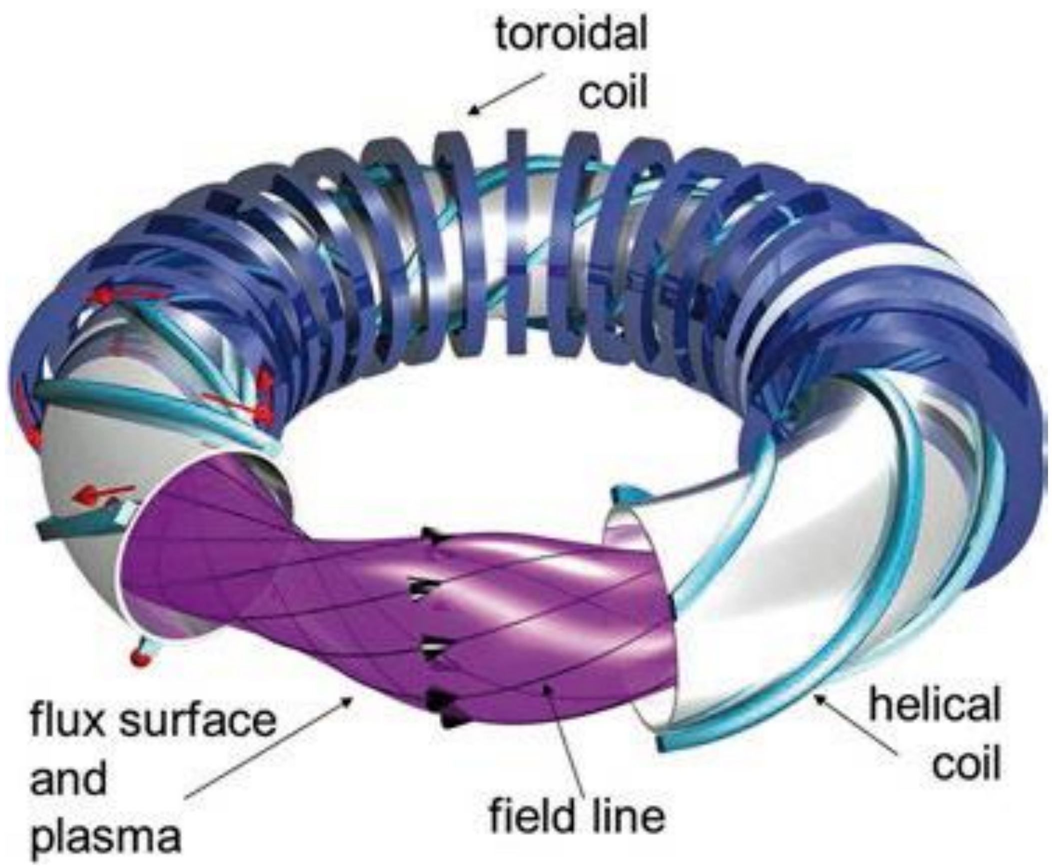






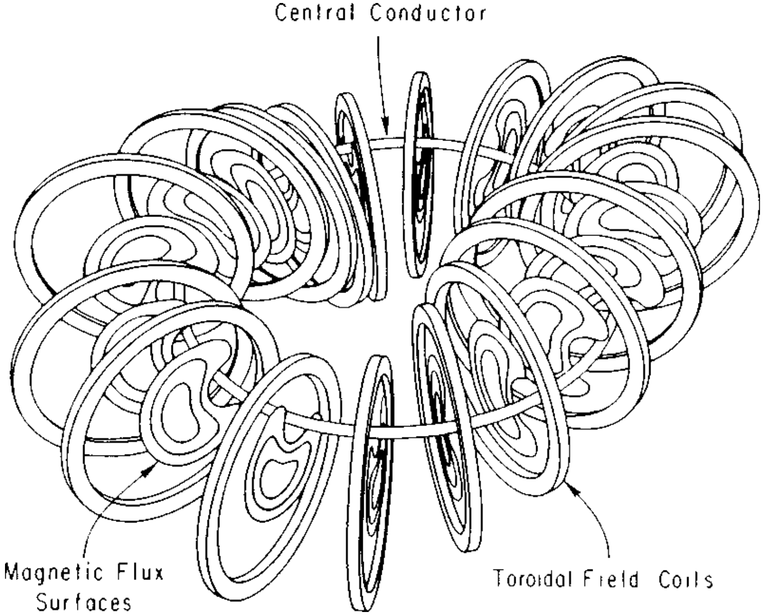
https://en.wikipedia.org/wiki/Compact_Toroidal_Hybrid
https://www.energyencyclopedia.com/en/glossary/heliotron
https://en.wikipedia.org/wiki/Large_Helical_Device

