Introduction to Nuclear Fusion as An Energy Source

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Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 5

2025 spring semester

Tuesday 9:00-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=mf1a33a5dab5eb71de9da4380ae888592

2025/3/18 updated

Course Outline

- Magnetic confinement fusion (MCF)
  - Gyro motion, MHD
  - 1D equilibrium (z pinch, theta pinch)
  - Drift: ExB drift, grad B drift, and curvature B drift
  - Tokamak, Stellarator (toroidal field, poloidal field)
  - Magnetic flux surface
  - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
  - Stability (Kink instability, sausage instability, Safety factor Q)
  - Central-solenoid (CS) start-up (discharge) and current drive
  - CS-free current drive: electron cyclotron current drive, bootstrap current.
  - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

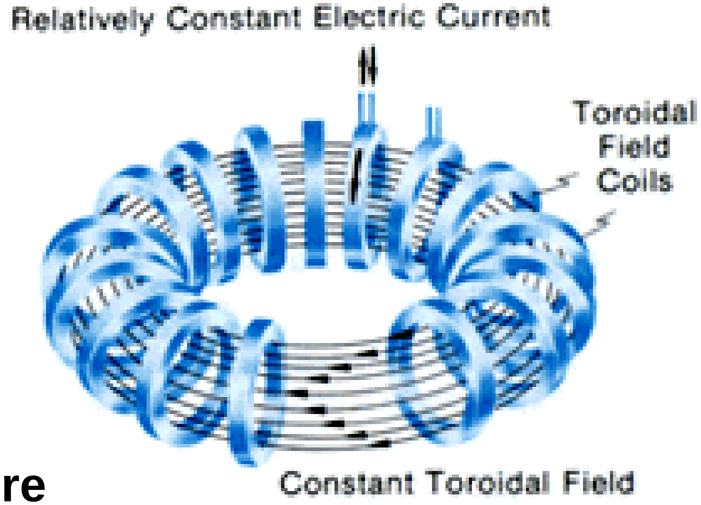
Quick summary of different drifts

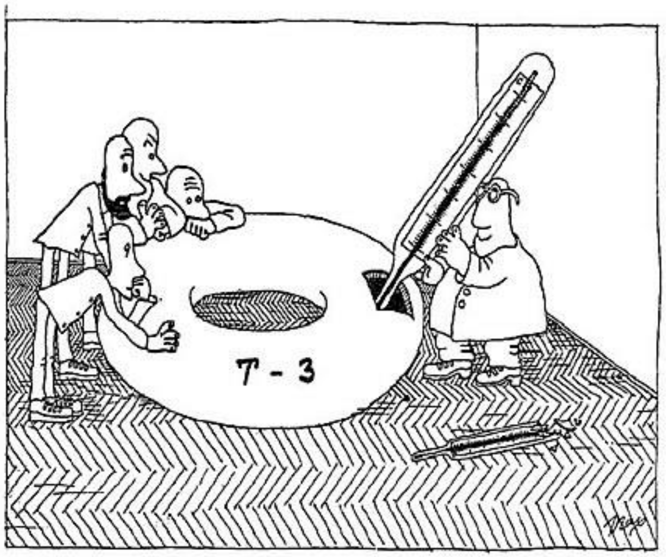
- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m {v_{\perp}}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_R = \frac{m {v_{||}}^2}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_c}^2 B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}} B^2} \left({v_{||}}^2 + \frac{1}{2} {v_{\perp}}^2 \right) = \frac{m}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2 B^2} \left({v_{||}}^2 + \frac{1}{2} {v_{\perp}}^2 \right)$$

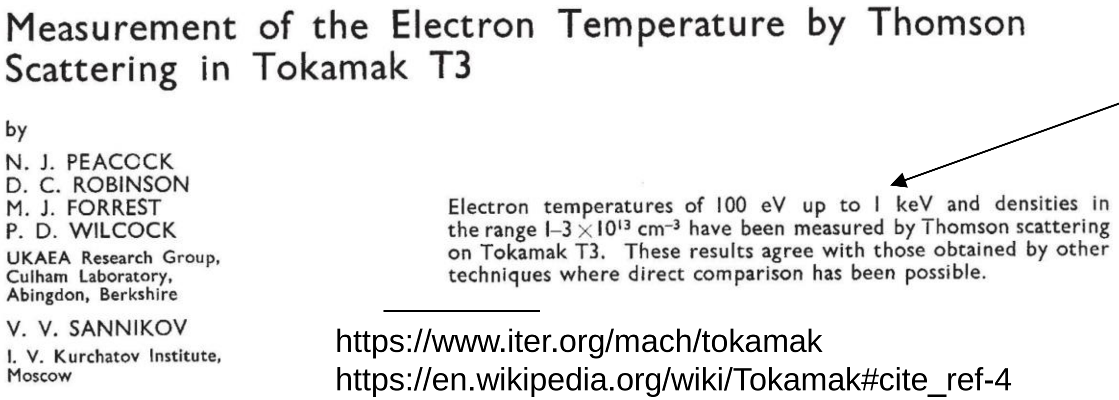
Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field

- **Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)**





Nature



Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

N. J. PEACOCK
D. C. ROBINSON
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Electron temperatures of 100 eV up to 1 keV and densities in the range 1–3 × 10^{13} cm^{-3} have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

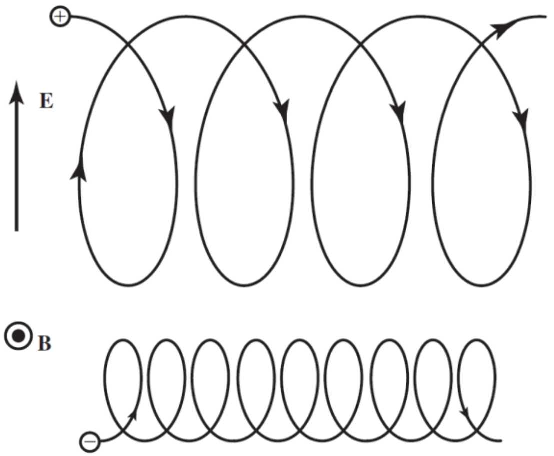
T_e **= 100 ~ 1 keV**

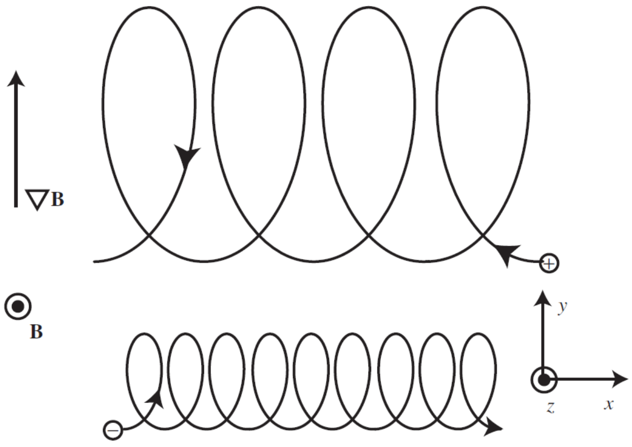
n_e **= 1-3 x 10^{13} cm^{-3}**

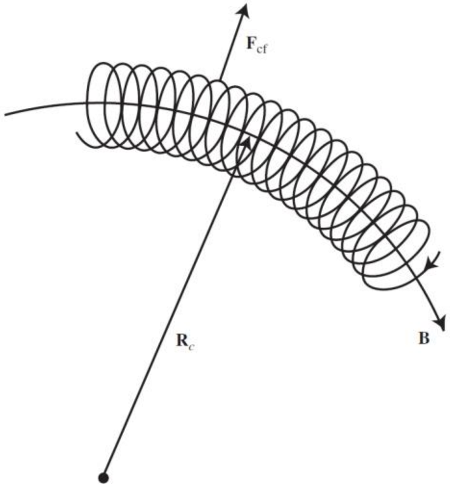
https://www.iter.org/mach/tokamak
https://en.wikipedia.org/wiki/Tokamak#cite_ref-4
Drawing from the talk “Evolution of the Tokamak” given in 1988 by B.B. Kadomtsev at Culham.
N. J. Peacock,et al., Nature **224**, 488 (1969)

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m v_{\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_R = \frac{m v_{||}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$ **Depended on charge**

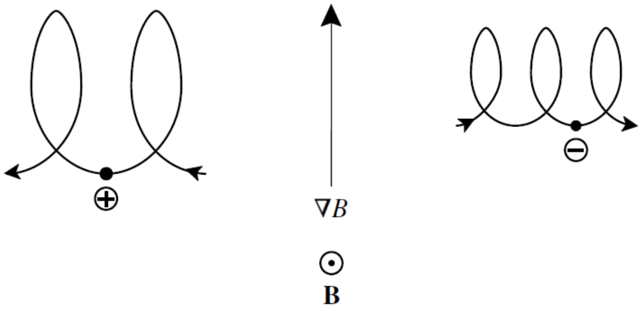




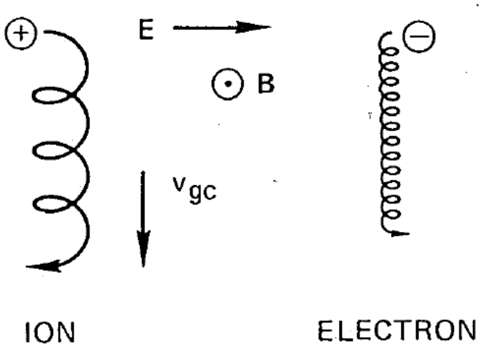


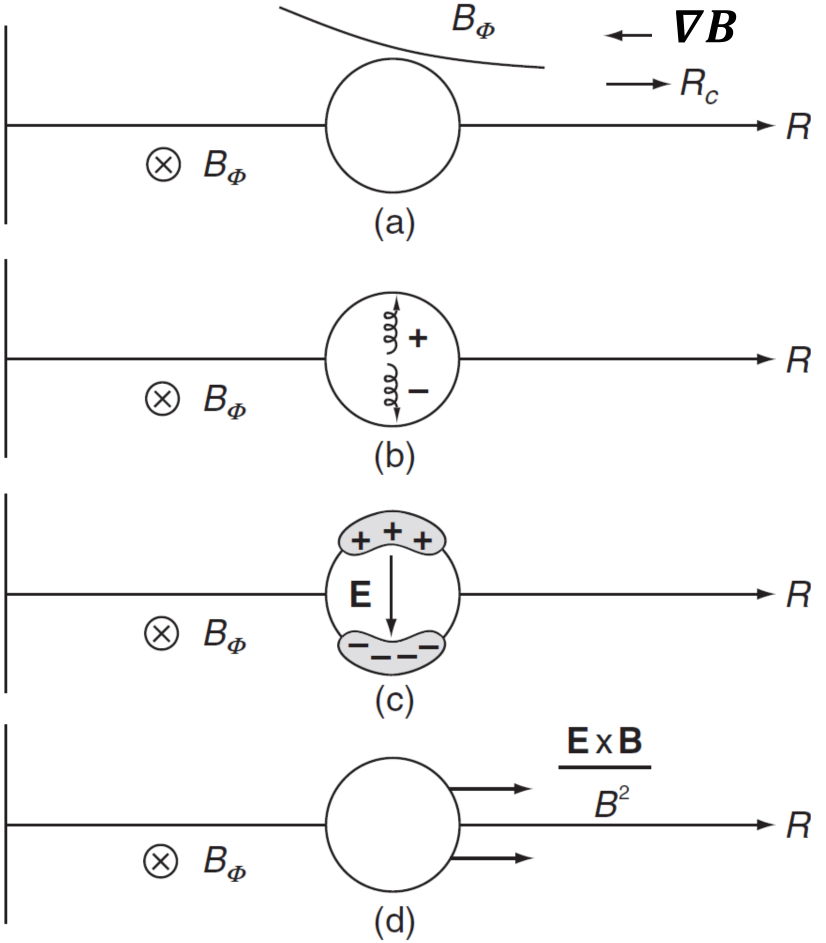
Charged particles drift across field lines

- **Grad-B drift**



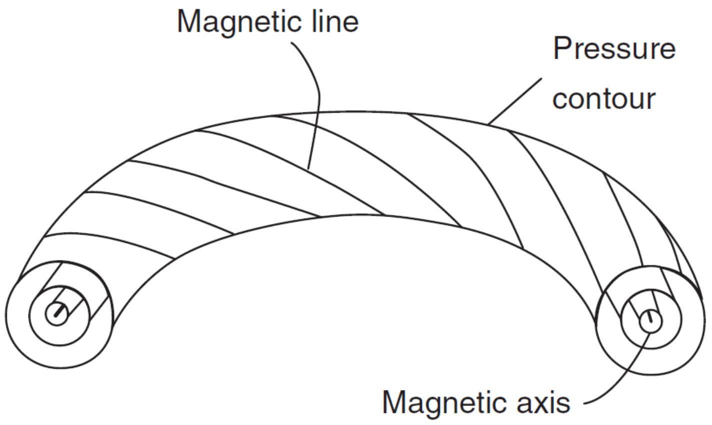
- **ExB drift**

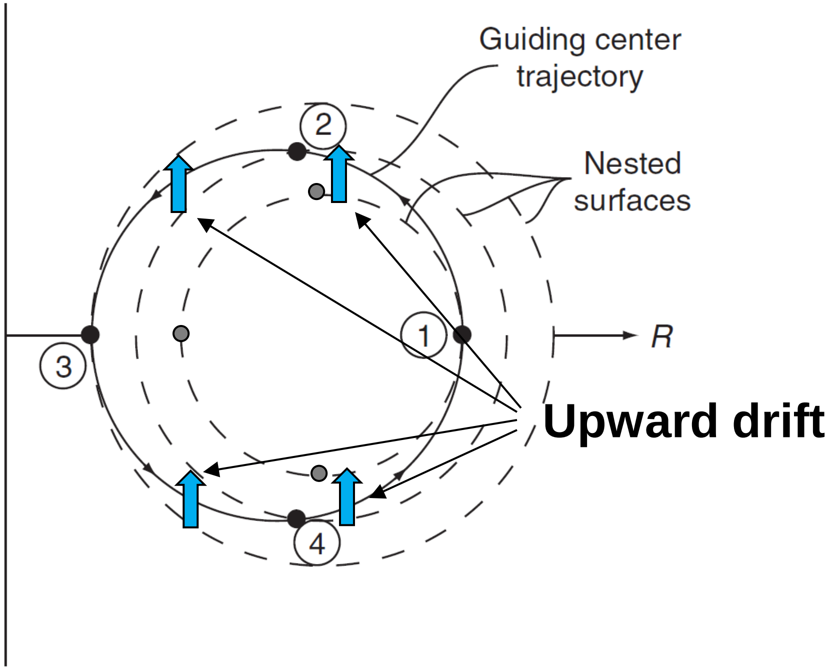




J. P. Freidberg, Ideal Magnetohydrodynamics

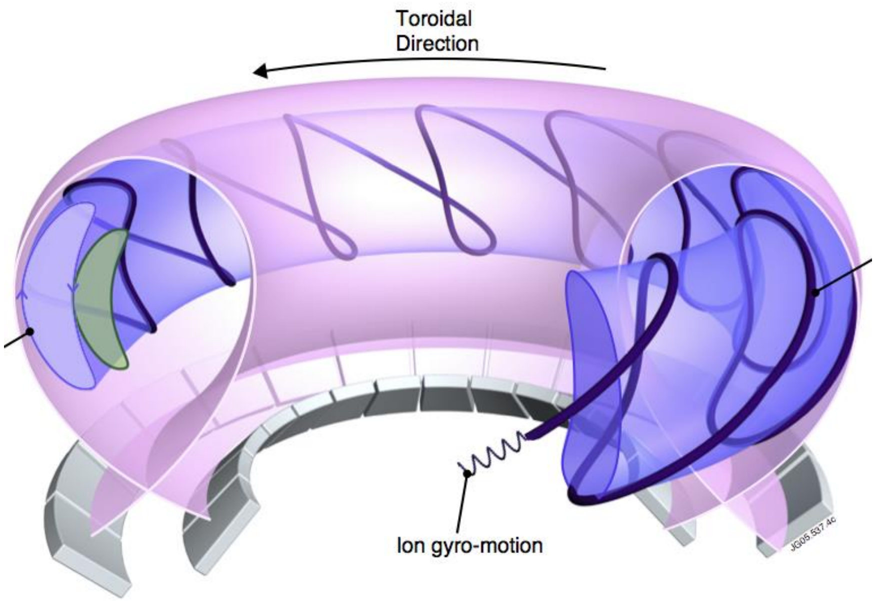
The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

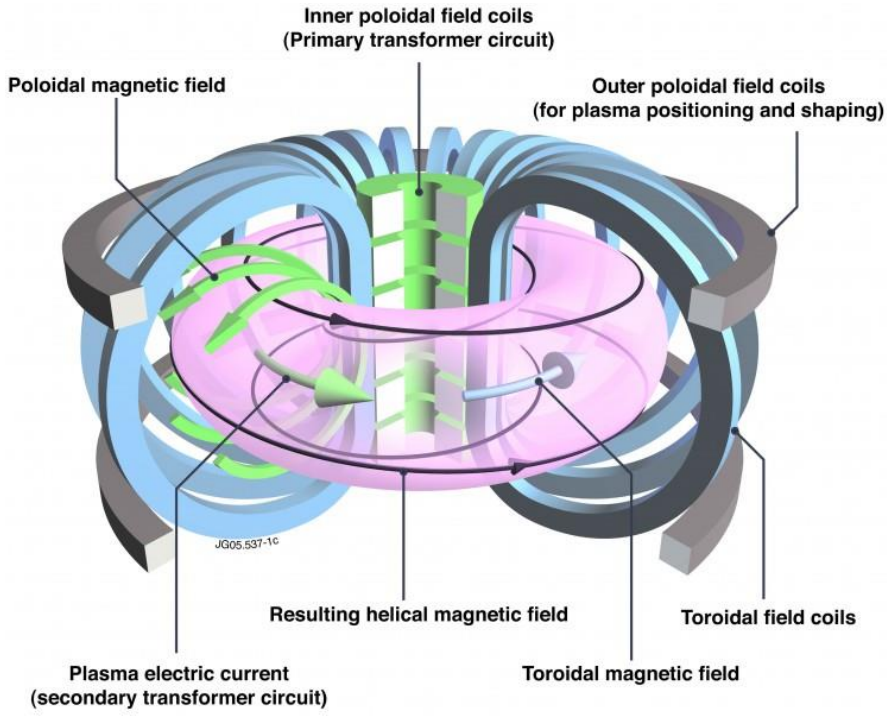




● **Points with no drift**

A poloidal magnetic field is required to reduce the drift across field lines





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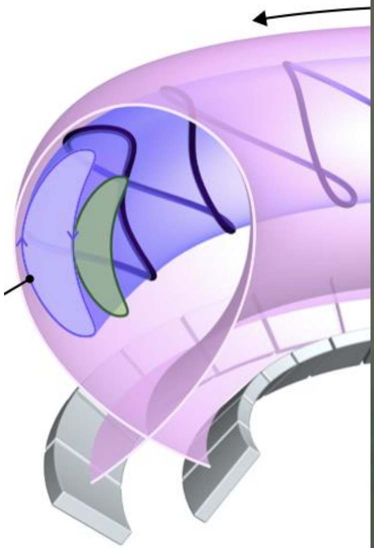
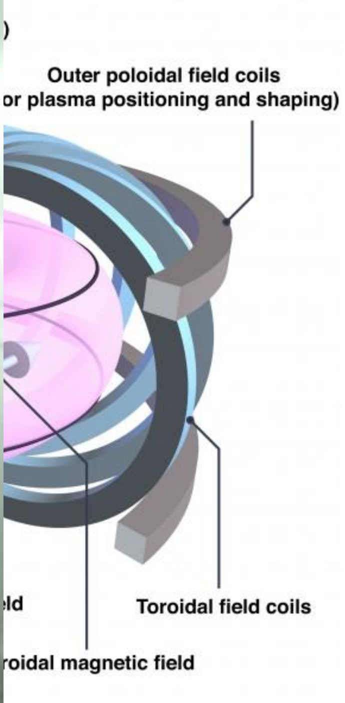
https://www.davidpace.com/keeping-fusion-plasmas-hot/
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

