Introduction to Nuclear Fusion as An Energy Source

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Lecture 5

2024 spring semester

Wednesday 9:10-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

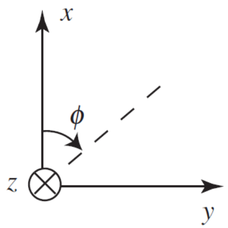
https://nckucc.webex.com/nckucc/j.php?MTID=ma76b50f97b1c6d72db61de9eaa9f0b27

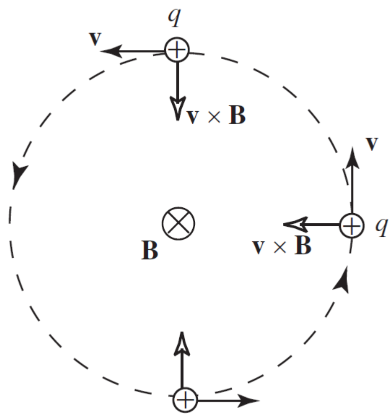
2024/4/03 updated

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Charged particles gyro around the magnetic field line





$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \qquad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

$$\omega_c \equiv \frac{|q|B}{m}$$ Cyclotron frequency or gyrofrequency

$$\ddot{v}_x + {\omega_c}^2 v_x = 0$$

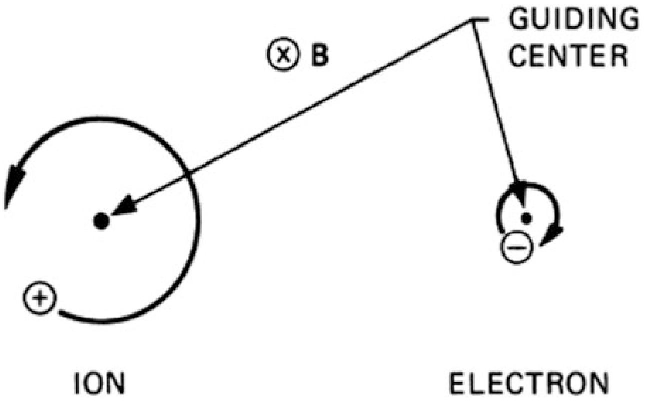
$$\ddot{v}_y + {\omega_c}^2 v_y = 0$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

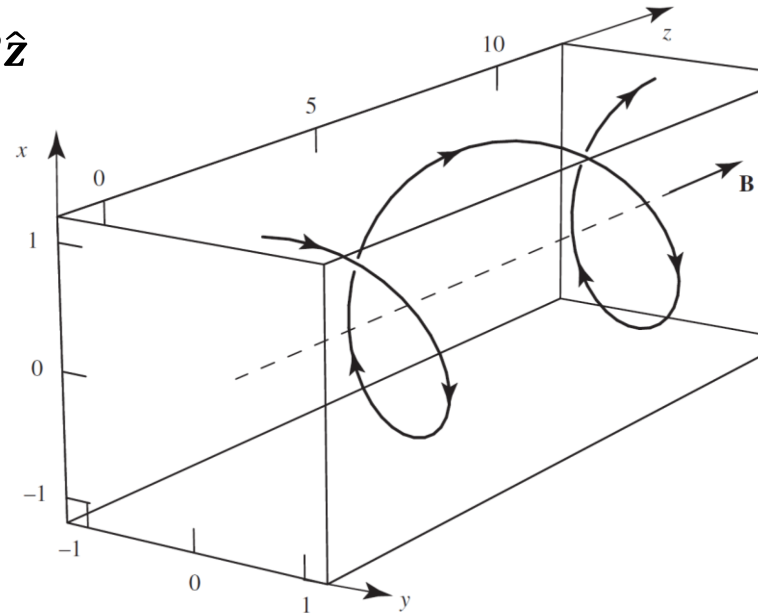
$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$\vec{B} = B\hat{z}$



$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left|\frac{m{v_\perp}^2}{r}\right| = \left|q\,\vec{v} \times \vec{B}\right| = |q v_\perp B|$$

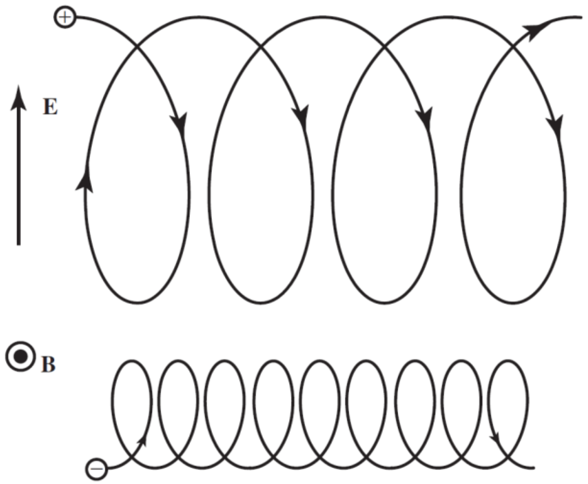
$$r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$$ **Larmor radius or gyroradius**