Twisted magnetic field lines can be provided by toroidal coils with helical coils

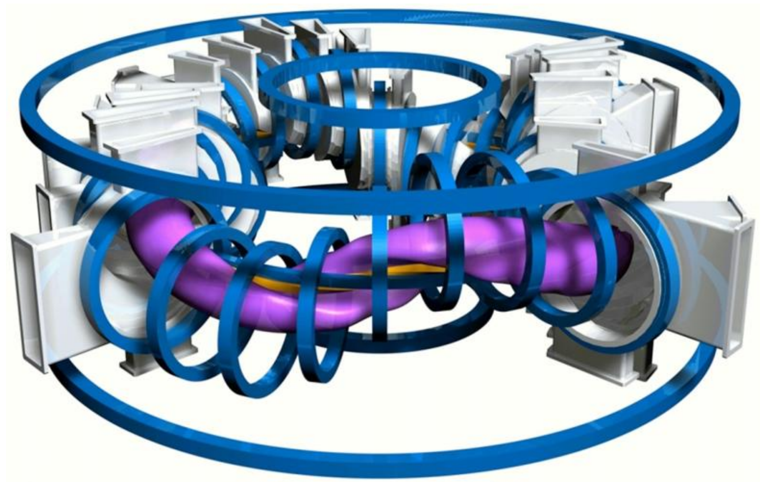


Wagner, F., Fusion energy. MRS Energy & Sustainability, 5, E8 (2018)

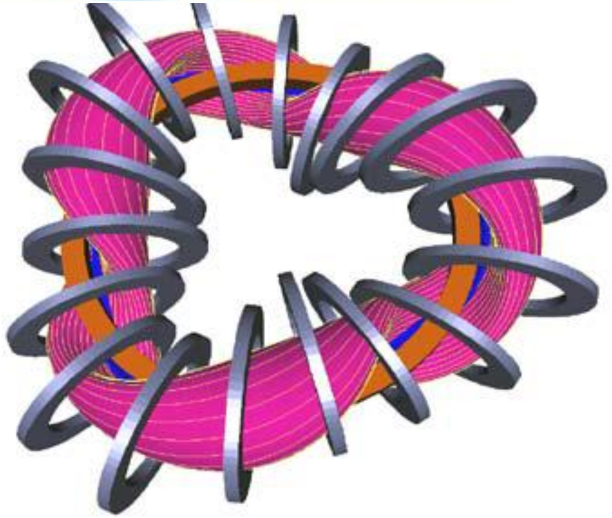
Heliac (Helical Axis stellarator)