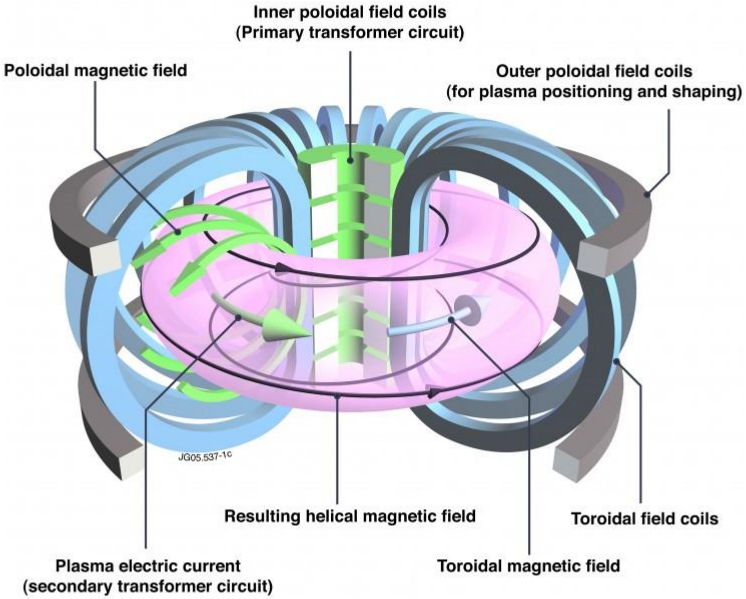
A poloidal magnetic field is required to reduce the drift across field lines



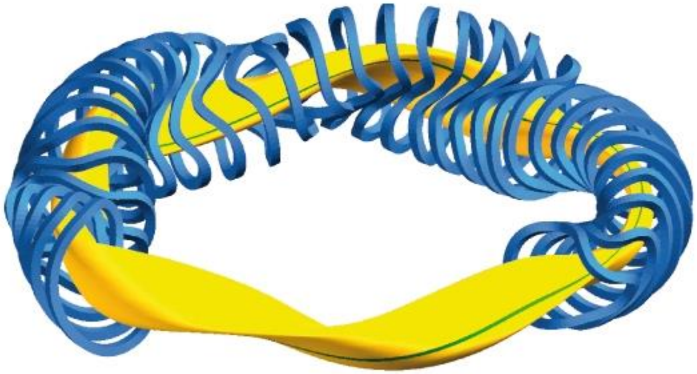


Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



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https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

When forces are balances, the system is in the equilibrium state, or called “Magnetohydrostatics”

- **Equilibrium state:**

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right]=\vec{j}\times\vec{B}-\nabla p\equiv 0$$

$$\vec{j}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_{\mathrm{o}}\,\vec{j}$$

$$\vec{j}\times\vec{B}=\frac{1}{\mu_{\mathrm{o}}}(\vec{\nabla}\times\vec{B})\times\vec{B}=\frac{1}{\mu_{\mathrm{o}}}\left[(\vec{B}\cdot\vec{\nabla})\,\vec{B}-\frac{1}{2}\,\vec{\nabla}\,B^{2}\right]=\nabla p$$

$$\nabla\left(P+\frac{B^{2}}{2\mu_{\mathrm{o}}}\right)=\frac{1}{\mu_{o}}(\vec{B}\cdot\vec{\nabla})\,\vec{B}$$

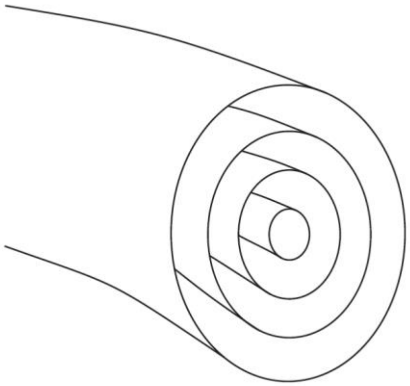
Magnetic pressure **Magnetic tension** ← **Forces caused by curvature of the field lines**

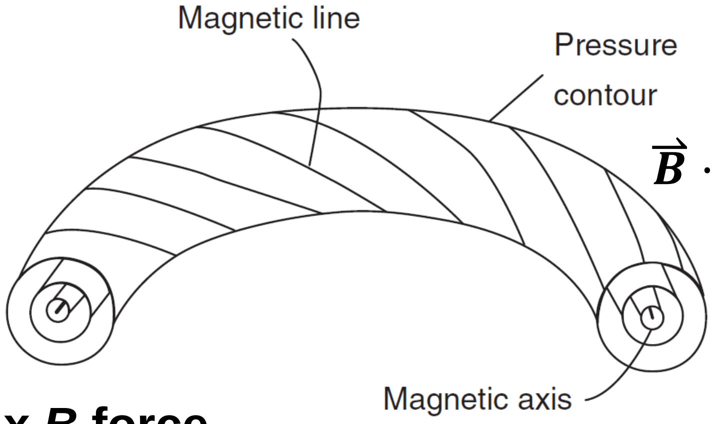
$$\vec{j}\perp\nabla p \qquad \vec{B}\perp\nabla p \quad\Rightarrow\quad \vec{j}\cdot\nabla p=0 \qquad \vec{B}\cdot\nabla p=0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

Magnetic lines lying on pressure contour

- **Contours of constant pressure**
- **Magnetic lines lying on pressure contour**

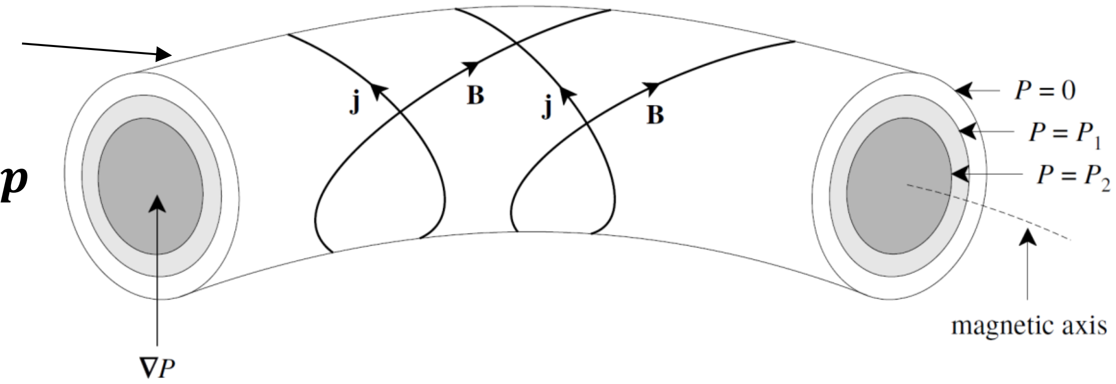




$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

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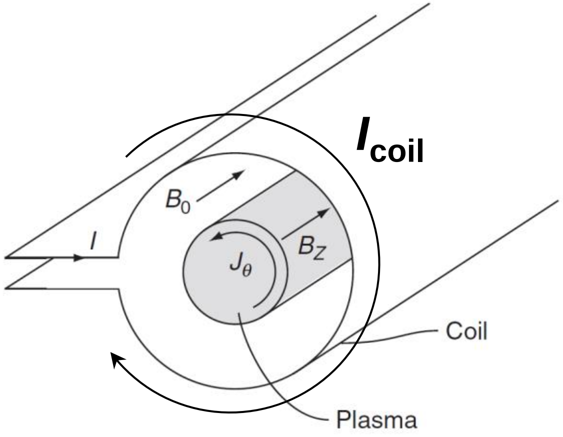
Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z \hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{\mathrm{r}}) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_{\mathrm{z}}}{\partial z} = 0$$

$$\frac{\partial B_{\mathrm{z}}}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_{\mathrm{r}} = \frac{1}{r}\frac{\partial B_{\mathrm{z}}}{\partial \theta} - \frac{\partial B_{\mathrm{z}}}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_{\theta} = \frac{\partial B_{\mathrm{r}}}{\partial z} - \frac{\partial B_{\mathrm{z}}}{\partial r} = -\frac{\partial B_{\mathrm{z}}}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_{\mathrm{z}} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_{\mathrm{r}}}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_{\mathrm{o}}}\frac{\partial B_{\mathrm{z}}}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_{\mathrm{o}}}\right) = \frac{1}{\mu_o}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} = 0$$

$$P + \frac{B_{\mathrm{z}}^2}{2\mu_{\mathrm{o}}} = \frac{B_{\mathrm{o}}^2}{2\mu_{\mathrm{o}}}$$

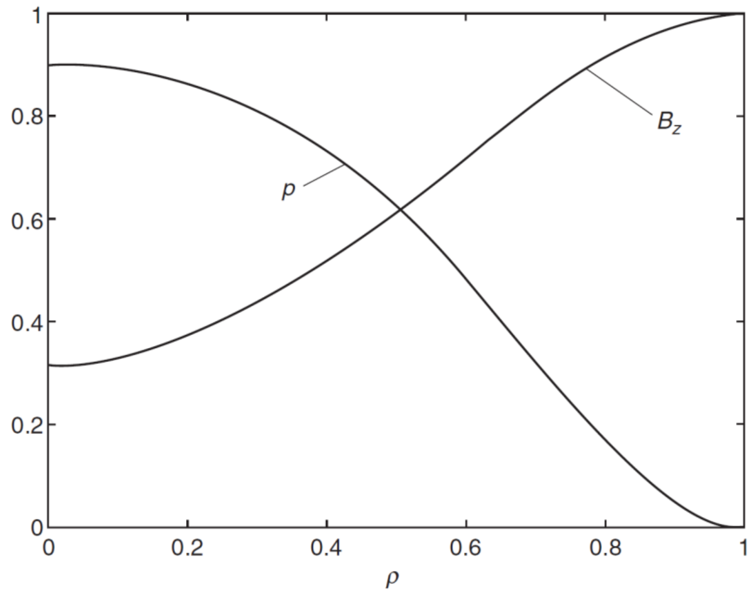
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_{\mathrm{z}} = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma

- **Example:**

$$\frac{2\mu_o p(r)}{B_o{}^2} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$



$$j_\theta B_z = \frac{dp}{dr} \quad ➡ \quad \frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{B_o{}^2} \qquad \rho = \frac{r}{a}$$

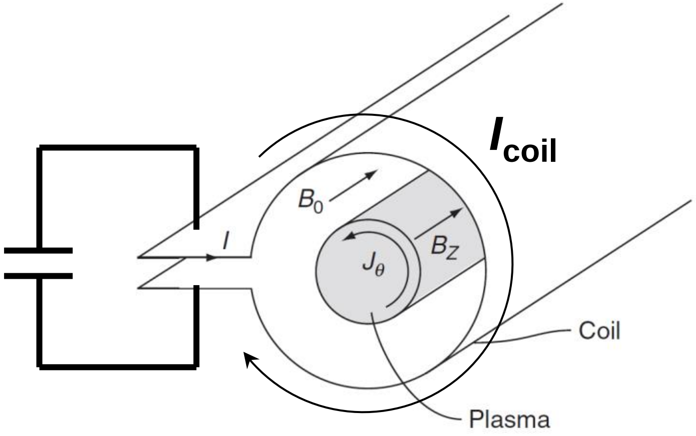
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{B_o{}^2} = \frac{4\mu_o}{a^2 B_o{}^2}\int_0^a prdr = 2\int_0^1\left(1-\frac{B_z{}^2}{B_o{}^2}\right)\rho d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_o}{2}\ , \beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad \Rightarrow \quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$$\boxed{0 < \beta < 1}$$

Theta pinches provide good radial confinement but NOT axially

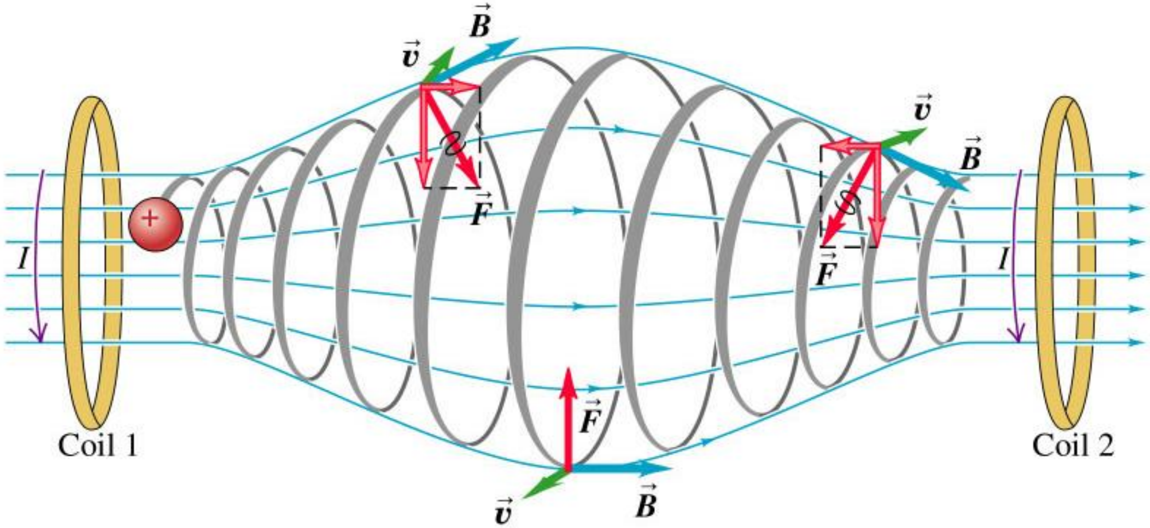


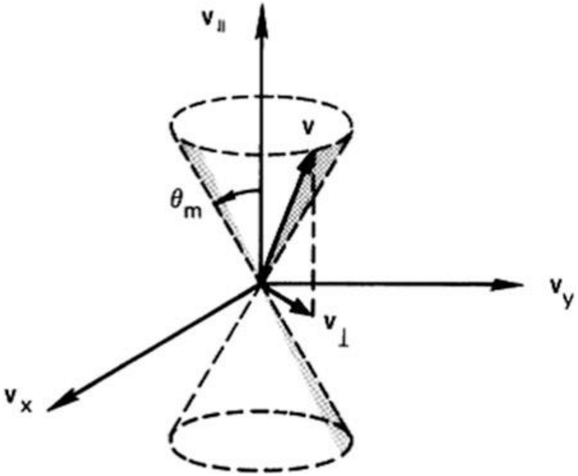
- The gas is initially preionized.
- The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.
- The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.
- This energy is ultimately converted to heat after repeated reflections off the converging piston.
- T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.
- The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{\mathrm{Ti}} \sim 10\mu s$ for L = 5 m.

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- **Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.**





$$v'_{\perp}{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_{\perp}{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv sin^2\theta$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_{\perp}{}^2$$

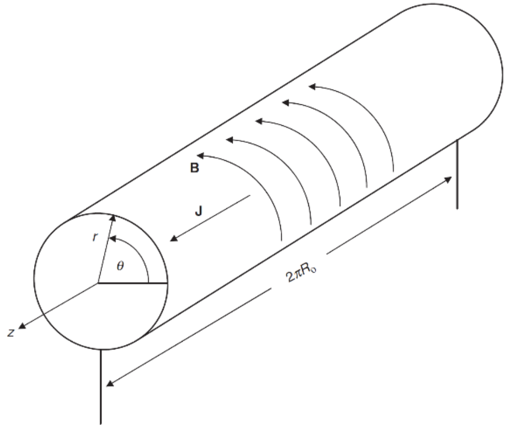
Invarient: $\mu \equiv \frac{1}{2}\frac{mv_{\perp}{}^2}{B}$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = sin^2\theta_m$$

- **Large $v_{||}$ may occur from collisions between particles.**
- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

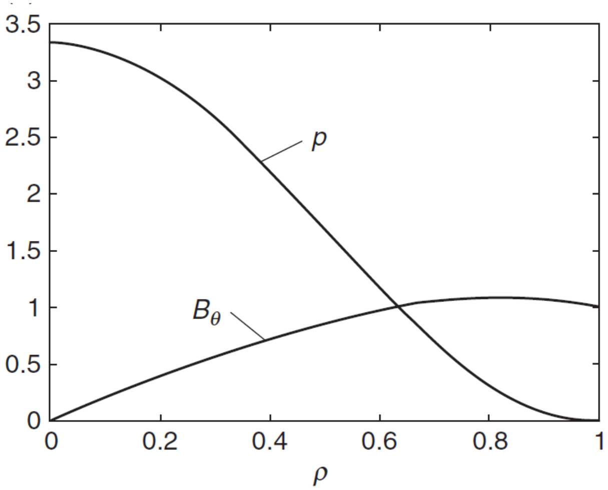
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

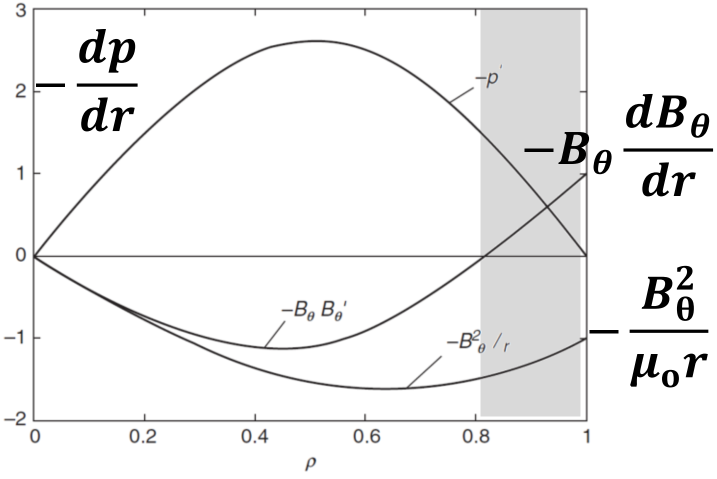
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β





$$\frac{d}{dr}\left(p+\frac{B_\theta^2}{2\mu_o}\right)+\frac{B_\theta^2}{\mu_o r}=0$$

$$\frac{2\mu_o p(r)}{{B_{\theta a}}^2}=\frac{2}{3}\left(5-2\rho^2\right)\left(1-\rho^2\right)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}}=2\rho\left(1-\frac{\rho^2}{2}\right)$$

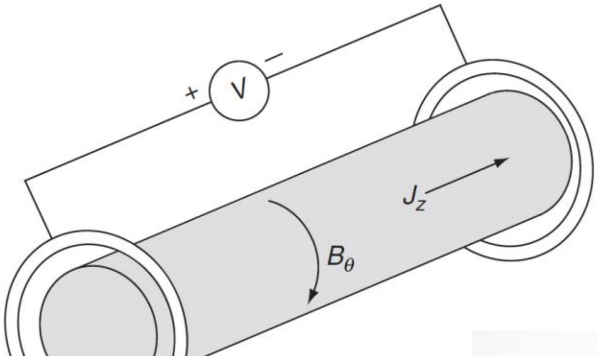
$$\frac{a\mu_o j_z(r)}{B_{\theta a}}=4\left(1-\rho^2\right)$$

$$B_{\theta a}\equiv B_\theta(a)=\frac{\mu_o I}{2\pi a}$$

$$\beta\equiv\beta_{\rm p}=\frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2}=\frac{4\mu_o}{a^2{B_{\theta a}}^2}\int_0^a prdr=1$$

Bennett pinch relation: $\beta=1$

Huge instabilities occur in a z pinch

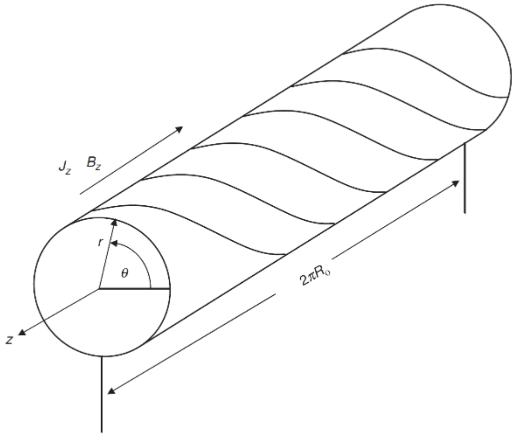


- A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.
- The gas is ionized by the high voltage and produces a z current flowing along the plasma.
- Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.