$$x = \mp r_c sin(\pm\omega_c t + \psi) + (x_o - r_c sin\psi)$$

$$y = \pm r_c cos(\pm\omega_c t + \psi) + (y_o + r_c cos\psi)$$

$$z = z_o + v_{||} t$$

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$v_{||}(t) = \frac{qE_{||}}{m}t + v_{||,o}$$

$$\vec{v}_\perp(t) = \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$m\frac{d}{dt}\left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) = q\left[\hat{x}E_\perp + \left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) \times \hat{z}B\right]$$

$$m\frac{d\,\vec{v}_{\mathrm{ac}}(t)}{dt} = q\left[\hat{x}E_\perp + \vec{v}_{\mathrm{E}} \times \hat{z}B + \vec{v}_{\mathrm{ac}}(t) \times \hat{z}B\right]$$

No E field case: $m\frac{d\,\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$

$$\hat{x}E_\perp + \vec{v}_{\mathrm{E}} \times \hat{z}B = 0$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$m\frac{d\,\vec{v}_{\mathrm{ac}}(t)}{dt} = q\,\vec{v}_{\mathrm{ac}}(t) \times \hat{z}B$$

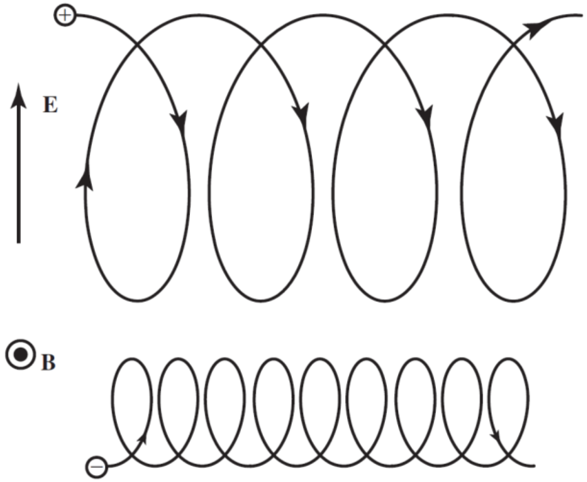
Gyro motion

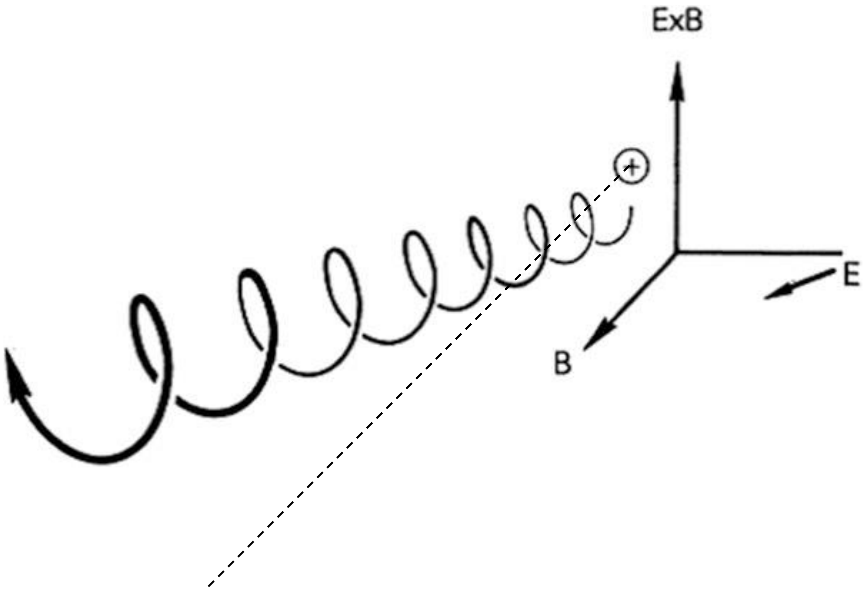
$$\vec{v}(t) = \hat{z}v_{||}(t) + \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$\langle\vec{v}(t)\rangle = \frac{1}{T_c}\int_0^{T_c}\vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

- **Electrons and ions drift in the same direction.**

No current is generated in ExB drift





$$\langle \vec{v}(t) \rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E}\times\vec{B}}{B^2}$$ **ExB drift velocity**

- **Electrons and ions drift in the same direction.**

Gravitational drift

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs

$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{||} = \hat{x}E_{\perp} + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_{\perp} + \hat{z}F_{||} = \hat{x}F_{\perp} + \hat{z}F_{||}$$

$$m\frac{dv_{||}}{dt} = F_{||}$$

$$m\frac{d\,\vec{v}_{\perp}}{dt} = q\left(\hat{x}\frac{F_{\perp}}{q} + \vec{v}_{\perp} \times \hat{z}B\right)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_F$$