- **TJ-II (Spain's National Fusion Laboratory):**

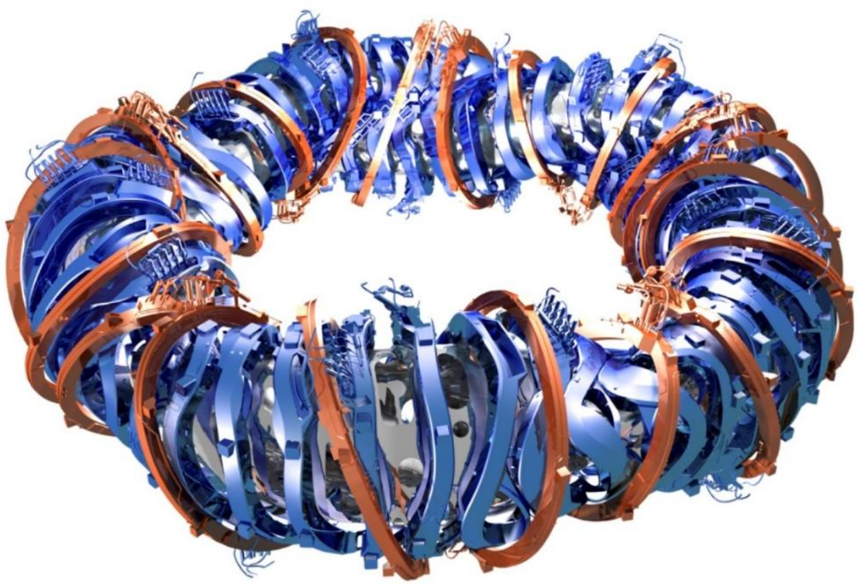


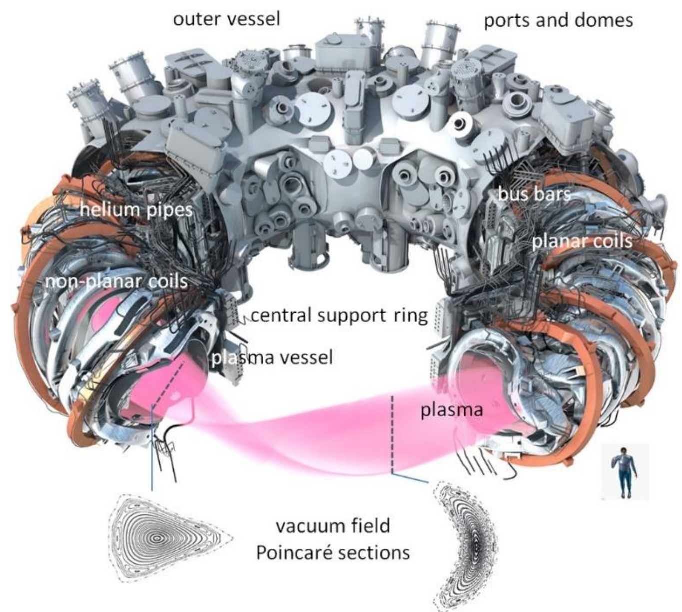
- **H-1 (Australian Plasma Fusion Research Facility):**



A. H. Boozer, Phys. Plasmas, 5, 1647 (1998)
https://wiki.fusion.ciemat.es/wiki/TJ-II
B. D. Blackwell, et. al, 23rd IAEA Fusion Energy Conference, 2010

Wendelstein 7-X is a (Helias) stellarator built by Max Planck Institute for Plasma Physics (IPP)





- **Wemdelstein 7-x is now installing new diverters.**

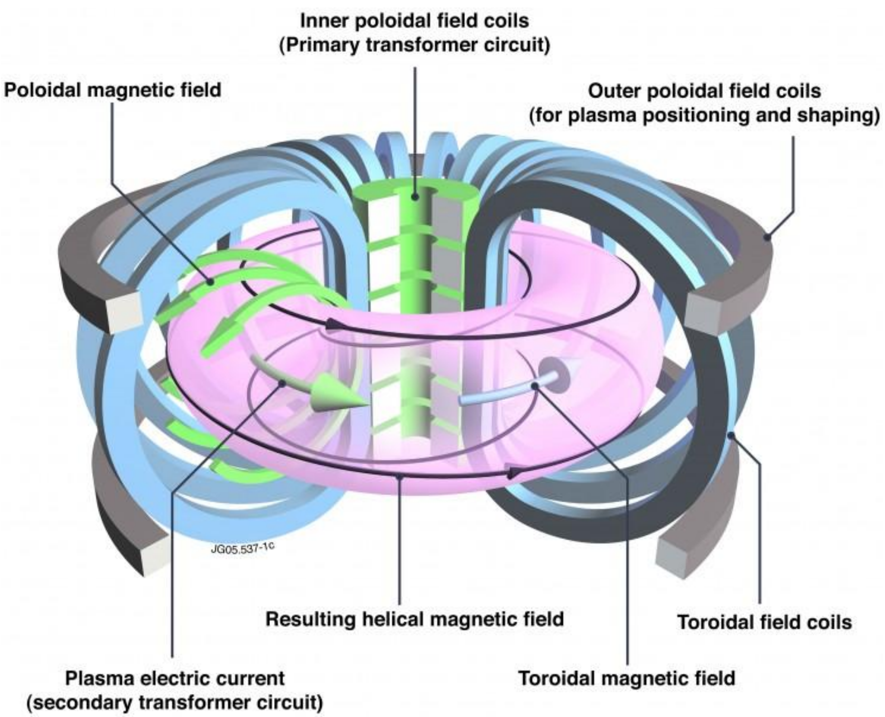
Advantages of Stellarator

- No need to drive plasma current. It is intrinsically steady state.
- With zero net current, one potentially dangerous class of MHD instabilities, the current-driven kink modes, is eliminated.
- Magnetic configuration is set by external coils, not by currents in the plasma. Stellarators do not suffer violent disruptions.
- Potential for greater range of designs and optimization of fusion performance.

Disadvantages of Stellarator

- **Complicated coil configurations. It's difficult to design. The precision requirement is high. It is expensive to build coils for stellarators.**
- **Achieving good particle confinement in stellarators is more difficult than that in tokamaks.**
- **Divertors and heat load geometry in stellarators is more complicated than those in tokamaks.**

2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p$$

$$\Downarrow$$

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p$$

$$\Downarrow$$

$$\vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j} \Rightarrow \nabla \cdot \vec{j} = 0$$