Main issue: unstable.

Facebook@鄭明典 (https://www.facebook.com/share/p/18N8x8kFmu/)
https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

$$\nabla\cdot\vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla\times\vec{B} = \mu_o\,\vec{j}$$

$$\left(\nabla\times\vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla\times\vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

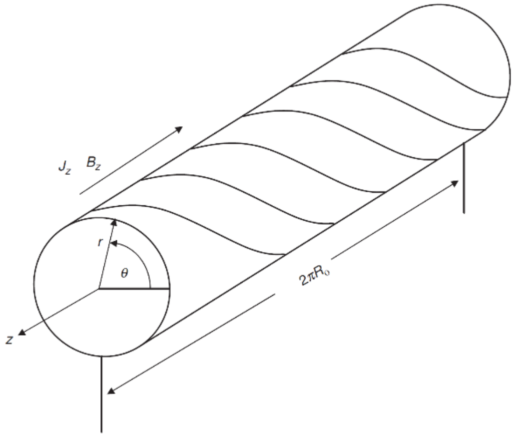
$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

$$\vec{j}\times\vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

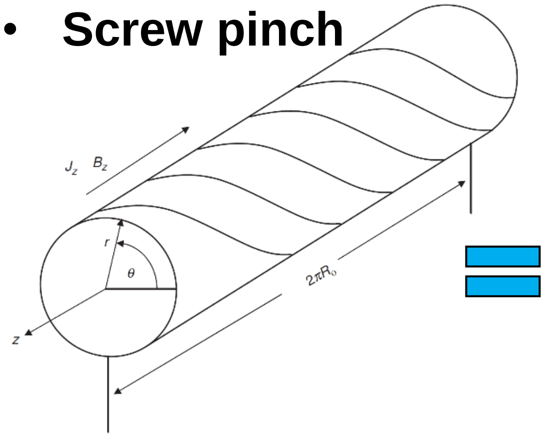
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

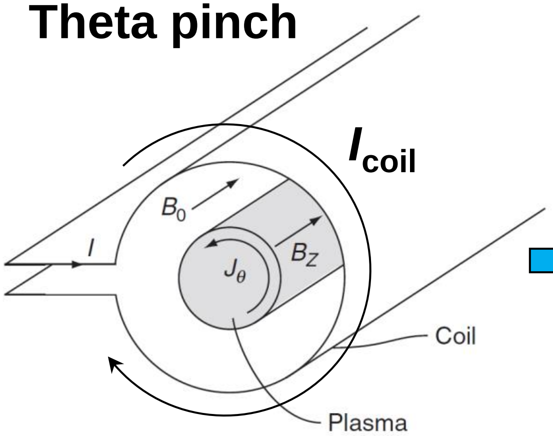
General screw pinch is flexible with varies range of *β*

- **Screw pinch**



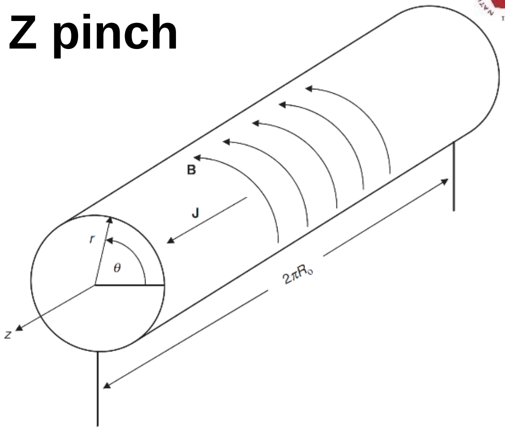
=

- **Theta pinch**



+

- **Z pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$\vec{B} = B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta}$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\vec{B} = B_\theta \hat{\theta}$$

$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

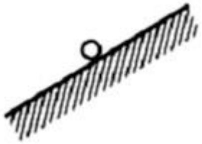
$$\int_0^a \pi r^2 dr \left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2 \rangle\right)$$

$$\beta_t = \frac{2\mu_o \langle p \rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o \langle p \rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_o \langle p \rangle}{B_o^2 + {B_{\theta a}}^2}$$

$$\boxed{0 \le \langle \beta \rangle \le 1}$$

An equilibrium state may not be stable

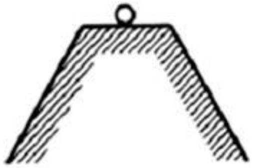


A

NO EQUILIBRIUM

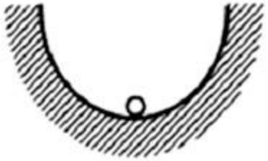
B

NEUTRALLY STABLE



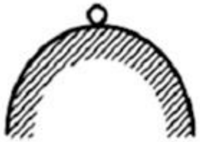
C

(METASTABLE)
EQUILIBRIUM



D

STABLE EQUILIBRIUM



E

UNSTABLE EQUILIBRIUM

F

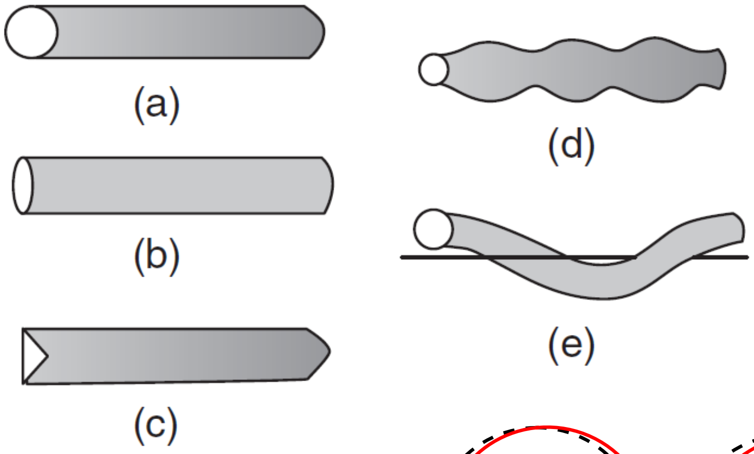
EQUILIBRIUM WITH LINEAR STABILITY AND NONLINEAR INSTABILITY

G

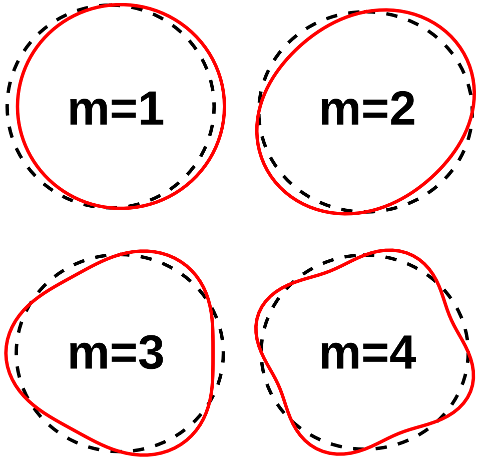
EQUILIBRIUM WITH LINEAR INSTABILITY AND NONLINEAR STABILITY

A cylindrical plasma column may not be stable

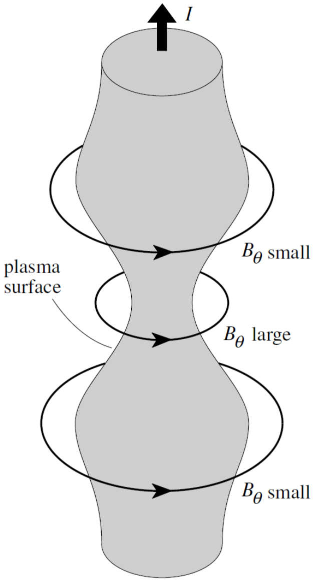
- **Instabilities of theta pinch**



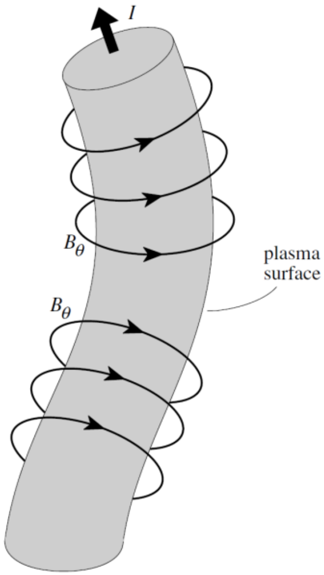
(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0



- **Instabilities of z pinch**



Sausage instability (m=0)

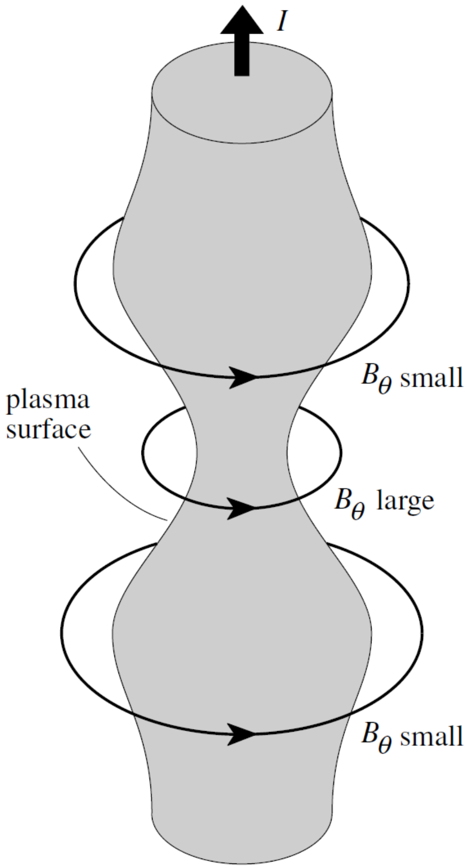


Kink instability (m=1)

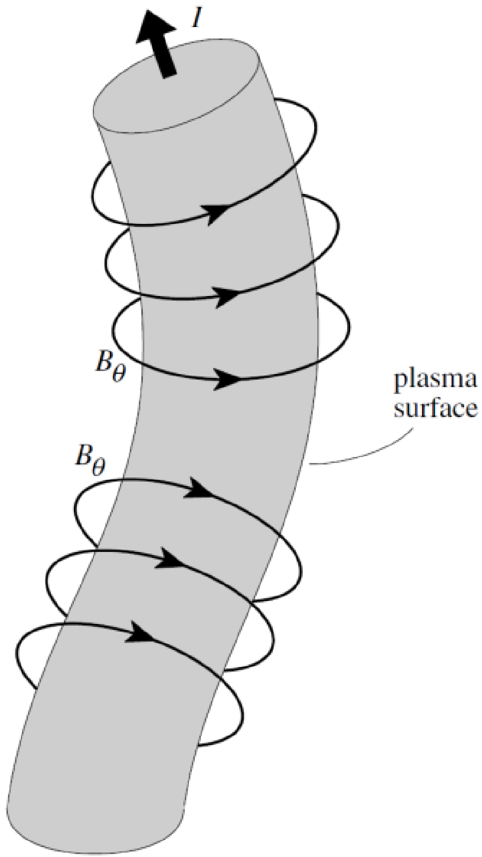
$$\boldsymbol{\zeta(\vec{r}) = \zeta(r)exp(im\theta + ikz)}$$

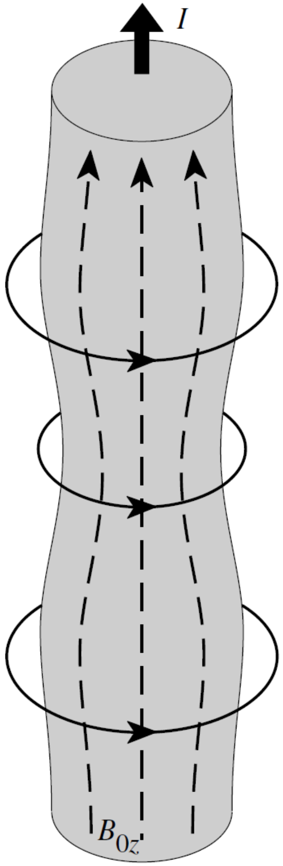
A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**



- **Kink instability**

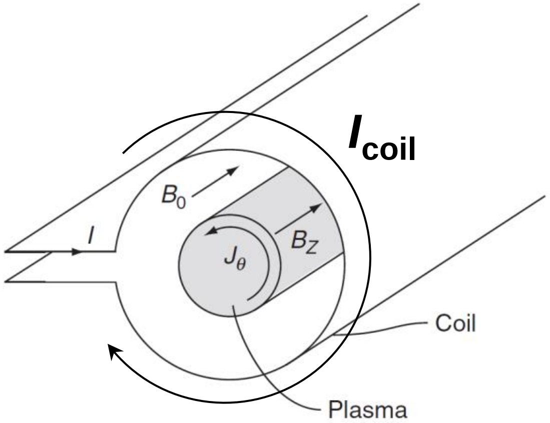




- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_oB_\theta(r)}$ **Kruskal–Shafranov limit**

Theta pinch is stable while z pinch is unstable

- **Theta pinch**

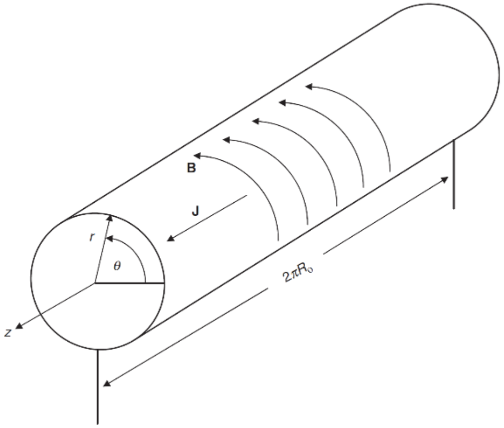


$$\vec{B} = B_z\hat{z}$$

$$q_\theta = \infty$$

Stable

- **Z pinch**

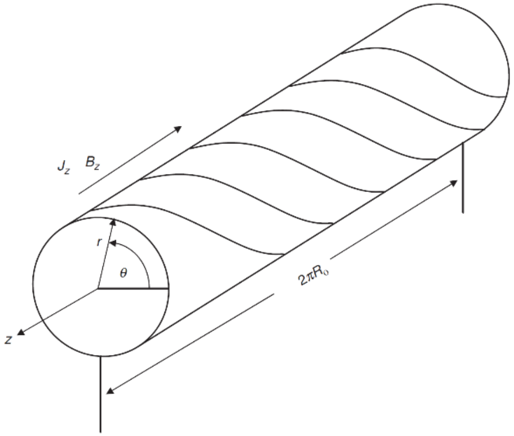


$$\vec{B} = B_\theta\hat{\theta}$$

$$q_z = 0$$

Unstable

- **Screw pinch**



$$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$$

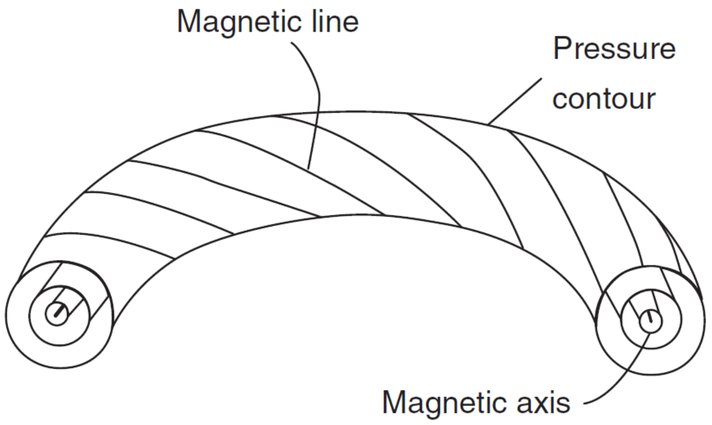
$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