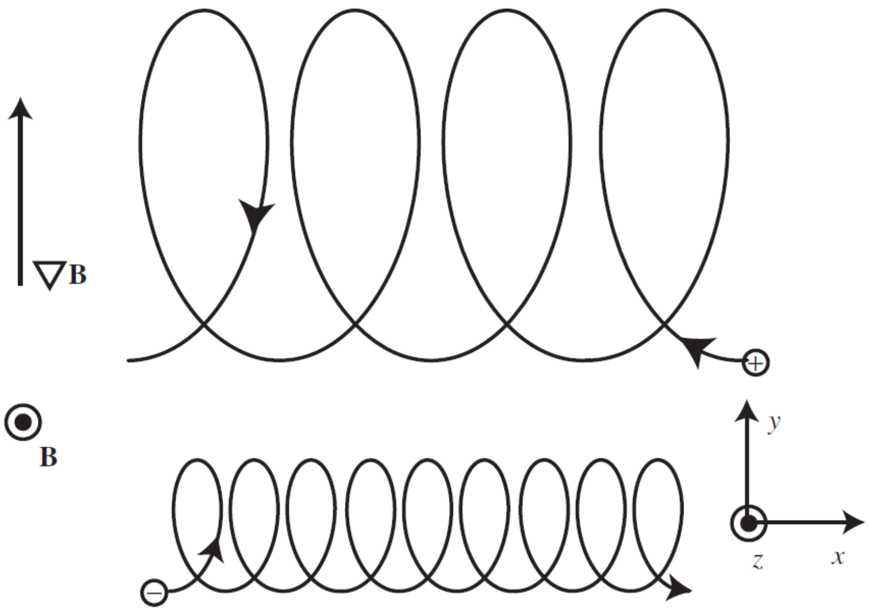
$$\vec{v}_{\mathrm{F}} = \frac{\hat{x}(F_{\perp}/q) \times \hat{z}B}{B^2} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

Gravitational drift velocity

- **Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.**

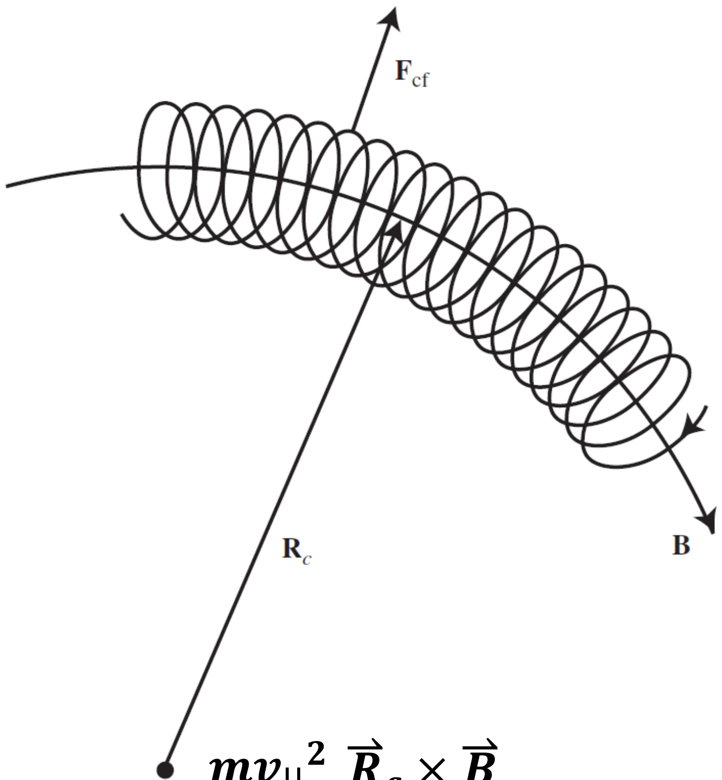
Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**



$$\vec{v}_{\nabla} = \frac{m{v_{\perp}}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$$

- **Curvature drift**

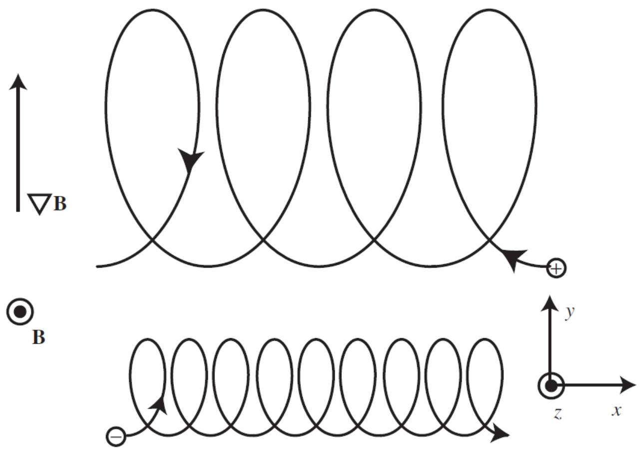


$$\vec{v}_{R} = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_{R} + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right)$$