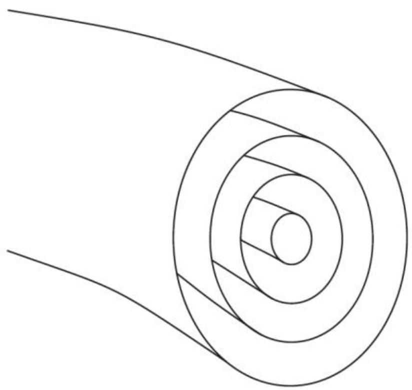
$$\nabla \cdot \vec{B} = 0$$

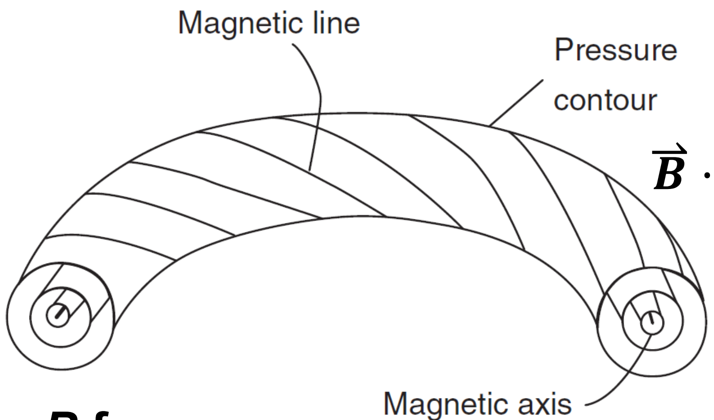
- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

https://www.euro-fusion.org/2011/09/tokamak-principle-2/

Magnetic lines lying on pressure contour

- **Contours of constant pressure**
- **Magnetic lines lying on pressure contour**

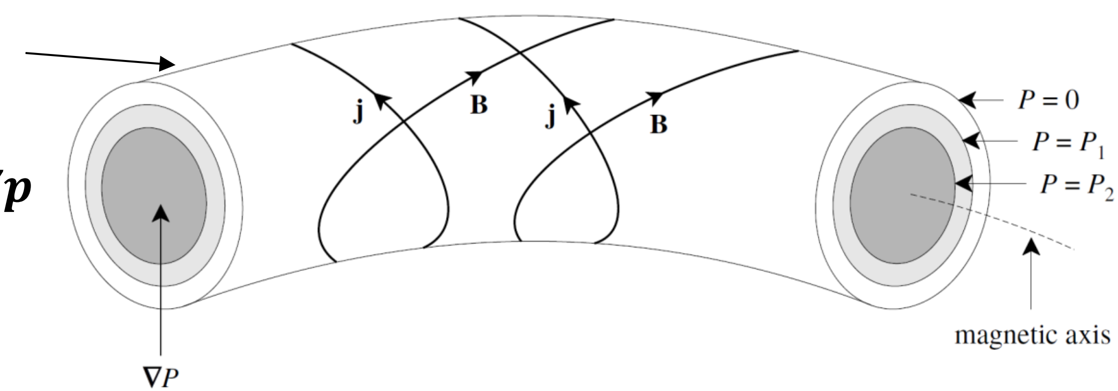




$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Derivation of Grad-Shafranov equation

$$\vec{j} \times \vec{B} = \nabla p \qquad \nabla \times \vec{B} = \mu_o \vec{j}$$

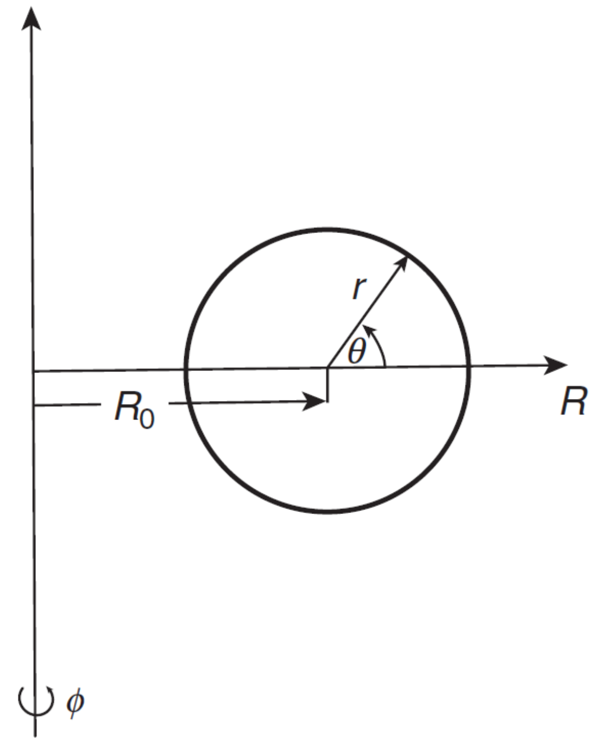
$$\nabla \cdot \vec{j} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = (B_R, B_\phi, B_z) \qquad \text{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\cancel{\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$