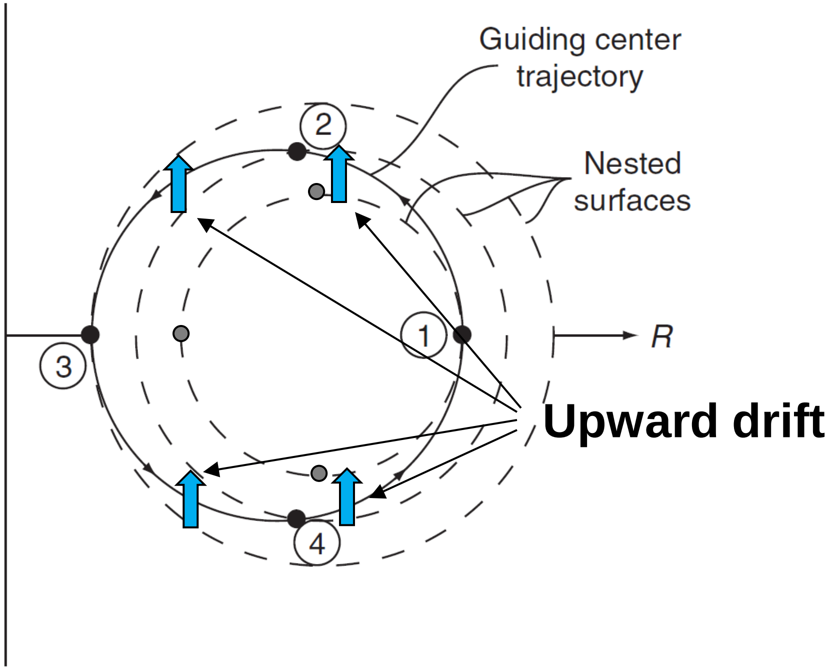
q **can be controlled.**

Stable/Unstable

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

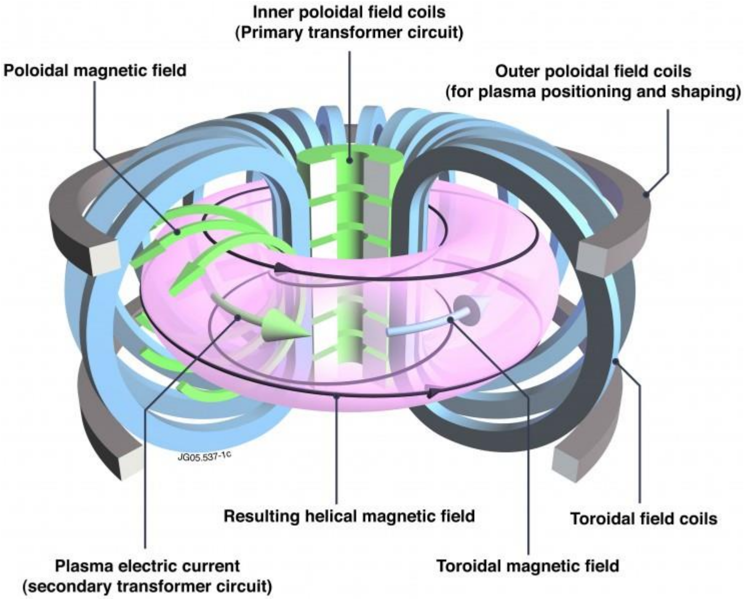




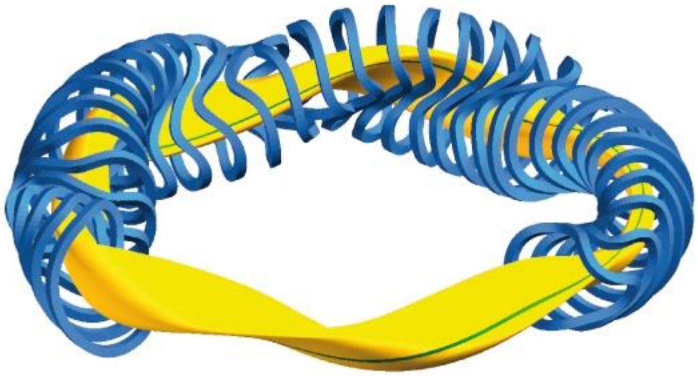
● Points with no drift

Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



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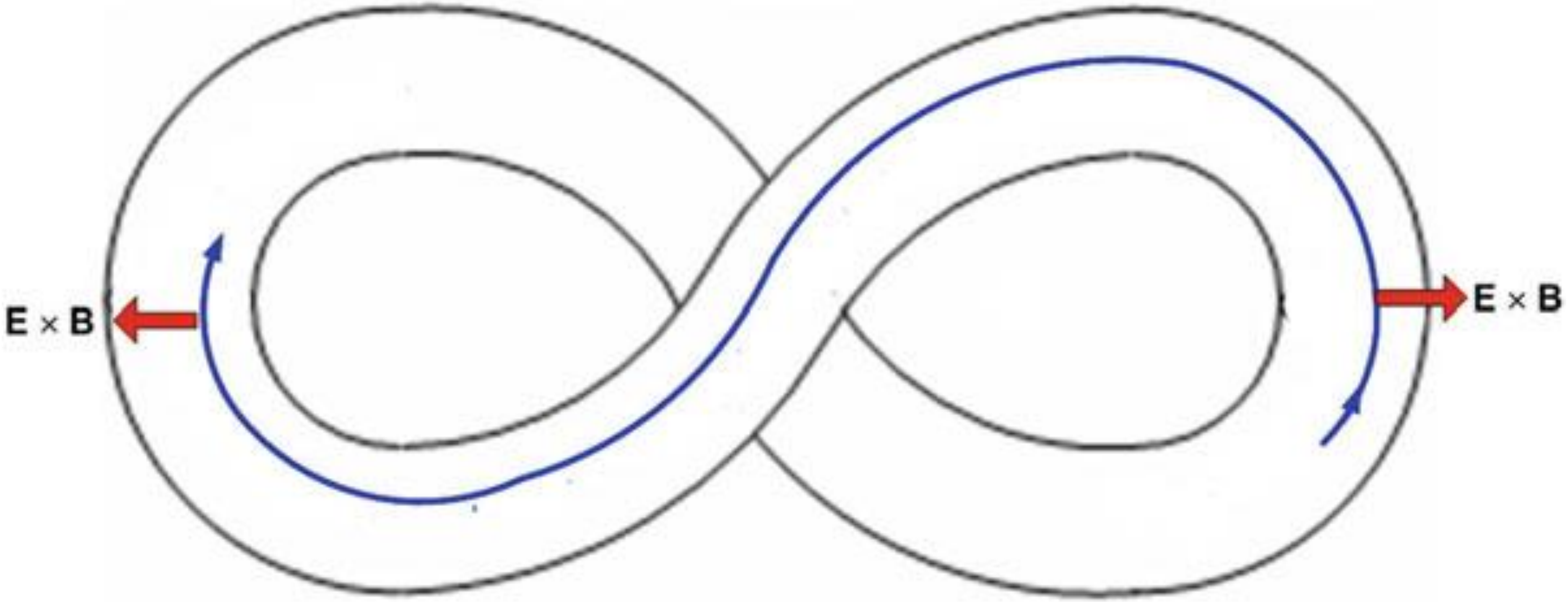
https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Stellerator

- **Figure eight shape**
- **Oval shape (racetrack)**
- **Torsatron**
- **Heliotron**
- **Heliac (Helical Axis stellarator)**
- **Helias (W7-x)**

A figure-8 stellarator solved the drift issues



Introduction to Plasma Physics and Controlled Fusion 3rd Edition, by Francis F. Chen

A figure-8 stellarator solved the drift issues

Lyman Spitzer, Jr. came out the idea during a long ride on a ski lift at Garmisch-Partenkirchen

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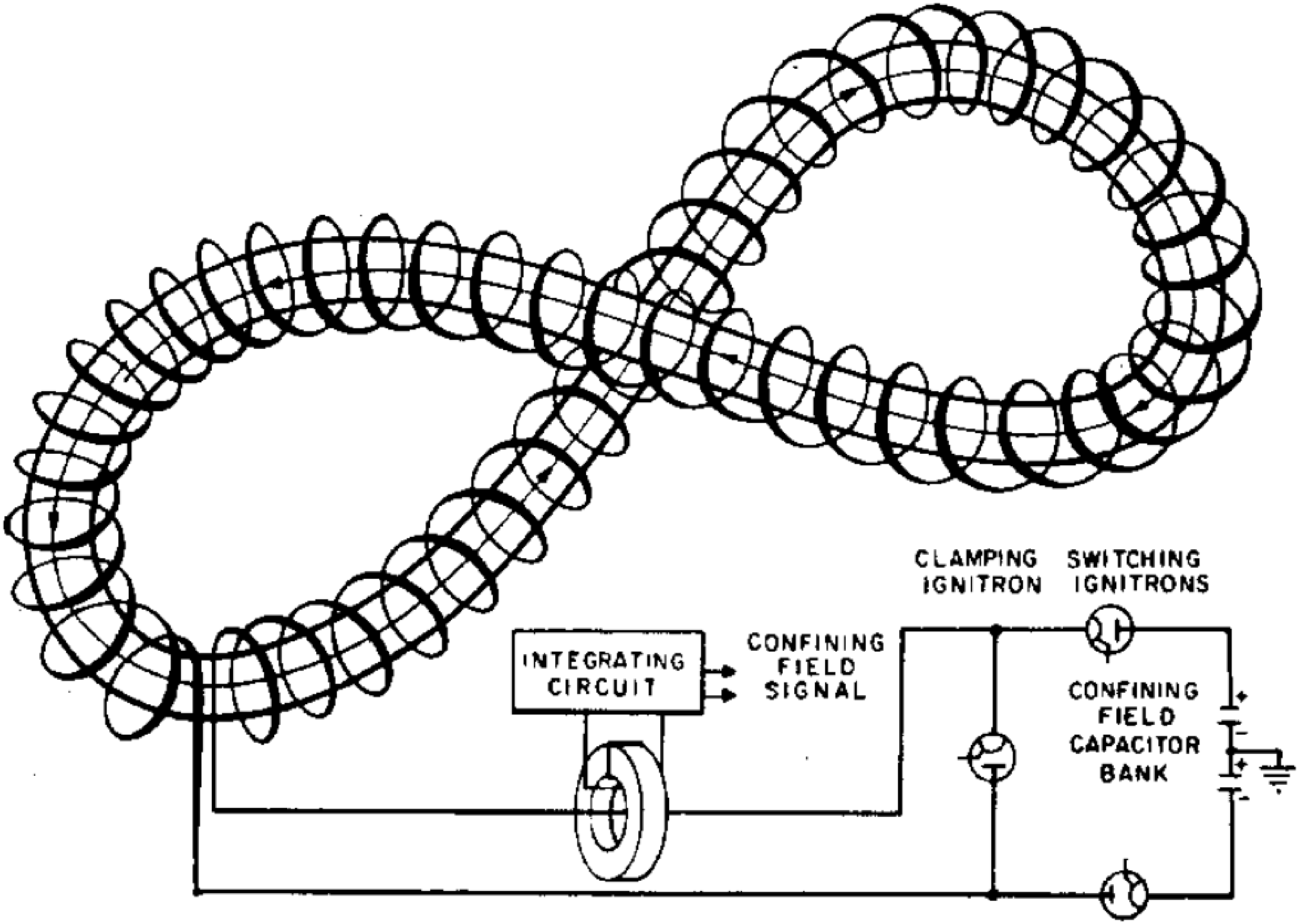
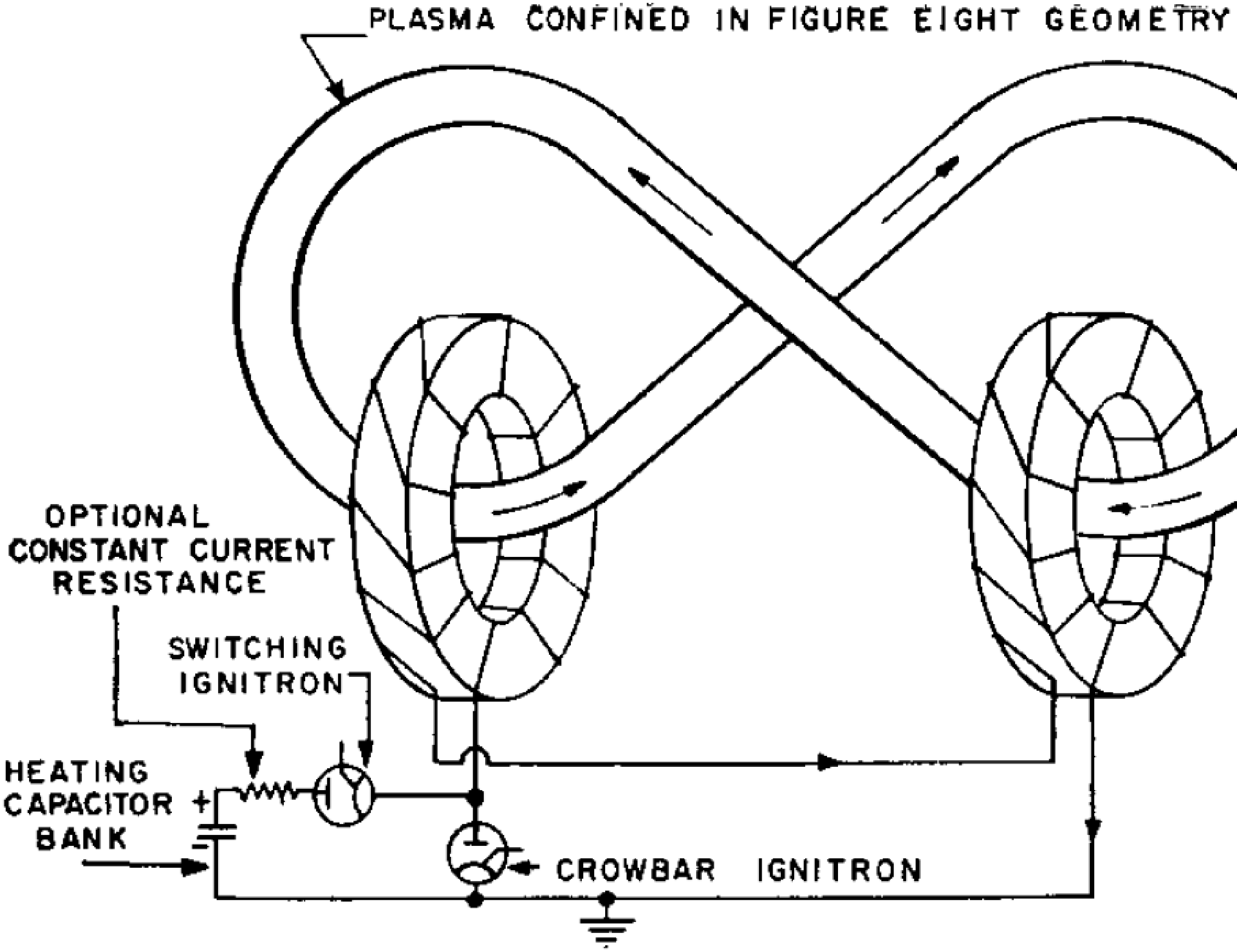
https://www.snowtrex.de/magazin/skigebiete/garmisch-classic-zugspitze/

Concept of figure-8 stellarator



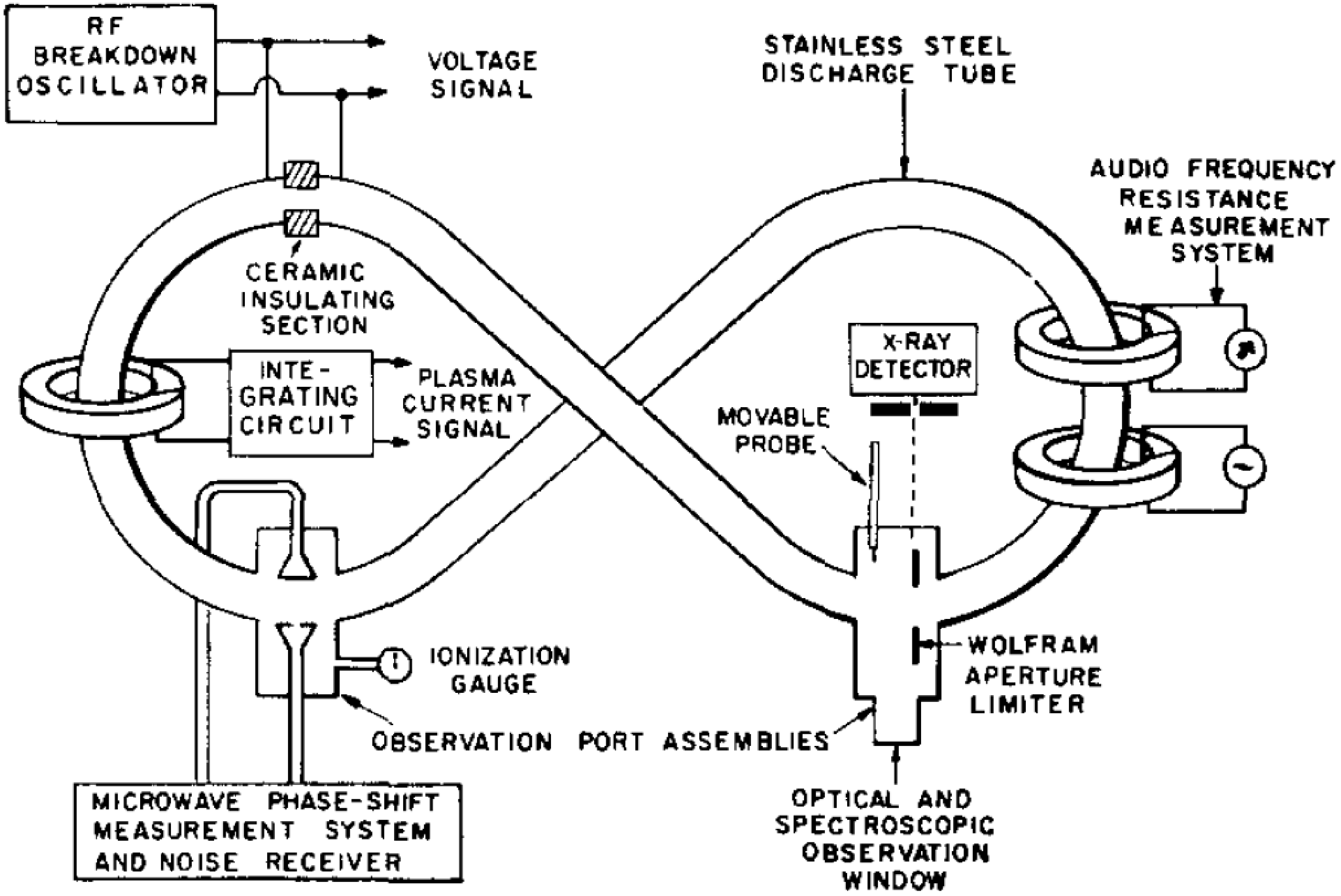
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Figure-8 stellarator with ohmic heating apparatus



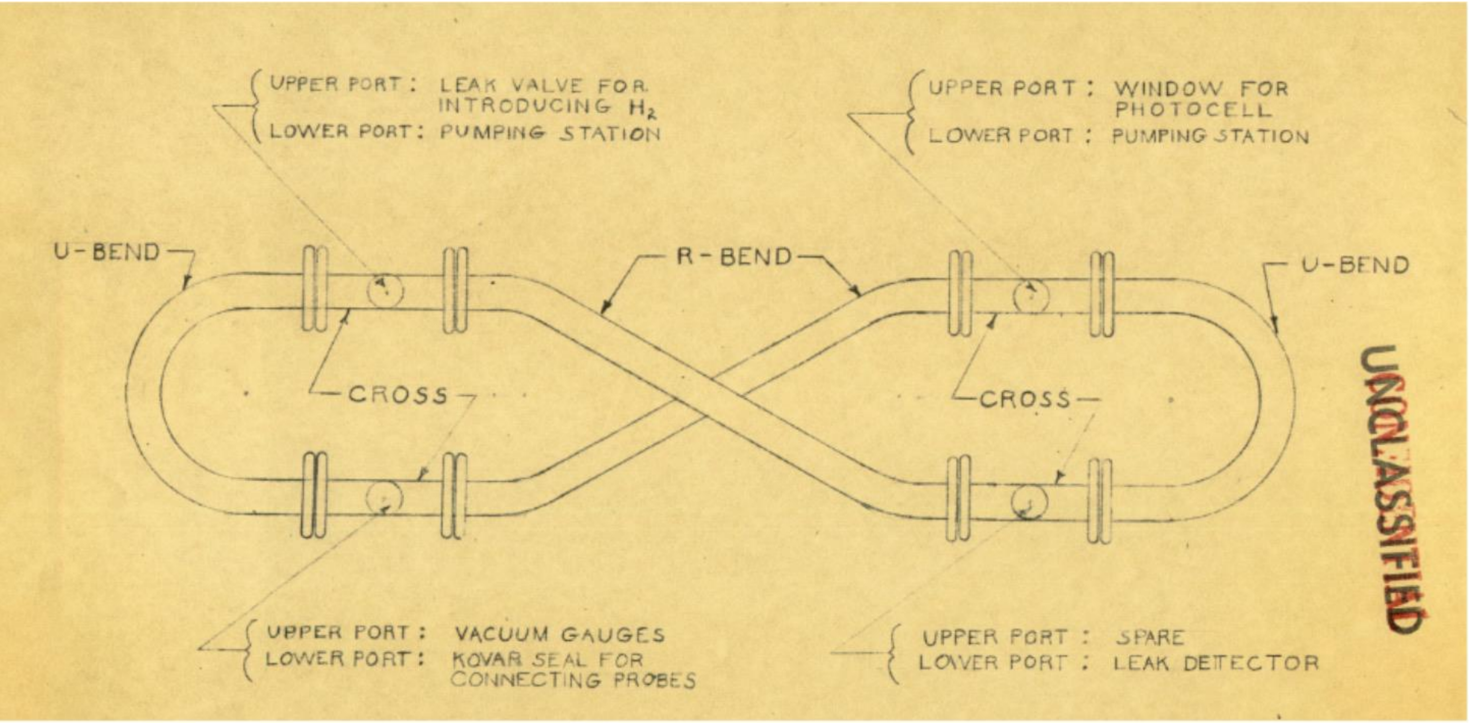
T. Coor, et al., Phys. Fluids **1**, 411 (1958)