Gradient-B drift

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm\omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm\omega_c t + \psi)$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$

$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x} q v_y B_z - \hat{y} q v_x B_z$$

$$\simeq \hat{x} q v_y \left(B_o + y \frac{\partial B_z}{\partial y}\right) - \hat{y} q v_x \left(B_o + y \frac{\partial B_z}{\partial y}\right)$$

$$B_z(y) = B_o + y \frac{\partial B_z}{\partial y} + y^2 \frac{1}{2} \frac{\partial^2 B_z}{\partial y^2} + \ldots$$

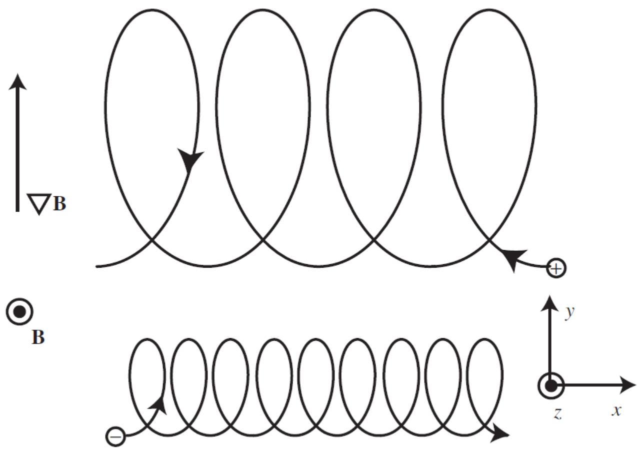
$$F_x = q v_y \left(B_o + y \frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_x \left(B_o + y \frac{\partial B_z}{\partial y}\right)$$

$$F_x \simeq -q v_\perp \sin(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_\perp \cos(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y}\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_\perp \sin(\pm\omega_c t + \psi)\left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$F_y \simeq -qv_\perp \cos(\pm\omega_c t + \psi)\left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

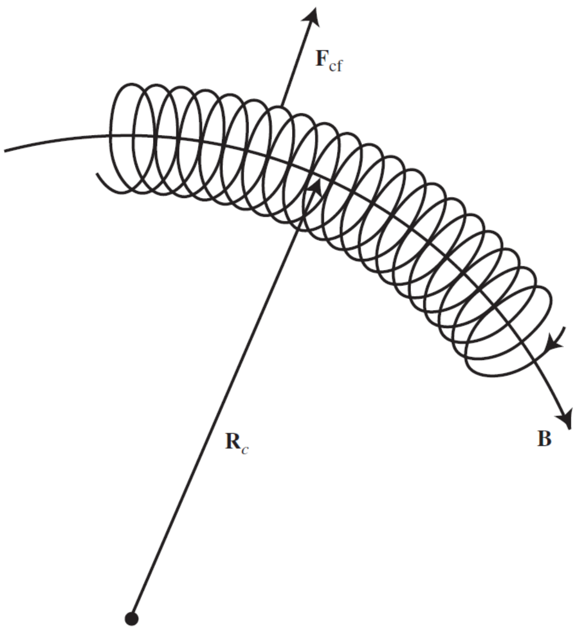
$$\langle F_x \rangle = 0$$

$$\langle F_y \rangle = \mp \frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} = -\frac{m{v_\perp}^2}{2B}\frac{\partial B_z}{\partial y} \qquad r_c = \frac{v_\perp}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \qquad \vec{v}_\nabla = \frac{1}{q}\frac{\langle F_y\rangle\hat{y}\times\hat{z}B_z}{{B_z}^2} = -\frac{m{v_\perp}^2}{2q{B_z}^2}\frac{\partial B_z}{\partial y}\hat{x}$$

- More general: $\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$

Charge particles drift across magnetic field lines when the magnetic field line is curved

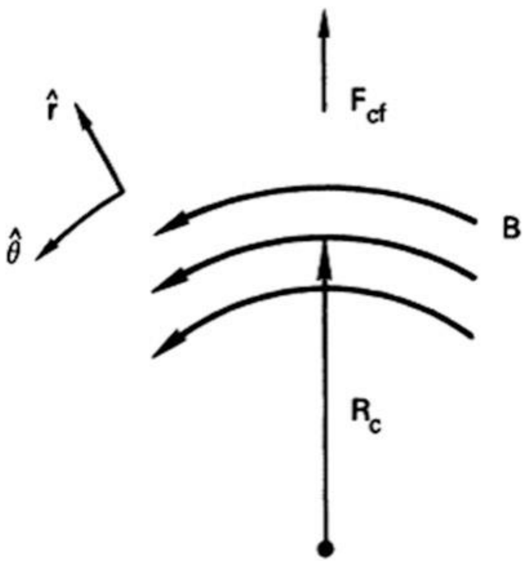


$$\vec{F}_{cf} = m{v_{||}}^2 \frac{\vec{R}_c}{{R_c}^2}$$

$$\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q}\frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m{v_{||}}^2}{q}\frac{\vec{R}_c \times \vec{B}}{{R_c}^2 B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B\hat{r}$$