- Represent the magnetic field using a vector potential *A*:

$$\vec{B} = \nabla \times \vec{A} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\cancel{\frac{\partial A_R}{\partial \phi}}\right)$$

$$= \hat{R}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\right)$$

$$\equiv \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z$$

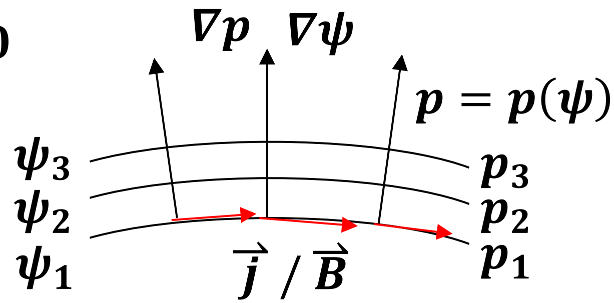
$$B_R = -\frac{\partial A_\phi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

Pressure can be written as a function of flux

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R)+\frac{\partial B_z}{\partial z}=0$$

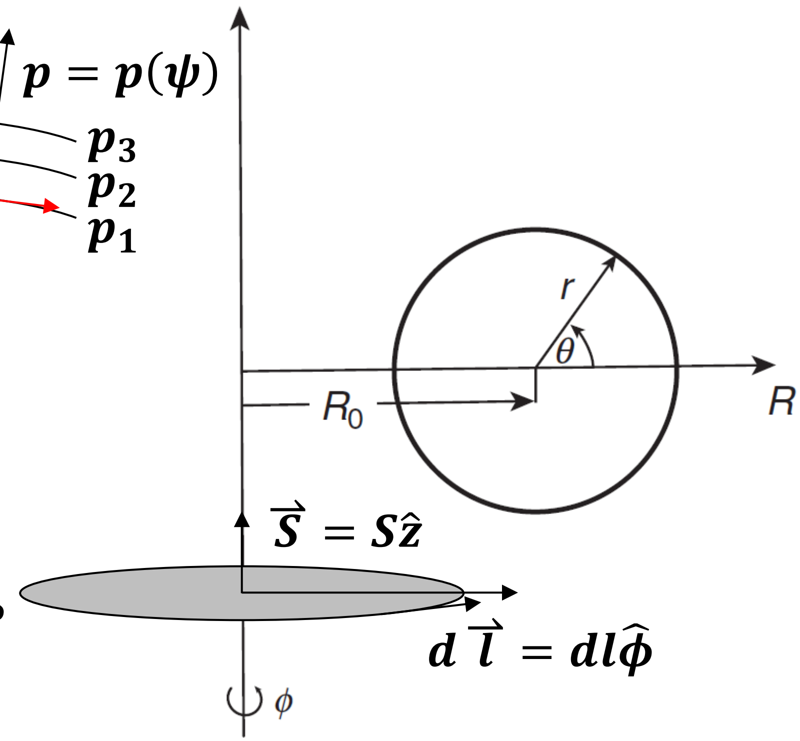
$$B_R=-\frac{\partial A_\phi}{\partial z}$$

$$B_z=\frac{1}{R}\frac{\partial}{\partial R}\left(RA_\phi\right)$$



$$\psi\equiv\frac{1}{2\pi}\int\vec{B}\cdot d\vec{S}=\frac{1}{2\pi}\int\left(\nabla\times\vec{A}\right)\cdot d\vec{S}$$

$$=\frac{1}{2\pi}\int\vec{A}\,2\pi R\cdot d\vec{l}=\int\vec{A}\cdot\hat{\phi}R\,dl=RA_\phi$$



$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial z}\qquad B_z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}\cdot\nabla\psi=B_R\frac{\partial\psi}{\partial R}+B_\phi\frac{1}{R}\cancel{\frac{\partial\psi}{\partial\phi}}+B_z\frac{\partial\psi}{\partial z}=B_R\frac{\partial\psi}{\partial R}+B_z\frac{\partial\psi}{\partial z}$$

$$=\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)\frac{\partial\psi}{\partial R}+\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\frac{\partial\psi}{\partial z}=0$$

$$\vec{B}\cdot\nabla\psi=0$$

$$\vec{B}\cdot\nabla p=0$$

for $\nabla p\neq 0$:

$$p=p(\psi)$$

Derivation of Grad-Shafranov equation

- **Let's see the $\hat{\phi}$ component of the force-balance equation:**

$$\left(\vec{j} \times \vec{B} = \nabla p\right)_{\phi} \qquad j_z B_R - j_R B_z = \frac{1}{R}\frac{\partial p}{\partial \phi} \equiv 0$$