Schematic diagram of B-1 stellarator

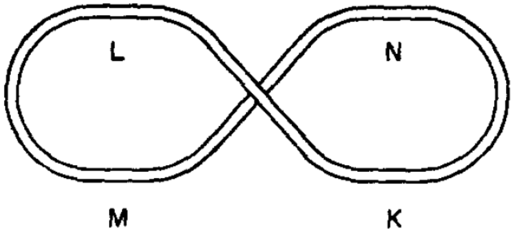


T. Coor, et al., Phys. Fluids **1**, 411 (1958)

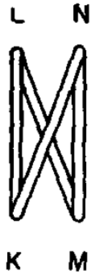
Figure-eight (Princeton Model A) – 1953-1958



- **Top view**



- **Side view**

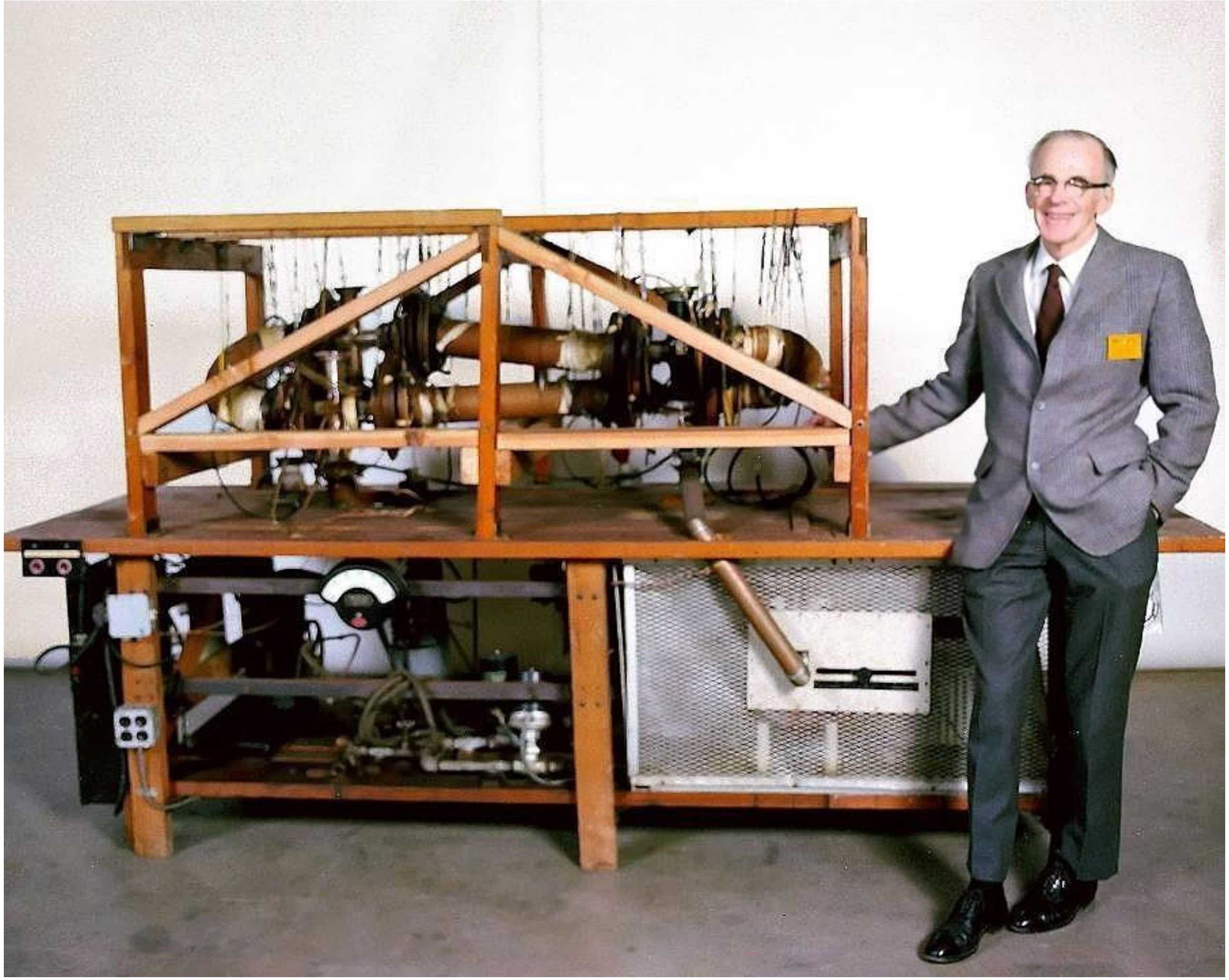


C. H. Willis, NJ Project Matterhorn (1953)
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Model A stellarator

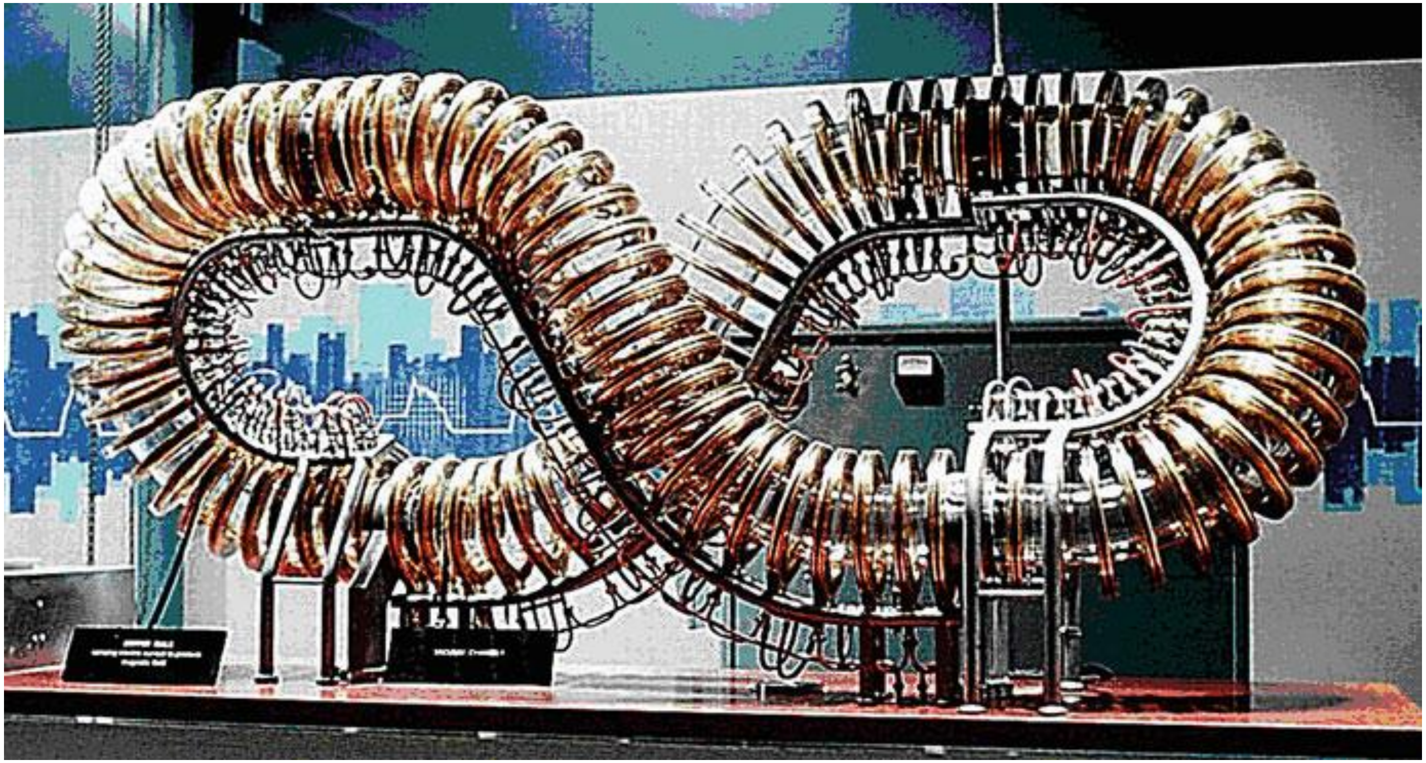
https://www.pppl.gov/timeline

Model A stellarator



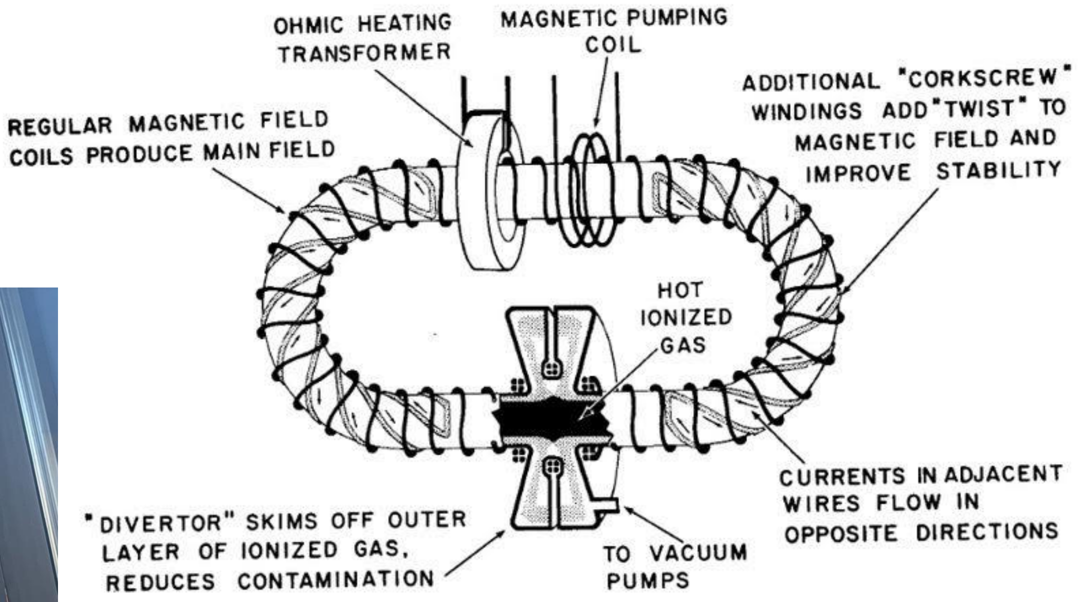
https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

Exhibit model of a figure-8 stellarator for the Atoms for Peace conference in Geneva in 1958

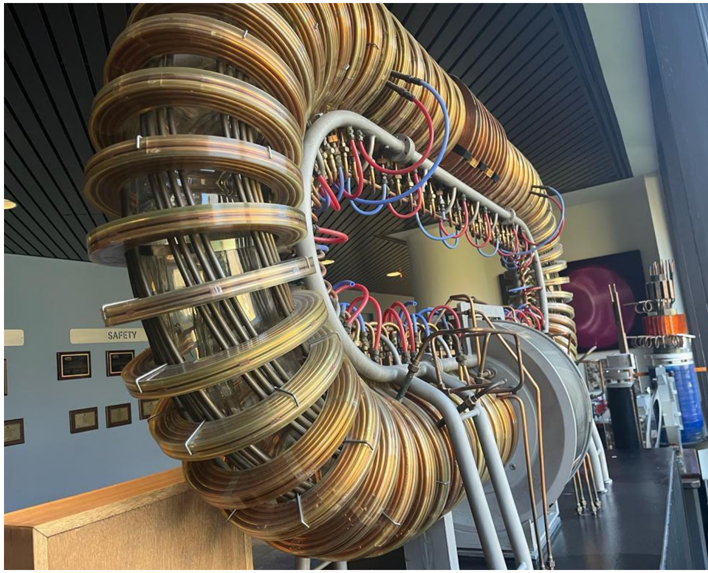


Racetrack Stellarator (Project Matterhorn)

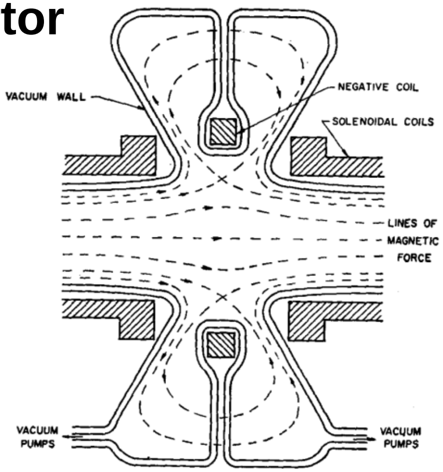
FIG. 4: SCHEMATIC "RACETRACK" STELLARATOR



SEPTEMBER 19, 1958 ★ 9



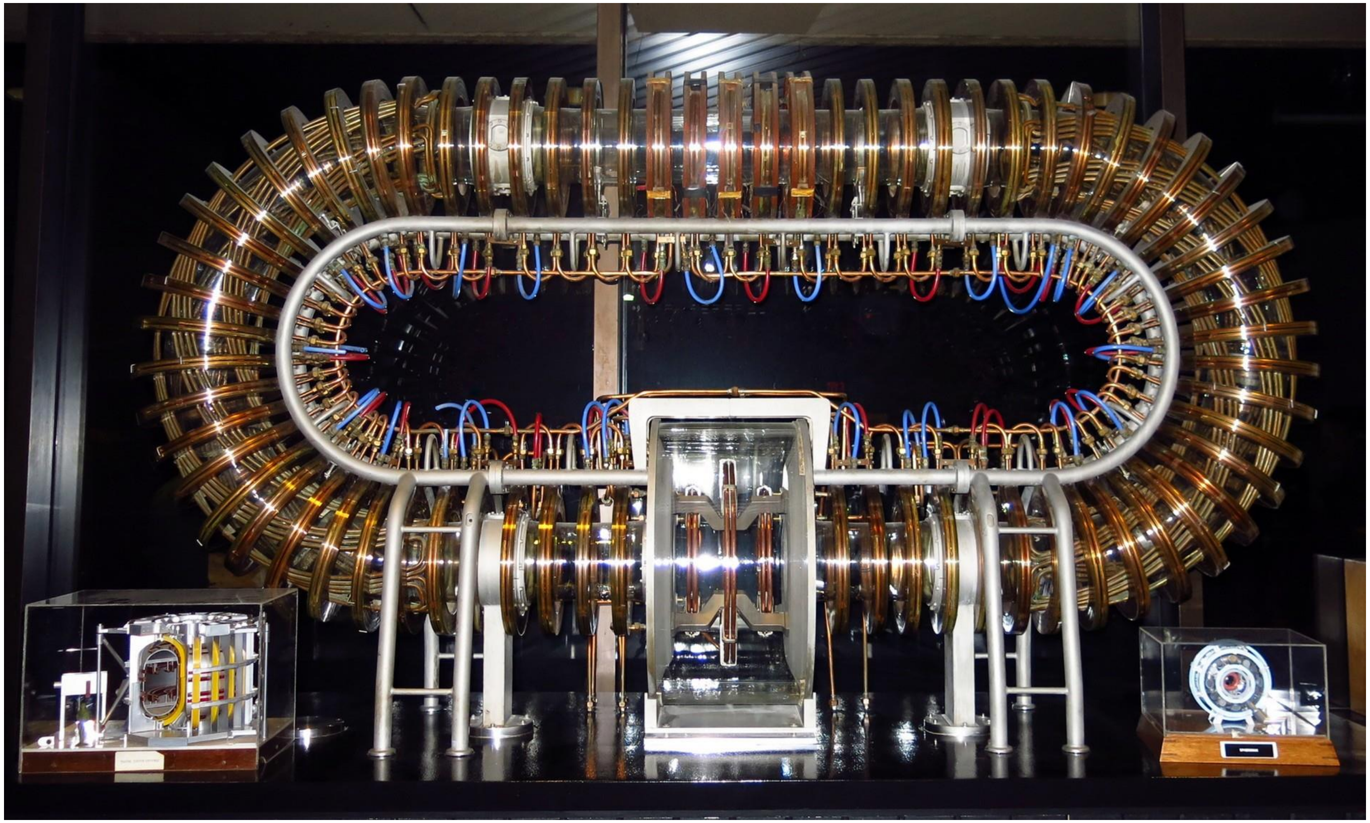
- **Divertor**



https://twitter.com/quantumwriters/status/1724959216966775270

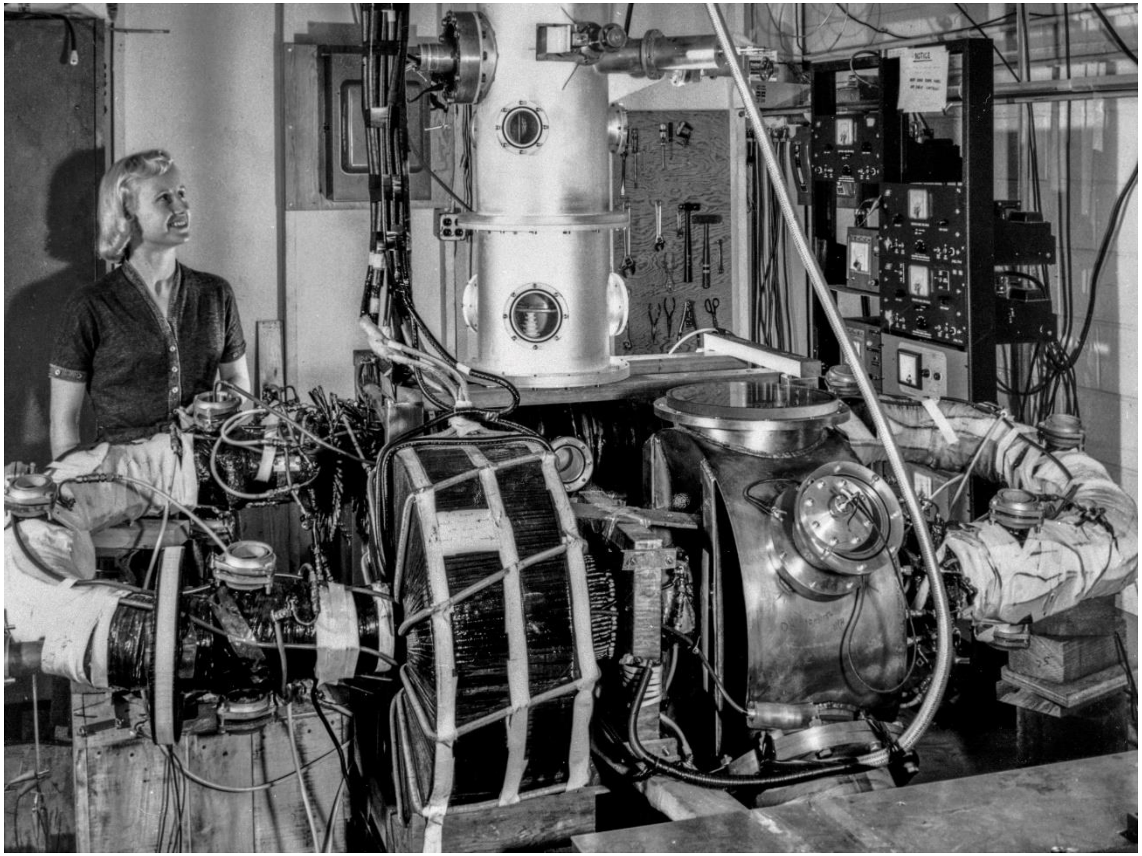
L. Spitzer, Jr., Phys. Fluids **1**, 253 (1958)