Cylindrical coordinate

$$\vec{v}_\nabla = \frac{m v_\perp^2}{2q}\frac{\vec{B}\times\nabla B}{B^3} \qquad \vec{v}_R = \frac{m v_{||}^2}{q}\frac{\vec{R}_c\times\vec{B}}{R_c^2 B^2}$$

- In a vacuum, $\nabla\times\vec{B} = 0$

$$(\nabla\times\vec{B})_r = (\nabla\times\vec{B})_\theta = 0$$

$$\nabla\times\vec{B} = (\nabla\times\vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \quad B_\theta \propto \frac{1}{r} \equiv \frac{\text{Const}}{r}$$

$$\nabla|B| = \nabla B_\theta\hat{r} = -\frac{\text{Const}}{r^2}\hat{r} = -\frac{B_\theta}{r}\hat{r} = -\frac{|B|}{R_c^2}\vec{R}_c$$

$$\frac{\nabla|B|}{|B|} = -\frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_{\text{R}} + \vec{v}_\nabla = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left(v_{||}^2 + \frac{1}{2}v_\perp^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{R_c^2 B^2}\left(v_{||}^2 + \frac{1}{2}v_\perp^2\right)$$

- **Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.**

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E}\times\vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_R = \frac{m{v_{||}}^2}{q}\frac{\vec{R}_{\mathrm{c}}\times\vec{B}}{{R_c}^2B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_{\mathrm{c}}B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_{\mathrm{c}}\times\vec{B}}{{R_{\mathrm{c}}}^2B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right)$$

Magnetohydrodynamics description of plasma

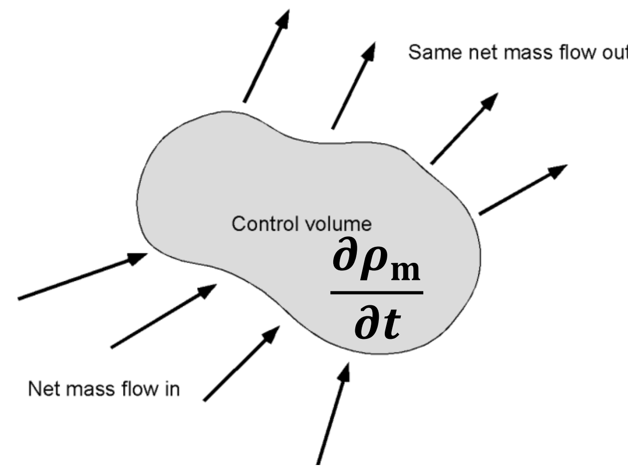
- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \rho_q \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

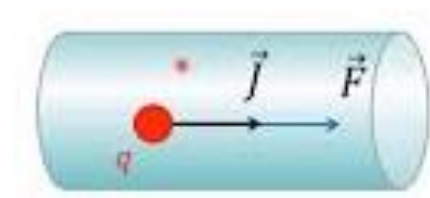
$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$





https://eaglepubs.erau.edu/introductiontoaerospaceflightvehicles/chapter/conservation-of-mass-continuity-equation/
https://www.youtube.com/watch?v=lu0Ep8_Gp8U

Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation

- **Continuity eq:** $\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ **w/ long wavelength ($\lambda >> \lambda_d$)**
- **Momentum eq:** $\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \cancel{\rho_{\mathrm{q}} \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$
- **Ohm's law:** $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt}\left(\frac{P}{\rho_{\mathrm{m}}{}^{\gamma}}\right) = 0$
- **Maxwell's eqs:**

$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{q}}}{\epsilon_o} \approx 0$ **w/ long wavelength ($\lambda >> \lambda_d$) => quasi neutral**

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j} + \cancel{\epsilon_{\mathrm{o}} \mu_{\mathrm{o}} \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega << \omega_{\mathrm{pe}}$)

Ideal MHD

- Continuity eq: $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- Momentum eq: $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- Ohm's law: $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- Equation of state: $$\frac{d}{dt} \left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$$
- Maxwell's eqs:

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

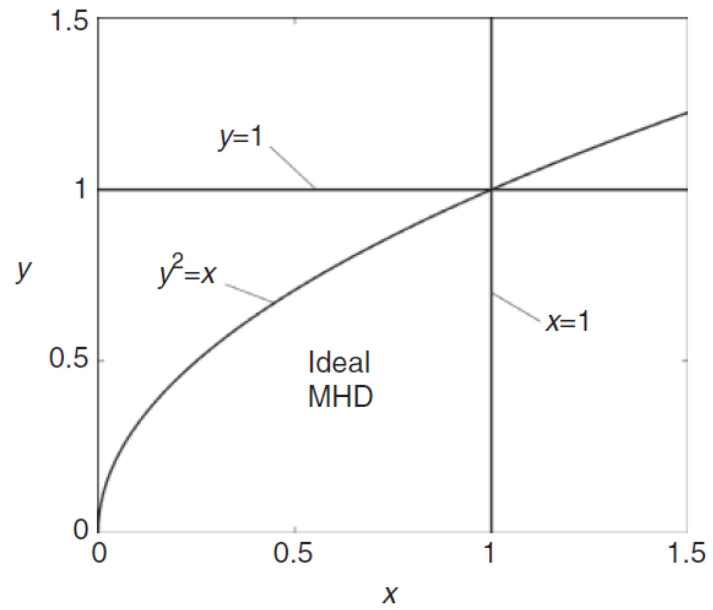
$$\nabla \cdot \vec{j} = 0$$

- Requirement:
  - High collisionality – fluid model
  - Small gyro radius – low frequency
  - Small resistivity – a perfect conductor

Conflict!

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a} \frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

Region of validity for ideal MHD



$$x = \left(\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad \lambda_i \sim v_{\mathrm{Ti}}\tau_{\mathrm{ii}}$$

$$y = \frac{r_{\mathrm{Li}}}{a}$$

- **Requirement:**
  - **High collisionality** $x \ll 1$
  - **Small gyro radius** $y \ll 1$
  - **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad \eta \vec{j} = \vec{E} + \vec{v} \times \vec{B} \qquad \frac{|\eta j|}{|v \times B|} \sim ?$$

$$|j \times B| \sim |\nabla p| \qquad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a}\frac{nT}{B} \sim \frac{1}{a}\frac{n m_i {v_{Ti}}^2}{B} \qquad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}}$$

$$\eta \sim \frac{m_e}{ne^2 \tau_{ei}} \qquad \tau_{ei} \sim \tau_{ee} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \tau_{ii} \qquad k \sim \nabla \sim \frac{1}{a}$$

$$\frac{|\eta j|}{|v \times B|} \sim \frac{\eta j}{v_{Ti} B} \sim \frac{m_e}{ne^2 \tau_{ei}} \frac{1}{a} \frac{n m_i {v_{Ti}}^2}{B} \frac{1}{v_{Ti} B} = \frac{m_e}{\tau_{ei}} \frac{v_{Ti}}{a} \frac{m_i}{e^2 B^2} = \frac{m_e}{m_i \tau_{ei}} \frac{v_{Ti}}{a} \frac{{m_i}^2}{e^2 B^2} = \frac{m_e}{m_i \tau_{ei}} \frac{v_{Ti}}{a {\omega_{ci}}^2}$$

$$\sim \frac{m_e}{m_i \tau_{ii}} \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}}{a {\omega_{ci}}^2} = \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{Ti}}{\tau_{ii} a} \frac{{r_{Li}}^2}{{v_{Ti}}^2} = \left(\frac{m_e}{m_i}\right)^{1/2} \frac{1}{\tau_{ii} a} \frac{{r_{Li}}^2}{v_{Ti}} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{1}{\omega \tau_{ii}} \left(\frac{r_{Li}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

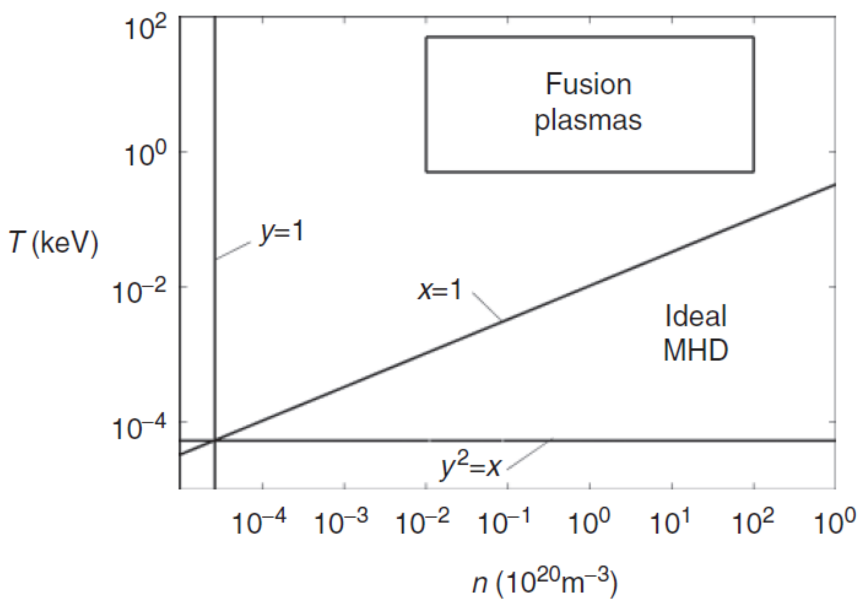
Fusion plasma is not in the ideal MHD region!