- **Ampére's law:**

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \times \vec{B} = \hat{R}\left(\frac{1}{R}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) + \hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right) - \frac{1}{R}\frac{\partial B_R}{\partial \phi}\right]$$

$$= \hat{R}\left(-\frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) + \hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)\right]$$

$$= \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z$$

$$j_R = -\frac{1}{\mu_o}\frac{\partial B_\phi}{\partial z} \qquad j_z = \frac{1}{\mu_o}\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)$$

$$\frac{B_R}{R}\frac{\partial}{\partial R}\left(RB_\phi\right) + B_z\frac{\partial B_\phi}{\partial z} = 0$$

Magnetic field can be decomposed into the poloidal component and the toroidal component

$$\frac{B_\mathrm{R}}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)+B_\mathrm{z}\frac{\partial B_\phi}{\partial z}=0 \;\Rightarrow\; B_\mathrm{R}\frac{\partial}{\partial R}\left(RB_\phi\right)+B_\mathrm{z}\frac{\partial}{\partial z}\left(RB_\phi\right)=0$$

$$F\equiv RB_\phi \;\Rightarrow\; B_\mathrm{R}\frac{\partial F}{\partial R}+B_\mathrm{z}\frac{\partial F}{\partial z}=0 \;\Rightarrow\; \vec{B}\cdot\nabla F=0$$

$$\left(\frac{\partial}{\partial\phi}=0\right)$$

$$\Downarrow$$

$$F=F(\psi)$$

$$\vec{B}\cdot\nabla\psi=0$$

$$\vec{B}\cdot\nabla p=0$$

$$p=p(\psi)$$

$$B_\mathrm{R}=-\frac{\partial A_\phi}{\partial z}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

$$B_\mathrm{z}=\frac{1}{R}\frac{\partial}{\partial R}\left(RA_\phi\right)=\frac{1}{R}\frac{\partial\psi}{\partial R} \qquad \left(\psi=RA_\phi\right)$$

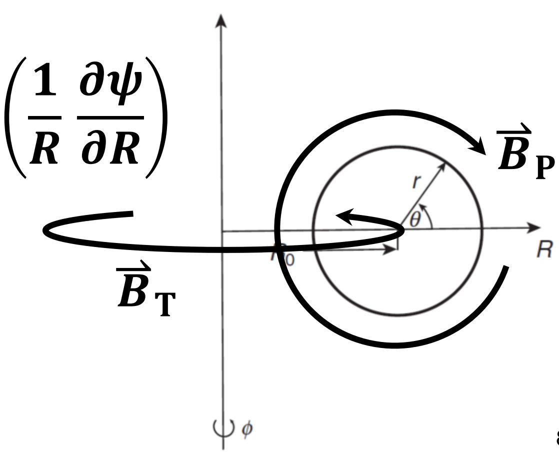
$$B_\phi=\frac{F(\psi)}{R}$$

$$\vec{B}=\hat{R}B_\mathrm{R}+\hat{\phi}B_\phi+\hat{z}B_\mathrm{z}=\hat{R}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)+\hat{\phi}\left(\frac{F(\psi)}{R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

$$\equiv\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}+\frac{F(\psi)}{R}\hat{\phi}$$

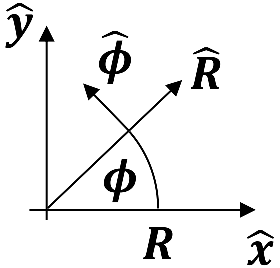
Poloidal component $\vec{B}_\mathrm{P}$

Toroidal component $\vec{B}_\mathrm{T}$



Alternative way to express magnetic field

$$\vec{B} = \underbrace{\left(\frac{\nabla\psi}{R}\right) \times \hat{\phi}}_{\vec{B}_{\mathrm{P}}} + \underbrace{\frac{F(\psi)}{R}\hat{\phi}}_{\vec{B}_{\mathrm{T}}} = \underbrace{\nabla\psi \times \nabla\hat{\phi}}_{\text{Poloidal component}} + \underbrace{F(\psi)\nabla\hat{\phi}}_{\text{Toroidal component}}$$



$$\nabla\hat{\phi} = \frac{1}{R}\hat{\phi}$$

$$x = R\cos\phi \qquad y = R\sin\phi \qquad z = z \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad R^2 = x^2 + y^2$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial z} = 0$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\cos\phi = \frac{x}{R} \qquad \sin\phi = \frac{y}{R}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z} = -\frac{y}{R^2}\hat{x} + \frac{x}{R^2}\hat{y} \Longleftrightarrow \frac{1}{R}\hat{\phi} = -\frac{y}{R^2}\hat{x} + \frac{x}{R^2}\hat{y}$$

Arbitrary integration constant associated with flux can be chosen such that flux equals to zero on the field axis

- **The poloidal flux of the area of a washer-shaped surface lying in the $z = 0$ plane from $R = R_a$ to an arbitrary ψ contour defined by $\psi = \psi(R_b, 0)$:**