Racetrack Stellarator



https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

B-65 stellarator



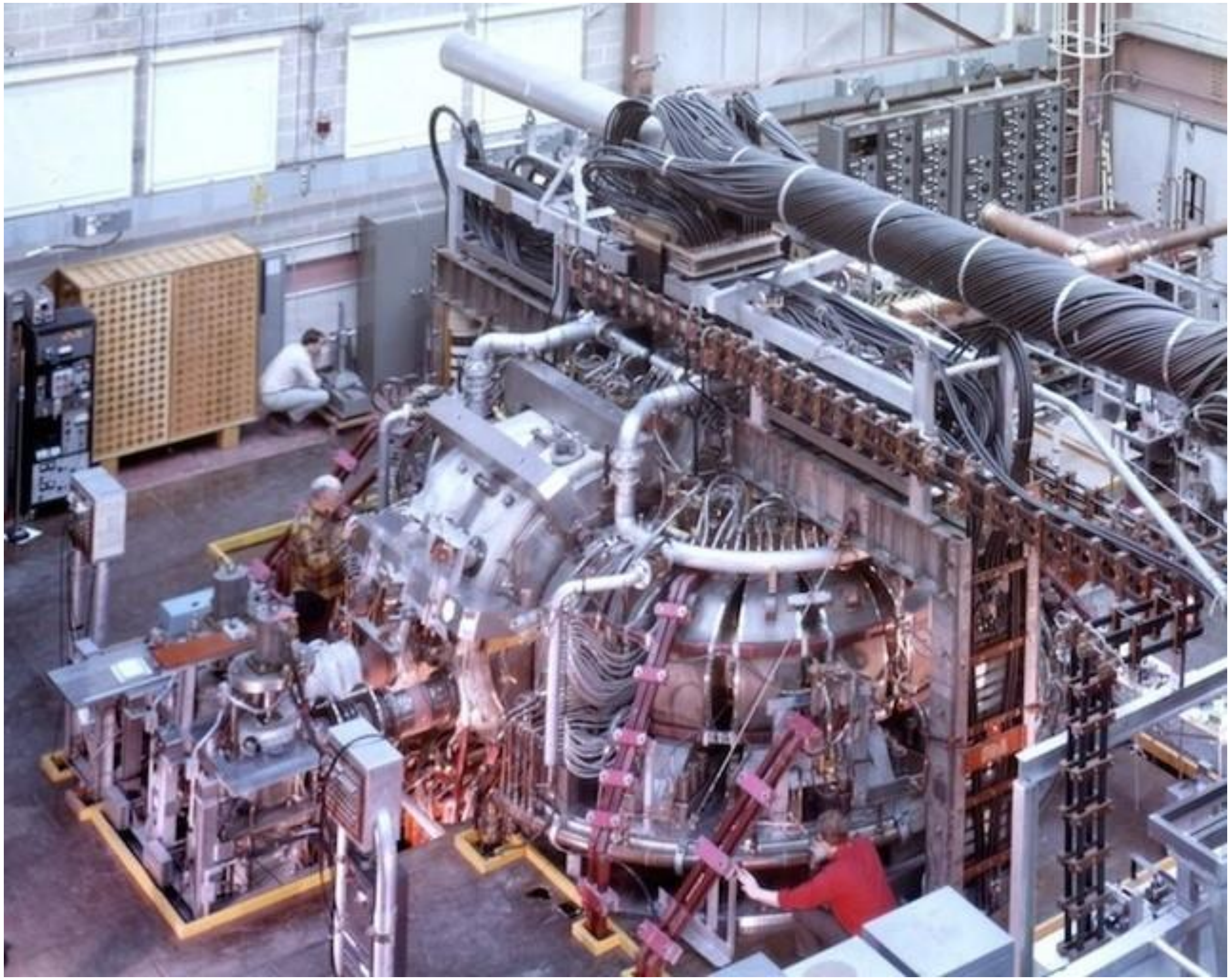
PRINCETON
ALUMNI WEEKLY

Vol. LIX · SEPTEMBER 19, 1958 · No. 1

STELLARATOR

https://www.pppl.gov/timeline
Elizabeth Paul, An introduction to stellarators,
Princeton Alumni Weekly, Sep. 19, 1958

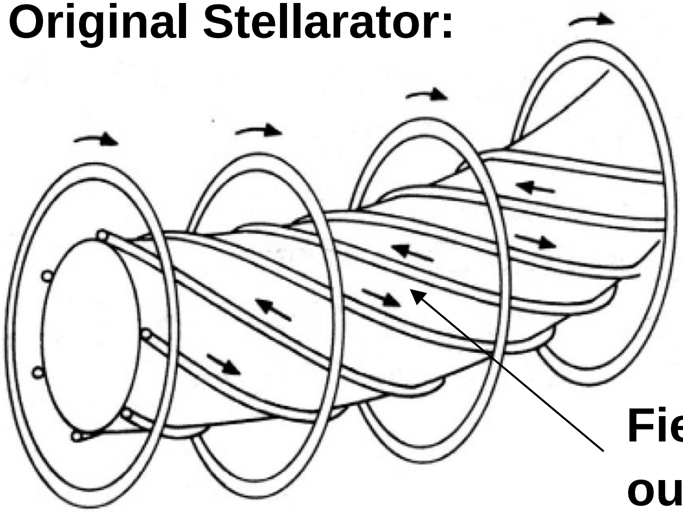
Racetrack (Princeton Model C) – 1962-1969



https://www.autoevolution.com/news/stellarator-reactors-the-once-forgotten-all-american-approach-to-nuclear-fusion-209478.html#agal_2

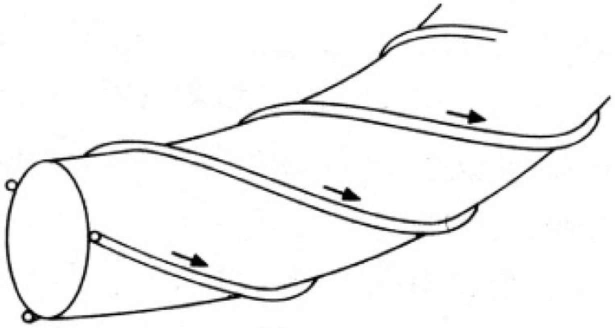
Different types of stellarators

- **Original Stellarator:**

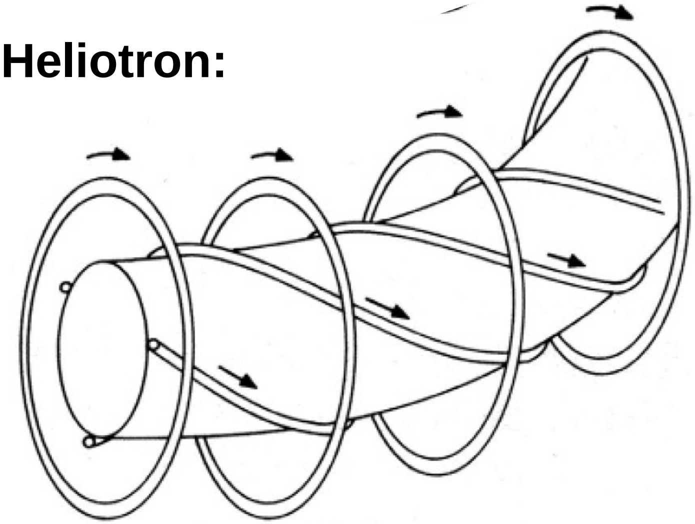


Field cancels out on axis

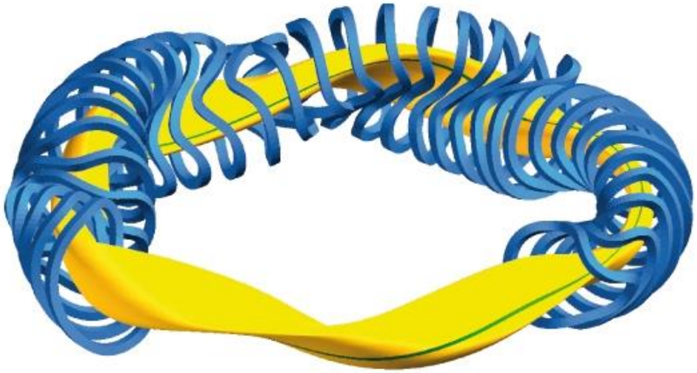
- **Torsatron:**



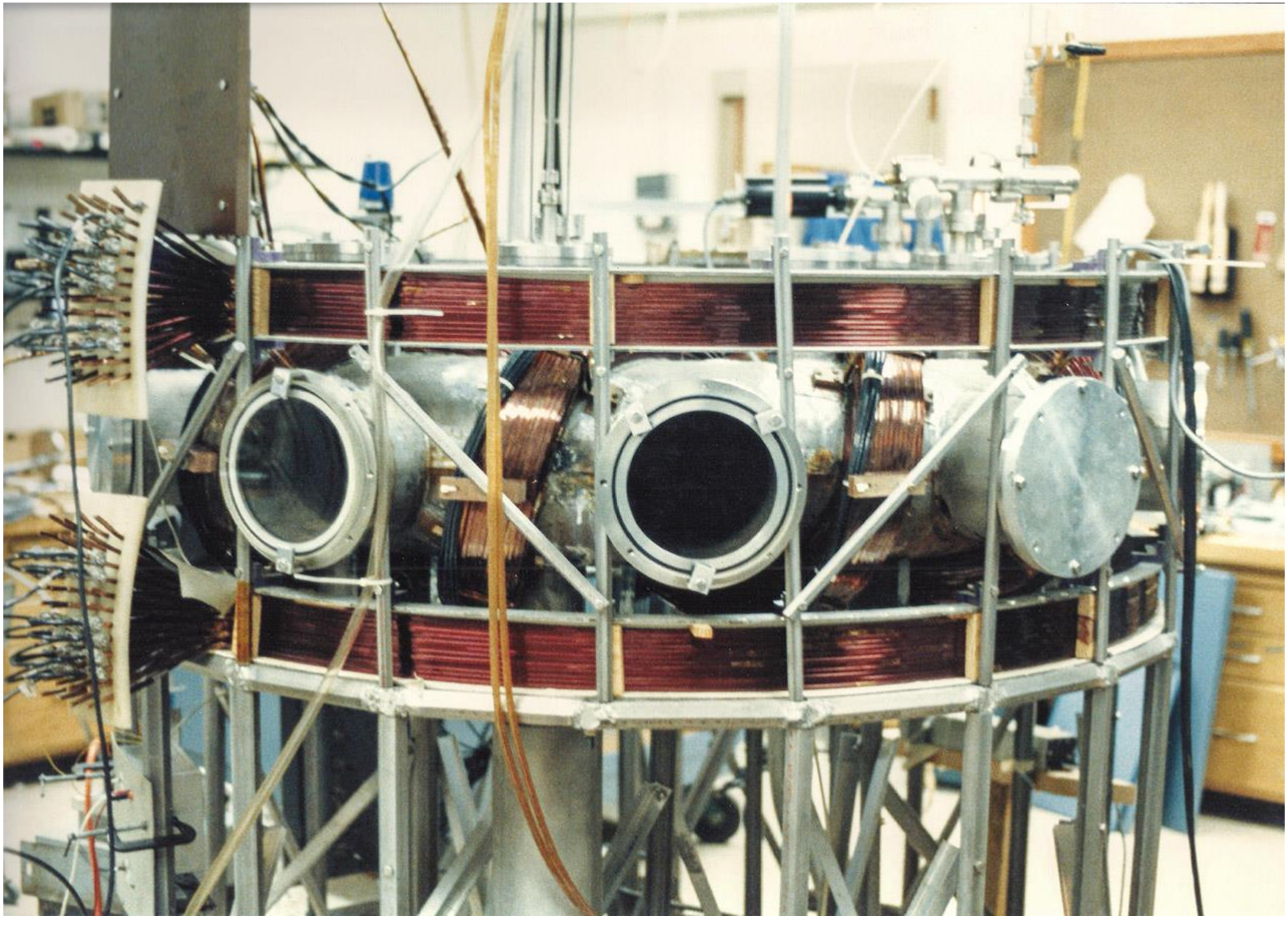
- **Heliotron:**



- **Helias:**

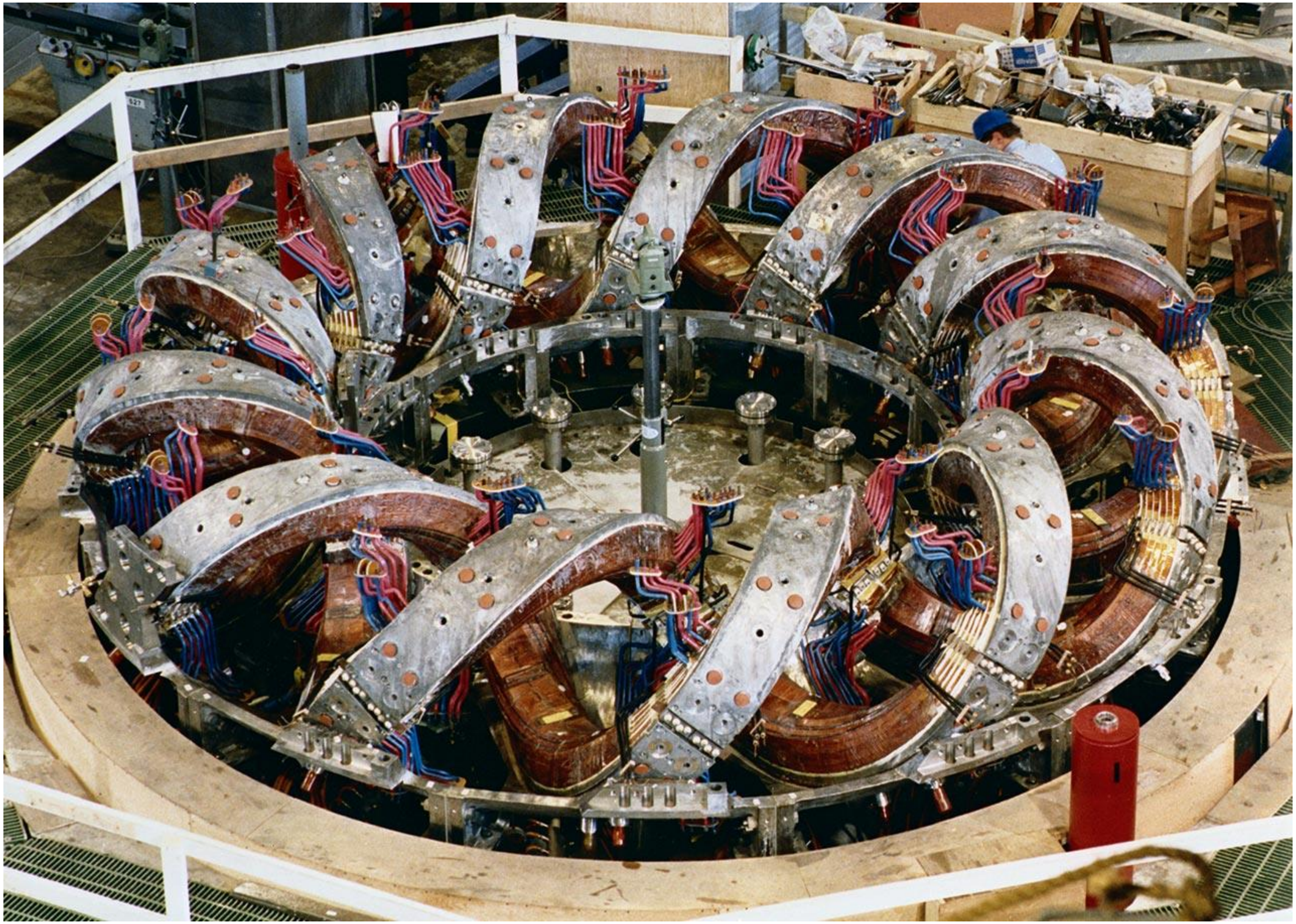


Auburn torsatron — winding of both helical and poloidal coils can be seen



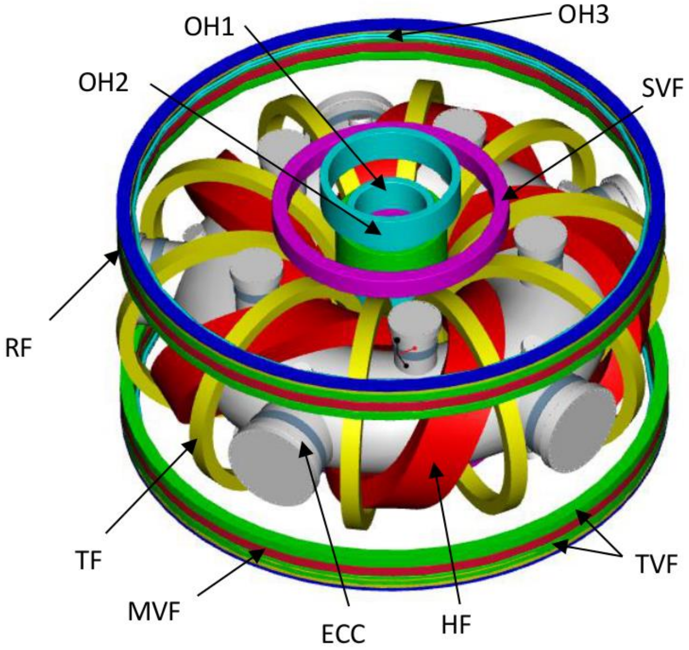
https://www.energyencyclopedia.com/en/glossary/torsatron

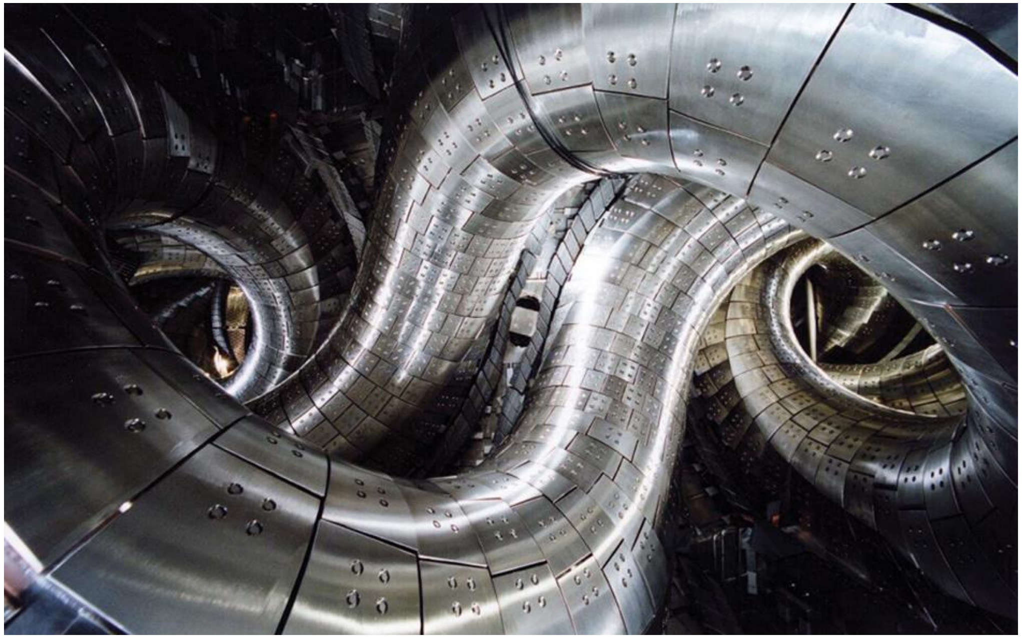
Construction of a pair of helical magnetic coils for the Advanced Toroidal Facility torsatron