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad y = \frac{r_{Li}}{a}$$

$$10^{18}\ \mathrm{m}^{-3} < n < 10^{22}\ \mathrm{m}^{-3}$$

$$0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$$

$$\beta \equiv \frac{2\mu_o n T}{B^2}$$



- Requirement:
  - High collisionality $x = 3\times 10^3 \frac{T^2}{an} << 1$
  - Small gyro radius $y = 2.3\times 10^{-2}\left(\frac{\beta}{na^2}\right)^{1/2} << 1$
  - Small resistivity $\frac{y^2}{x} = 1.8\times 10^{-7}\frac{\beta}{aT^2} << 1$

- **With strong B, the gyromotion mimic the collisional characteristics.**

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Additional simplification of the momentum equation

- **Momentum eq:** $\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] = \vec{j}\times\vec{B} - \nabla\cdot\overleftrightarrow{P}$

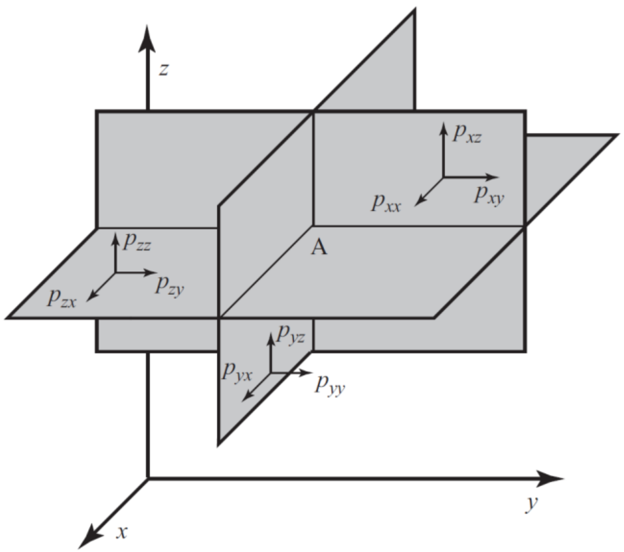
$$\nabla\cdot\overleftrightarrow{P} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{\mathrm{xx}} & p_{\mathrm{xy}} & p_{\mathrm{xz}}\\ p_{\mathrm{yx}} & p_{\mathrm{yy}} & p_{\mathrm{yz}}\\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & p_{\mathrm{zz}}\end{pmatrix} = \begin{pmatrix}\frac{\partial p_{\mathrm{xx}}}{\partial x}+\frac{\partial p_{\mathrm{yx}}}{\partial y}+\frac{\partial p_{\mathrm{zx}}}{\partial z}\\ \frac{\partial p_{\mathrm{xy}}}{\partial x}+\frac{\partial p_{\mathrm{yy}}}{\partial y}+\frac{\partial p_{\mathrm{zy}}}{\partial z}\\ \frac{\partial p_{\mathrm{xz}}}{\partial x}+\frac{\partial p_{\mathrm{yz}}}{\partial y}+\frac{\partial p_{\mathrm{zz}}}{\partial z}\end{pmatrix}$$

$$\nabla\cdot\overleftrightarrow{P} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{\mathrm{xx}} & 0 & 0\\ 0 & p_{\mathrm{yy}} & 0\\ 0 & 0 & p_{\mathrm{zz}}\end{pmatrix} + \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}0 & p_{\mathrm{xy}} & p_{\mathrm{xz}}\\ p_{\mathrm{yx}} & 0 & p_{\mathrm{yz}}\\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & 0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\partial p_{\mathrm{xx}}}{\partial x}\\ \frac{\partial p_{\mathrm{yy}}}{\partial y}\\ \frac{\partial p_{\mathrm{zz}}}{\partial z}\end{pmatrix} + \begin{pmatrix}\frac{\partial p_{\mathrm{yx}}}{\partial y}+\frac{\partial p_{\mathrm{zx}}}{\partial z}\\ \frac{\partial p_{\mathrm{xy}}}{\partial x}+\frac{\partial p_{\mathrm{zy}}}{\partial z}\\ \frac{\partial p_{\mathrm{xz}}}{\partial x}+\frac{\partial p_{\mathrm{yz}}}{\partial y}\end{pmatrix}$$