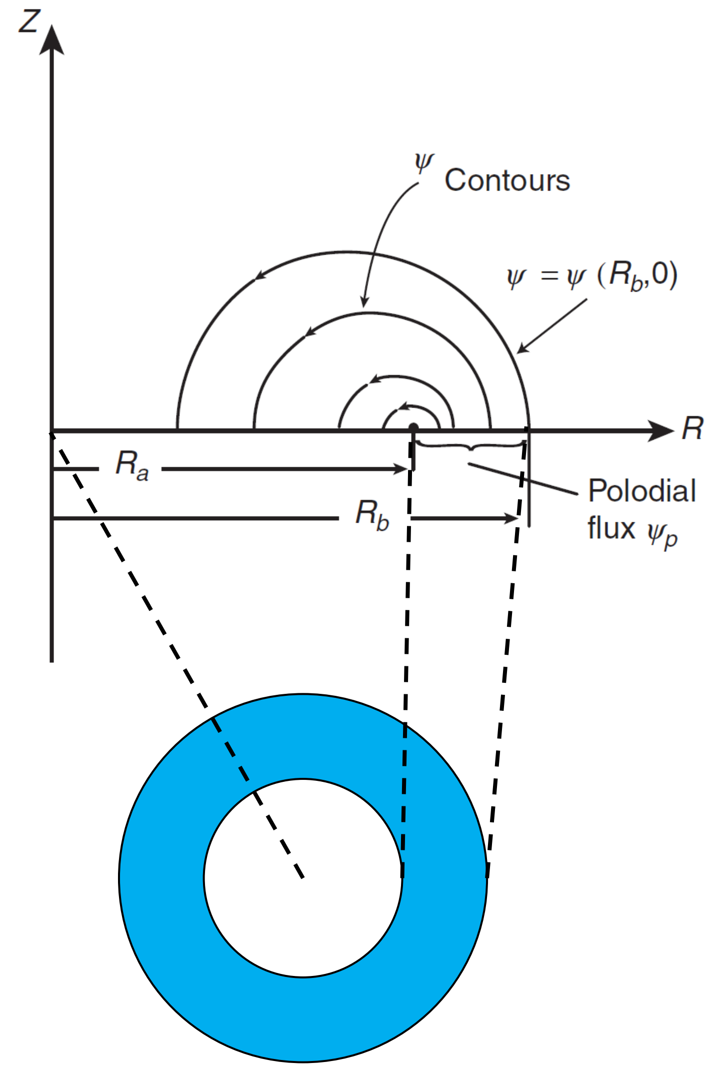
$$\psi_P \equiv \frac{1}{2\pi}\int \vec{B}_P \cdot d\,\vec{S}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{R_a}^{R_b} dR R B_z(R, 0)$$

$$= \psi(R_b, 0) - \psi(R_a, 0)$$

$$\equiv \psi(R_b, 0)$$

where $\psi(R_a, 0) \equiv 0$ is chosen.



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{R}$ component of the force-balance equation:**

$$\left(\vec{j} \times \vec{B} = \nabla p\right)_{\mathrm{R}} \qquad j_{\phi} B_{z} - j_{z} B_{\phi} = \frac{\partial p}{\partial R}$$

$$B_{\mathrm{R}} = -\frac{1}{R}\frac{\partial \psi}{\partial z}$$

$$B_{z} = \frac{1}{R}\frac{\partial \psi}{\partial R}$$

$$B_{\phi} = \frac{F(\psi)}{R}$$

- **Ampére's law:**

$$\nabla \times \vec{B} = \mu_{o}\, \vec{j}$$

$$\nabla \times \vec{B} = \hat{R}\left(-\frac{\partial B_{\phi}}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_{\mathrm{R}}}{\partial z} - \frac{\partial B_{z}}{\partial R}\right) + \hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R B_{\phi}\right)\right] = \hat{R}\mu_{o} j_{\mathrm{R}} + \hat{\phi}\mu_{o} j_{\phi} + \hat{z}\mu_{o} j_{z}$$

$$\mu_{o} j_{\phi} = \frac{\partial B_{\mathrm{R}}}{\partial z} - \frac{\partial B_{z}}{\partial R} = \frac{\partial}{\partial z}\left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) = -\frac{1}{R}\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{R}\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial \psi}{\partial R}$$

$$\equiv -\frac{1}{R}\Delta^{*}\psi \qquad \text{where } \Delta^{*}\psi \equiv \frac{\partial^2 \psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) = R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2}\right)$$

$$\mu_{o} j_{z} = \frac{1}{R}\frac{\partial}{\partial R}\left(R B_{\phi}\right) = \frac{1}{R}\frac{\partial F}{\partial R} = \frac{1}{R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