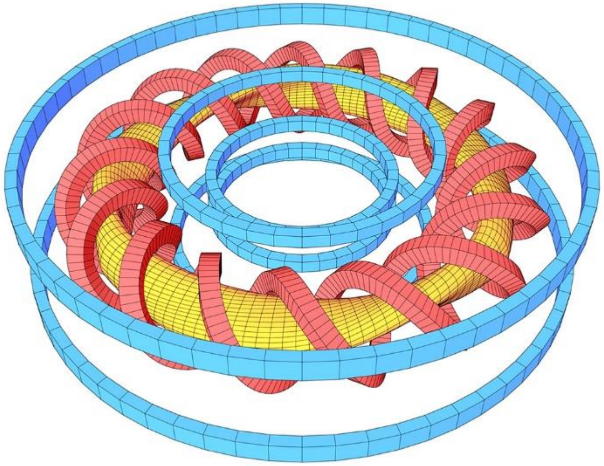


https://www.energyencyclopedia.com/en/glossary/torsatron

LHD stellarator in Japan (Heliotron)

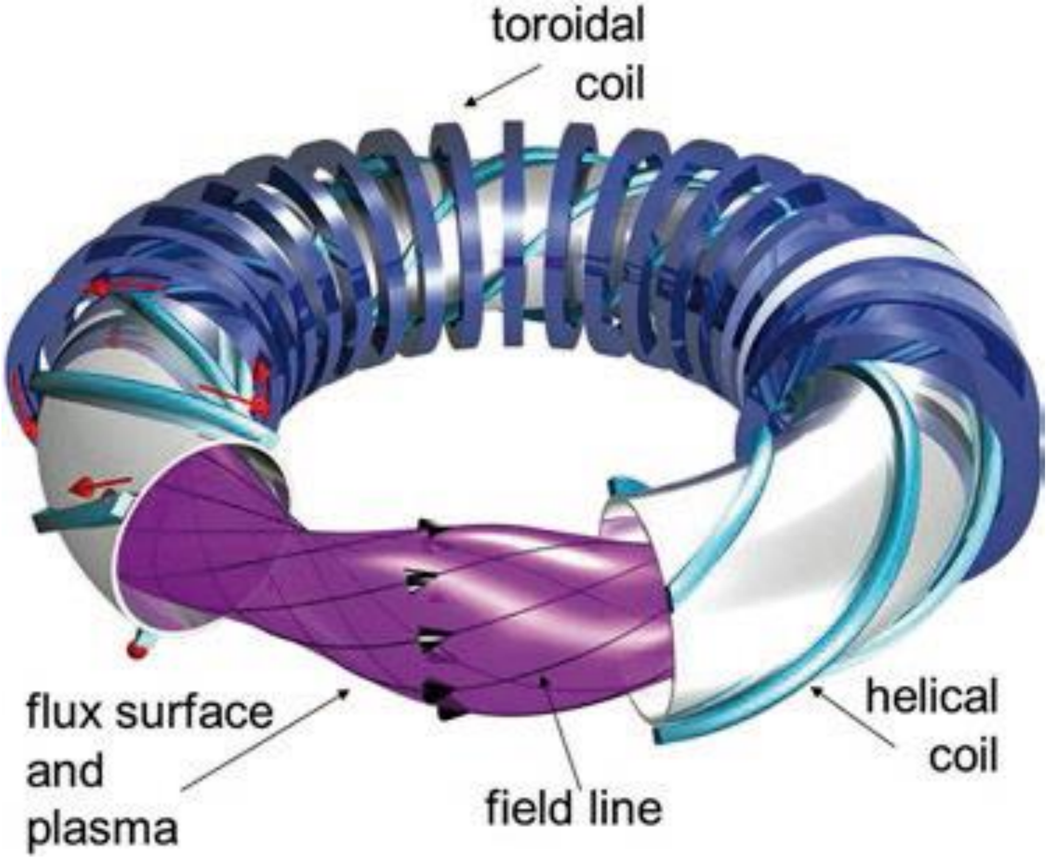






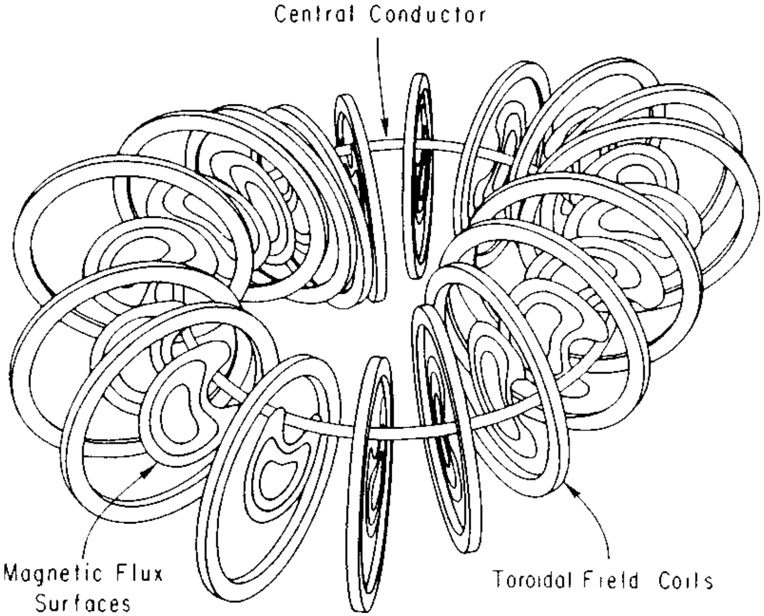
https://en.wikipedia.org/wiki/Compact_Toroidal_Hybrid
https://www.energyencyclopedia.com/en/glossary/heliotron
https://en.wikipedia.org/wiki/Large_Helical_Device

Twisted magnetic field lines can be provided by toroidal coils with helical coils

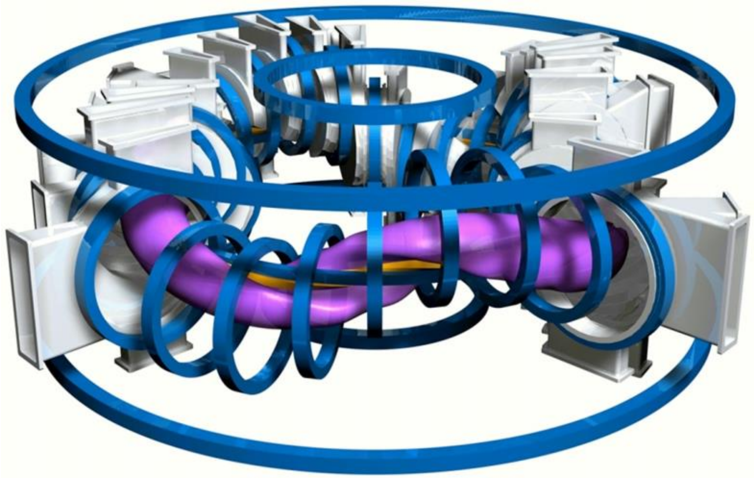


Wagner, F., Fusion energy. MRS Energy & Sustainability, 5, E8 (2018)

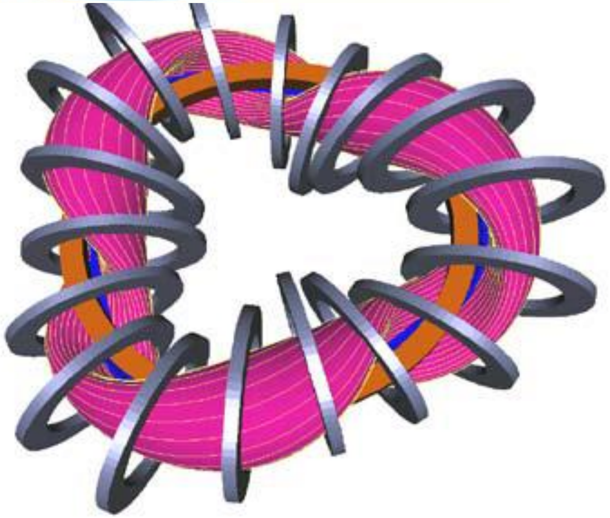
Heliac (Helical Axis stellarator)



- **TJ-II (Spain's National Fusion Laboratory):**

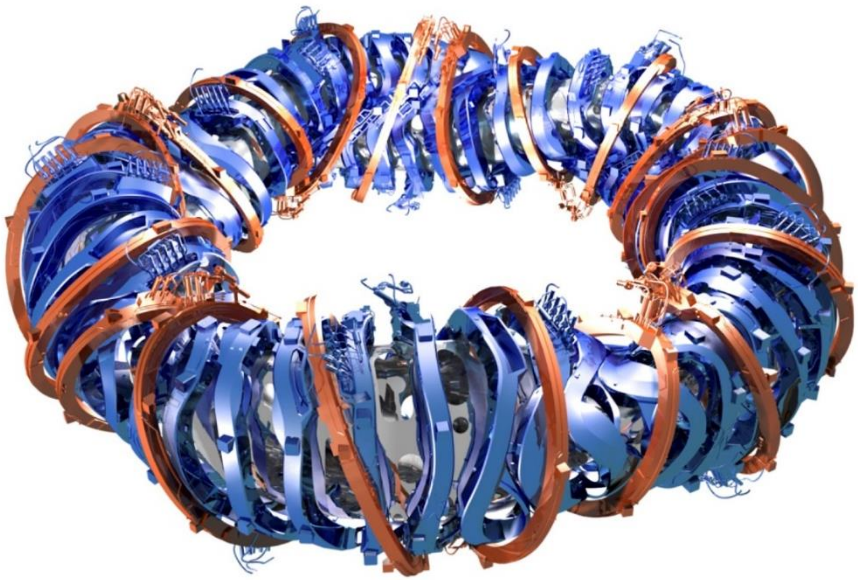


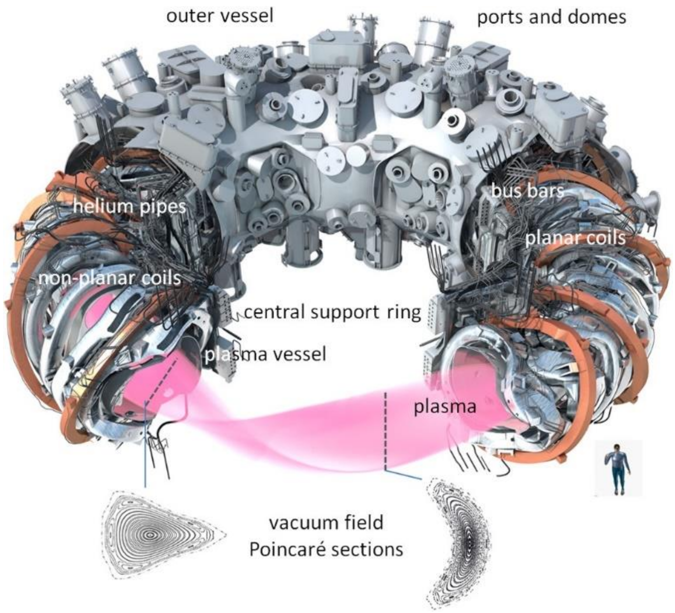
- **H-1 (Australian Plasma Fusion Research Facility):**



A. H. Boozer, Phys. Plasmas, 5, 1647 (1998)
https://wiki.fusion.ciemat.es/wiki/TJ-II
B. D. Blackwell, et. al, 23rd IAEA Fusion Energy Conference, 2010

Wendelstein 7-X is a (Helias) stellarator built by Max Planck Institute for Plasma Physics (IPP)





- **Wemdelstein 7-x is now installing new diverters.**

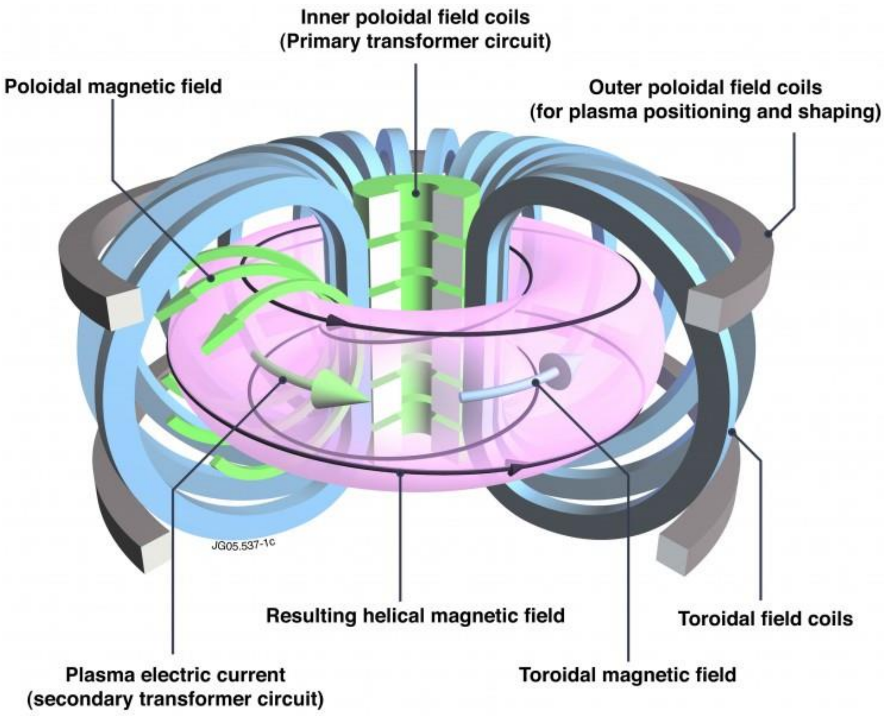
Advantages of Stellarator

- **No need to drive plasma current. It is intrinsically steady state.**
- **With zero net current, one potentially dangerous class of MHD instabilities, the current-driven kink modes, is eliminated.**
- **Magnetic configuration is set by external coils, not by currents in the plasma. Stellarators do not suffer violent disruptions.**
- **Potential for greater range of designs and optimization of fusion performance.**

Disadvantages of Stellarator

- **Complicated coil configurations. It’s difficult to design. The precision requirement is high. It is expensive to build coils for stellarators.**
- **Achieving good particle confinement in stellarators is more difficult than that in tokamaks.**
- **Divertors and heat load geometry in stellarators is more complicated than those in tokamaks.**

2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation



$$\vec{j} \times \vec{B} = \nabla p$$

$$\Downarrow$$

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p$$

$$\Downarrow$$

$$\vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j} \Rightarrow \nabla \cdot \vec{j} = 0$$

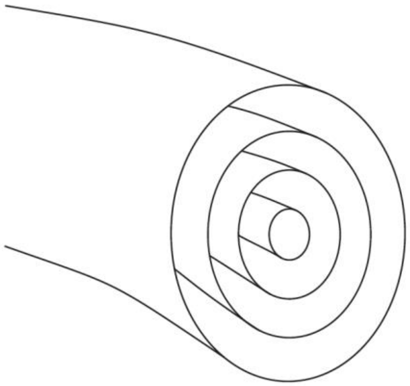
$$\nabla \cdot \vec{B} = 0$$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

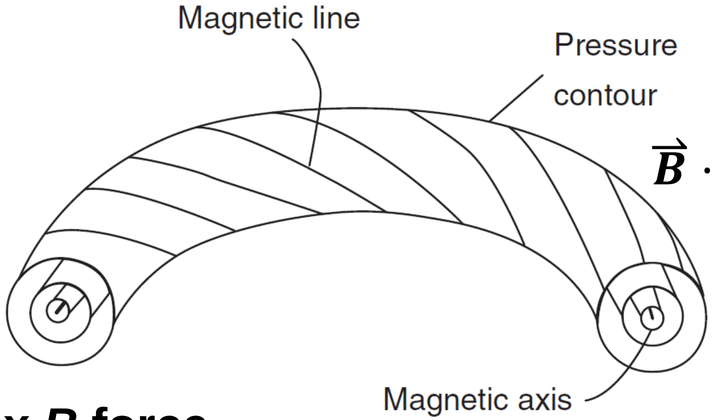
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

Magnetic lines lying on pressure contour

- **Contours of constant pressure**



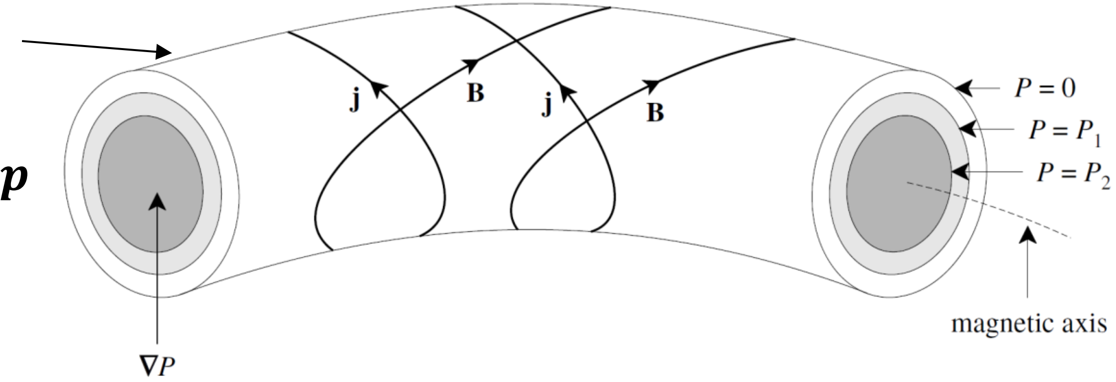
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Derivation of Grad-Shafranov equation

$$\vec{j} \times \vec{B} = \nabla p \qquad \nabla \times \vec{B} = \mu_o \vec{j}$$

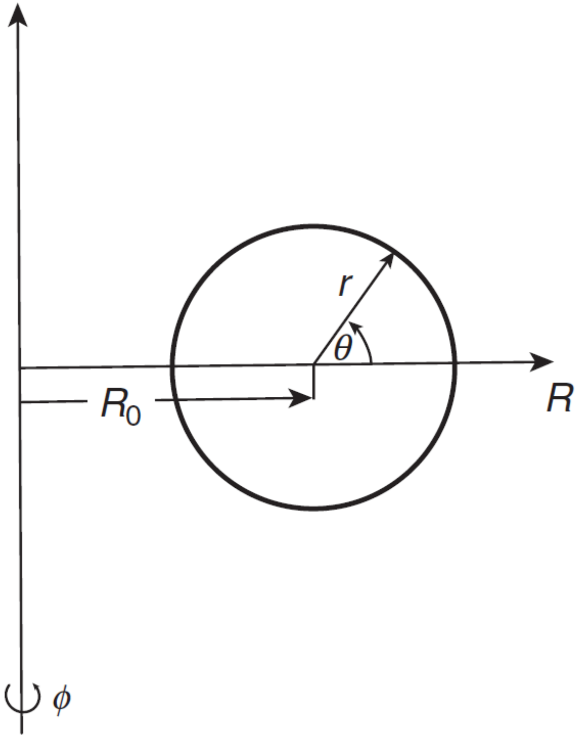
$$\nabla \cdot \vec{j} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\vec{B} = \left(B_R, B_\phi, B_z\right) \qquad \textbf{Axisymmetric: } \frac{\partial}{\partial \phi} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\cancel{\frac{\partial B_\phi}{\partial \phi}} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$



- **Represent the magnetic field using a vector potential *A*:**

$$\vec{B} = \nabla \times \vec{A} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\cancel{\frac{\partial A_R}{\partial \phi}}\right)$$

$$= \hat{R}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\right)$$

$$\equiv \hat{R}B_R + \hat{\phi}B_\phi + \hat{z}B_z$$

$$B_R = -\frac{\partial A_\phi}{\partial z} \qquad B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)$$

Pressure can be written as a function of flux

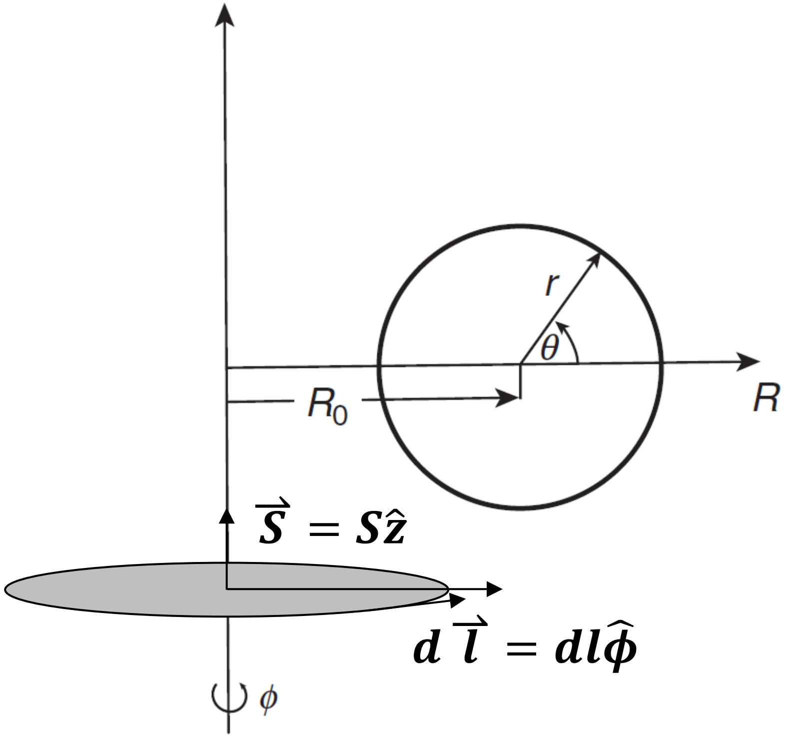
$$\frac{1}{R}\frac{\partial}{\partial R}(RB_{\mathrm{R}})+\frac{\partial B_{\mathrm{z}}}{\partial z}=0$$

$$B_{\mathrm{R}}=-\frac{\partial A_{\phi}}{\partial z}$$

$$B_{\mathrm{z}}=\frac{1}{R}\frac{\partial}{\partial R}(RA_{\phi})$$

$$\psi\equiv\frac{1}{2\pi}\int\vec{B}\cdot d\vec{S}=\frac{1}{2\pi}\int(\nabla\times\vec{A})\cdot d\vec{S}$$

$$=\frac{1}{2\pi}\int\vec{A}\,2\pi R\cdot d\vec{l}=\int\vec{A}\cdot\hat{\phi}R\,dl=RA_{\phi}$$



$$\vec{S}=S\hat{z}$$

$$d\vec{l}=dl\hat{\phi}$$

$$B_{\mathrm{R}}=-\frac{1}{R}\frac{\partial\psi}{\partial z}\qquad B_{\mathrm{z}}=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}\cdot\nabla\psi=B_{\mathrm{R}}\frac{\partial\psi}{\partial R}+B_{\phi}\frac{1}{R}\cancel{\frac{\partial\psi}{\partial\phi}}+B_{\mathrm{z}}\frac{\partial\psi}{\partial z}=B_{\mathrm{R}}\frac{\partial\psi}{\partial R}+B_{\mathrm{z}}\frac{\partial\psi}{\partial z}$$

$$=\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)\frac{\partial\psi}{\partial R}+\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\frac{\partial\psi}{\partial z}=0$$

$$\vec{B}\cdot\nabla\psi=0$$

$$\vec{B}\cdot\nabla p=0$$

for $\nabla p\neq0$:

$$p=p(\psi)$$

Derivation of Grad-Shafranov equation

- Let's see the $\hat{\phi}$ component of the force-balance equation:

$$\left(\vec{j}\times\vec{B}=\nabla p\right)_{\phi} \qquad\qquad j_z B_R - j_R B_z = \frac{1}{R}\frac{\partial p}{\partial \phi} \equiv 0$$

- Ampére's law:

$$\nabla\times\vec{B} = \mu_o\,\vec{j}$$

$$\nabla\times\vec{B} = \hat{R}\left(\frac{1}{R}\cancel{\frac{\partial B_z}{\partial\phi}} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z}-\frac{\partial B_z}{\partial R}\right)+\hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right) - \frac{1}{R}\cancel{\frac{\partial B_R}{\partial\phi}}\right]$$

$$= \hat{R}\left(-\frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_R}{\partial z}-\frac{\partial B_z}{\partial R}\right)+\hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)\right]$$

$$= \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z$$

$$j_R = -\frac{1}{\mu_o}\frac{\partial B_\phi}{\partial z} \qquad j_z = \frac{1}{\mu_o}\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)$$

$$\frac{B_R}{R}\frac{\partial}{\partial R}\left(RB_\phi\right) + B_z\frac{\partial B_\phi}{\partial z} = 0$$

Magnetic field can be decomposed into the poloidal component and the toroidal component

$$\frac{B_\mathrm{R}}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)+B_\mathrm{z}\frac{\partial B_\phi}{\partial z}=0 \quad\Rightarrow\quad B_\mathrm{R}\frac{\partial}{\partial R}\left(RB_\phi\right)+B_\mathrm{z}\frac{\partial}{\partial z}\left(RB_\phi\right)=0$$

$$F\equiv RB_\phi \quad\Rightarrow\quad B_\mathrm{R}\frac{\partial F}{\partial R}+B_\mathrm{z}\frac{\partial F}{\partial z}=0 \quad\Rightarrow\quad \vec{B}\cdot\nabla F=0$$

$$\left(\frac{\partial}{\partial \phi}=0\right)$$

$$\Downarrow$$

$$F=F(\psi)$$

$$\vec{B}\cdot\nabla\psi=0$$

$$\vec{B}\cdot\nabla p=0$$

$$p=p(\psi)$$

$$B_\mathrm{R}=-\frac{\partial A_\phi}{\partial z}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

$$B_\mathrm{z}=\frac{1}{R}\frac{\partial}{\partial R}\left(RA_\phi\right)=\frac{1}{R}\frac{\partial\psi}{\partial R} \qquad \left(\psi=RA_\phi\right)$$

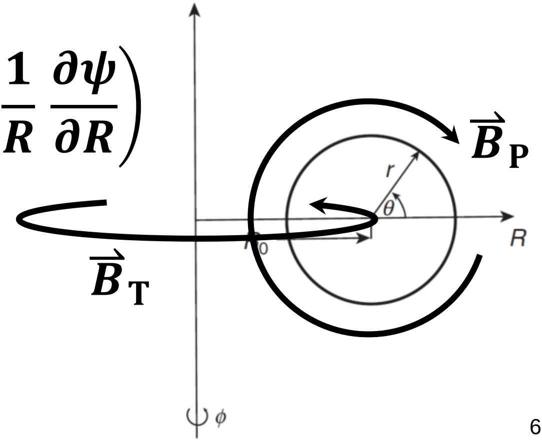
$$B_\phi=\frac{F(\psi)}{R}$$

$$\vec{B}=\hat{R}B_\mathrm{R}+\hat{\phi}B_\phi+\hat{z}B_\mathrm{z}=\hat{R}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)+\hat{\phi}\left(\frac{F(\psi)}{R}\right)+\hat{z}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

$$\equiv\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}+\frac{F(\psi)}{R}\hat{\phi}$$

Poloidal component $\vec{B}_\mathrm{P}$

Toroidal component $\vec{B}_\mathrm{T}$



Arbitrary integration constant associated with flux can be chosen such that flux equals to zero on the field axis

- **The poloidal flux of the area of a washer-shaped surface lying in the *z* = 0 plane from *R* = R_a to an arbitrary ψ contour defined by $\psi = \psi(R_b, 0)$:**

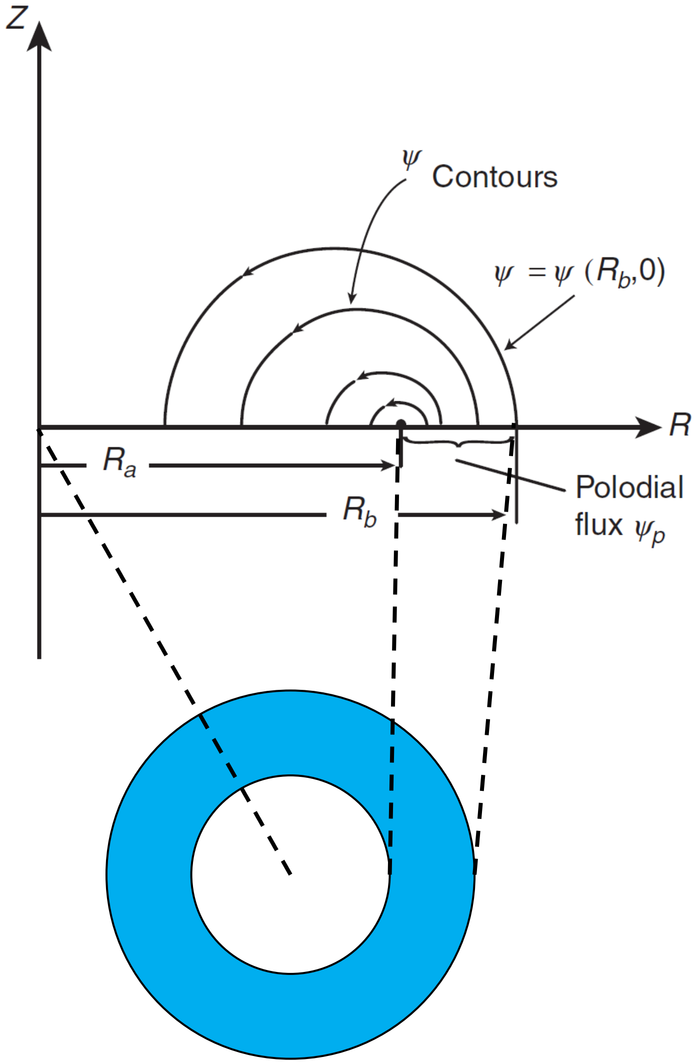
$$\psi_P \equiv \frac{1}{2\pi}\int \vec{B}_P \cdot d\,\vec{S}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{R_a}^{R_b} dR R B_z(R,0)$$

$$= \psi(R_b, 0) - \psi(R_a, 0)$$

$$\equiv \psi(R_b, 0)$$

where $\psi(R_a, 0) \equiv 0$ is chosen.



Derivation of Grad-Shafranov equation

- **Let's see the $\hat{R}$ component of the force-balance equation:**

$$\left(\vec{j}\times\vec{B}=\nabla p\right)_{\mathrm{R}} \qquad j_{\phi}B_{\mathrm{z}}-j_{\mathrm{z}}B_{\phi}=\frac{\partial p}{\partial R}$$

$$B_{\mathrm{R}}=-\frac{1}{R}\frac{\partial\psi}{\partial z}$$

$$B_{\mathrm{z}}=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$B_{\phi}=\frac{F(\psi)}{R}$$

- **Ampére's law:**

$$\nabla\times\vec{B}=\mu_{\mathrm{o}}\,\vec{j}$$

$$\nabla\times\vec{B}=\hat{R}\left(-\frac{\partial B_{\phi}}{\partial z}\right)+\hat{\phi}\left(\frac{\partial B_{\mathrm{R}}}{\partial z}-\frac{\partial B_{\mathrm{z}}}{\partial R}\right)+\hat{z}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(RB_{\phi}\right)\right]=\hat{R}\mu_{\mathrm{o}}j_{\mathrm{R}}+\hat{\phi}\mu_{\mathrm{o}}j_{\phi}+\hat{z}\mu_{\mathrm{o}}j_{z}$$

$$\mu_{\mathrm{o}}j_{\phi}=\frac{\partial B_{\mathrm{R}}}{\partial z}-\frac{\partial B_{\mathrm{z}}}{\partial R}=\frac{\partial}{\partial z}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)-\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)=-\frac{1}{R}\frac{\partial^{2}\psi}{\partial z^{2}}-\frac{1}{R}\frac{\partial^{2}\psi}{\partial R^{2}}+\frac{1}{R^{2}}\frac{\partial\psi}{\partial R}$$

$$\equiv-\frac{1}{R}\Delta^{*}\psi \qquad \text{where }\Delta^{*}\psi\equiv\frac{\partial^{2}\psi}{\partial z^{2}}+R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)=R^{2}\nabla\cdot\left(\frac{\nabla\psi}{R^{2}}\right)$$

$$\mu_{\mathrm{o}}j_{\mathrm{z}}=\frac{1}{R}\frac{\partial}{\partial R}\left(RB_{\phi}\right)=\frac{1}{R}\frac{\partial F}{\partial R}=\frac{1}{R}\frac{dF}{d\psi}\frac{\partial\psi}{\partial R}$$

Derivation of Grad-Shafranov equation

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial R}$$

$$B_\phi = \frac{F}{R} \qquad j_\phi = -\frac{1}{\mu_o R}\Delta^* \psi$$

$$\frac{\partial p}{\partial R} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial R}$$

$$B_z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad j_z = \frac{1}{\mu_o R}\frac{dF}{d\psi}\frac{\partial \psi}{\partial R}$$

$$-\frac{1}{\mu_o R}\Delta^* \psi \frac{1}{R}\cancel{\frac{\partial \psi}{\partial R}} - \frac{1}{\mu_o R}\frac{dF}{d\psi}\cancel{\frac{\partial \psi}{\partial R}}\frac{F}{R} = \frac{dp}{d\psi}\cancel{\frac{\partial \psi}{\partial R}}$$

$$-\Delta^* \psi \frac{1}{\mu_o}\frac{1}{R^2} - \frac{1}{\mu_o}\frac{F}{R^2}\frac{dF}{d\psi} = \frac{dp}{d\psi}$$

$$\boxed{\text{Grad} - \text{Shafranov equation: } \Delta^* \psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}}$$

$$F \equiv RB_\phi$$

$$\text{where } \Delta^* \psi = R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2}\right) \qquad \vec{B} = \left(\frac{\nabla \psi}{R}\right) \times \hat{\phi} + \frac{F(\psi)}{R}\hat{\phi}$$

Derivation of Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \quad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \quad \vec{B} = \underbrace{\left(\frac{\nabla\psi}{R}\right)\times\hat{\phi}}_{B_\mathrm{P}} + \underbrace{\frac{F(\psi)}{R}\hat{\phi}}_{B_\mathrm{T}}$$

$$\mu_o j_\phi = -\frac{1}{R}\Delta^*\psi \qquad \mu_o j_z = \frac{1}{R}\frac{\partial F}{\partial R}$$

$$F \equiv RB_\phi$$

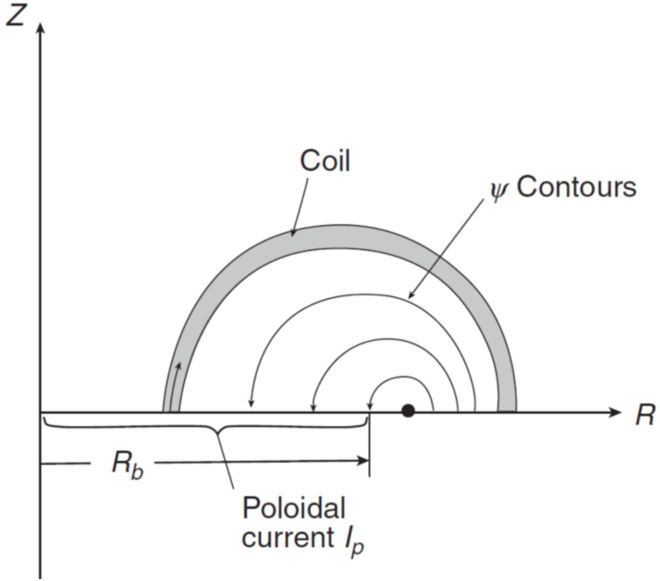
$$\mu_o j_R = -\frac{\partial B_\phi}{\partial z} = -\frac{1}{R}\frac{\partial}{\partial z}\left(RB_\phi\right) = -\frac{1}{R}\frac{\partial F}{\partial z}$$

$$\mu_o \vec{j} = \hat{R}\mu_o j_R + \hat{\phi}\mu_o j_\phi + \hat{z}\mu_o j_z = \hat{R}\left(-\frac{1}{R}\frac{\partial F}{\partial z}\right) + \hat{\phi}\left(-\frac{1}{R}\Delta^*\psi\right) + \hat{z}\left(\frac{1}{R}\frac{\partial F}{\partial R}\right)$$

$$\equiv \underbrace{\left(\frac{\nabla F}{R}\right)\times\hat{\phi}}_{j_\mathrm{P}} + \underbrace{\left(-\frac{1}{R}\Delta^*\psi\right)\hat{\phi}}_{j_\mathrm{T}}$$

$$I_\mathrm{P} = \int \vec{j}_\mathrm{P}\cdot d\vec{S} = -\int_0^{2\pi} d\phi \int_0^{R_\mathrm{b}} dR R j_z(R,0)$$

$$= -2\pi \int_0^{R_\mathrm{b}} dR R \frac{1}{R}\frac{\partial F(R,0)}{\partial R} = -2\pi F(\psi)$$



Plasma condition can be obtained by solving Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi} \qquad \text{where } \Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

- **The usual strategy to solve the Grad-Shafranov equation:**

  1. **Specify two free functions, the plasma pressure $p = p(\psi)$ and the toroidal field function $F = F(\psi)$.**
  2. **Solve the equation with specified boundary conditions to determine the flux function $\psi(R, z)$.**
  3. **Calculation the magnetic field using the following equations:**

$$B_\mathrm{R} = -\frac{1}{R}\frac{\partial\psi}{\partial z} \qquad B_\phi = \frac{F(\psi)}{R} \qquad B_\mathrm{z} = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

  4. **The pressure profile can then be obtained from $p = p\big(\psi(R, z)\big)$.**

Example of the analytical solution of the Grad-Shafranov equation

$$\Delta^*\psi = -\mu_o R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$

- For $\quad \mu_o \frac{dp}{d\psi} = -C_2 \qquad \frac{1}{2}\frac{dF^2}{d\psi} = C_1$

$$\psi(R,z) = -\frac{C_1}{2}z^2 + \frac{C_2}{8}R^4 + C_3 + C_4 R^2 + C_5\left(R^4 - 4R^2 z^2\right)$$

$$B_R(R,z) = -\frac{1}{R}\left(-C_1 z - 8C_5 R^2 z\right) \qquad B_z(R,z) = \frac{1}{R}\left(\frac{C_2}{2}R^3 + 2C_4 R + C_5\left(4R^3 - 8Rz^2\right)\right)$$

$C_1 = 1$

$C_2 = -8$

$C_3 = -20$

$C_4 = 20$

$C_5 = 0.2$