Derivation of Grad-Shafranov equation

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial R}$$

$$B_\phi = \frac{F}{R} \qquad j_\phi = -\frac{1}{\mu_o R}\Delta^*\psi \qquad \frac{\partial p}{\partial R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$B_z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad j_z = \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\frac{1}{\mu_o R}\Delta^*\psi\frac{1}{R}\frac{\partial \psi}{\partial R} - \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}\frac{F}{R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\Delta^*\psi\frac{1}{\mu_o}\frac{1}{R^2} - \frac{1}{\mu_o}\frac{F}{R^2}\frac{dF}{d\psi} = \frac{dp}{d\psi}$$

Grad – Shafranov equation: $\Delta^*\psi = -\mu_o R^2\frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$

$$F \equiv RB_\phi$$

where $\Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$ $\qquad \vec{B} = \left(\frac{\nabla\psi}{R}\right)\times\hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$

Derivation of Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \quad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \qquad \vec{B} = \underbrace{\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}}_{B_P} + \underbrace{\frac{F(\psi)}{R}\hat{\phi}}_{B_T}$$

$$\mu_o j_\phi = -\frac{1}{R}\Delta^*\psi \qquad \mu_o j_z = \frac{1}{R}\frac{\partial F}{\partial R}$$

$$F \equiv RB_\phi$$

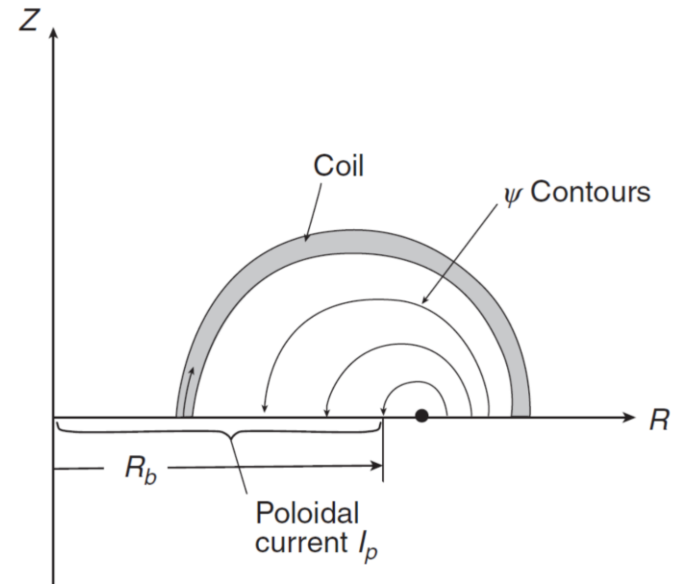
$$\mu_o j_R = -\frac{\partial B_\phi}{\partial z} = -\frac{1}{R}\frac{\partial}{\partial z}(RB_\phi) = -\frac{1}{R}\frac{\partial F}{\partial z}$$

$$\mu_o \vec{j} = \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z = \hat{R}\left(-\frac{1}{R}\frac{\partial F}{\partial z}\right) + \hat{\phi}\left(-\frac{1}{R}\Delta^*\psi\right) + \hat{z}\left(\frac{1}{R}\frac{\partial F}{\partial R}\right)$$

$$\equiv \underbrace{\left(\frac{\nabla F}{R}\right)\times\hat{\phi}}_{j_P} + \underbrace{\left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi}}_{j_T}$$

$$I_P = \int \vec{j}_P \cdot d\vec{S} = -\int_0^{2\pi} d\phi \int_0^{R_b} dR R j_z(R,0)$$

$$= -2\pi\int_0^{R_b} dR R \frac{1}{R}\frac{\partial F(R,0)}{\partial R} = -2\pi F(\psi)$$



Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2 \nabla \cdot \left(\frac{\nabla\psi}{R^2}\right)$$

- **The usual strategy to solve the Grad-Shafranov equation:**
  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Example of the analytical solution of the Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$

- For $\mu_o \frac{dp}{d\psi} = -C_2$ $\frac{1}{2}\frac{dF^2}{d\psi} = C_1$

$$\psi(R,z) = -\frac{C_1}{2}z^2 + \frac{C_2}{8}R^4 + C_3 + C_4R^2 + C_5\left(R^4 - 4R^2z^2\right)$$

$$B_R(R,z) = -\frac{1}{R}\left(-C_1 z - 8C_5R^2 z\right) \quad B_z(R,z) = \frac{1}{R}\left(\frac{C_2}{2}R^3 + 2C_4R + C_5\left(4R^3 - 8Rz^2\right)\right)$$

$C_1 = 1$

$C_2 = -8$

$C_3 = -20$

$C_4 = 20$

$C_5 = 0.2$