Additional simplification of the momentum equation

$$\nabla\cdot\overleftrightarrow{P}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{xx} & 0 & 0\\ 0 & p_{yy} & 0\\ 0 & 0 & p_{zz}\end{pmatrix}+\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}0 & p_{xy} & p_{xz}\\ p_{yx} & 0 & p_{yz}\\ p_{zx} & p_{zy} & 0\end{pmatrix}$$



$$=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{xx} & 0 & 0\\ 0 & p_{yy} & 0\\ 0 & 0 & p_{zz}\end{pmatrix}+\underbrace{\begin{pmatrix}\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\\ \frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{zy}}{\partial z}\\ \frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}\end{pmatrix}}_{\text{Viscosity}\quad \nabla\cdot\overleftrightarrow{\Pi}}$$

- **Isotropic plasma:** $p_{xx}=p_{yy}=p_{zz}\equiv p$

$$\begin{pmatrix}p_{xx} & 0 & 0\\ 0 & p_{yy} & 0\\ 0 & 0 & p_{zz}\end{pmatrix}\equiv p\overleftrightarrow{1}$$

$$\nabla\cdot\overleftrightarrow{P}=\nabla p+\nabla\cdot\overleftrightarrow{\Pi}$$

$$\rho_m\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=\vec{j}\times\vec{B}-\nabla p-\nabla\cdot\overleftrightarrow{\Pi}$$

Viscosity is negligible in a collision-dominated plasma

$$\overleftrightarrow{\boldsymbol{\Pi}} \sim \mu \left(2\nabla_{\parallel} \cdot \vec{v}_{\parallel} - \frac{2}{3} \nabla \cdot \vec{v} \right) \sim \mu \frac{v_{\mathrm{Ti}}}{a} \qquad \mu \sim n T_{\mathrm{i}} \tau_{\mathrm{ii}}$$

$$\left| \frac{\nabla \cdot \overleftrightarrow{\boldsymbol{\Pi}}}{\nabla p} \right| \sim \frac{\overleftrightarrow{\boldsymbol{\Pi}} a}{ap} \sim \frac{n T_{\mathrm{i}} \tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{ap} \sim \frac{\tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{a} \sim \frac{\lambda_i}{a} \ll 1 \qquad \lambda_i \sim v_{\mathrm{Ti}} \tau_{\mathrm{ii}}$$

$$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\boldsymbol{\Pi}}$$

⇓

$$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt} \left(\frac{P}{\rho_{\mathrm{m}}{}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"

- **Equilibrium state:**

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} \right] = \vec{j} \times \vec{B} - \nabla p \equiv 0$$

$$\vec{j} \times \vec{B} = \nabla p$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{\mu_o}\left[(\vec{B} \cdot \vec{\nabla})\,\vec{B} - \frac{1}{2}\vec{\nabla} B^2 \right] = \nabla p$$

$$\nabla \left(P + \frac{B^2}{2\mu_o} \right) = \frac{1}{\mu_o}(\vec{B} \cdot \vec{\nabla})\,\vec{B}$$

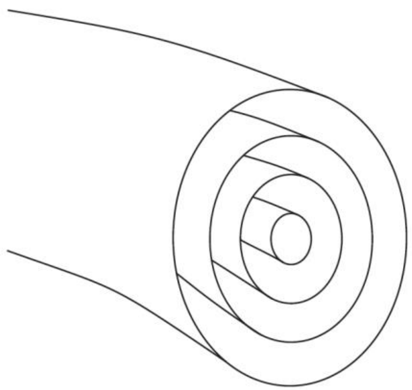
Magnetic pressure   **Magnetic tension** ← **Forces caused by curvature of the field lines**

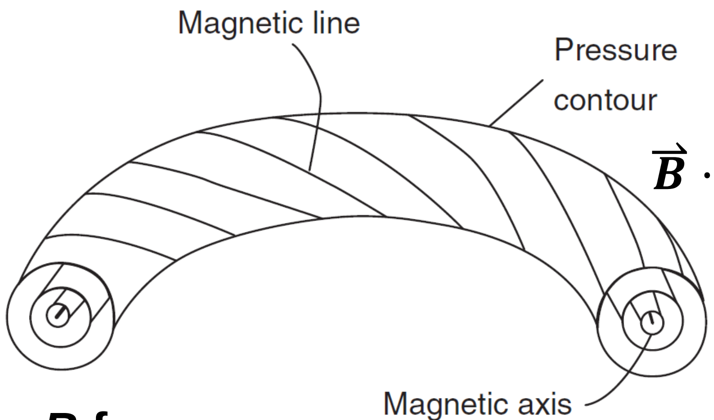
$\vec{j} \perp \nabla p$ $\vec{B} \perp \nabla p$ ⇒ $\vec{j} \cdot \nabla p = 0$ $\vec{B} \cdot \nabla p = 0$

- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

Magnetic lines lying on pressure contour

- **Contours of constant pressure**
- **Magnetic lines lying on pressure contour**

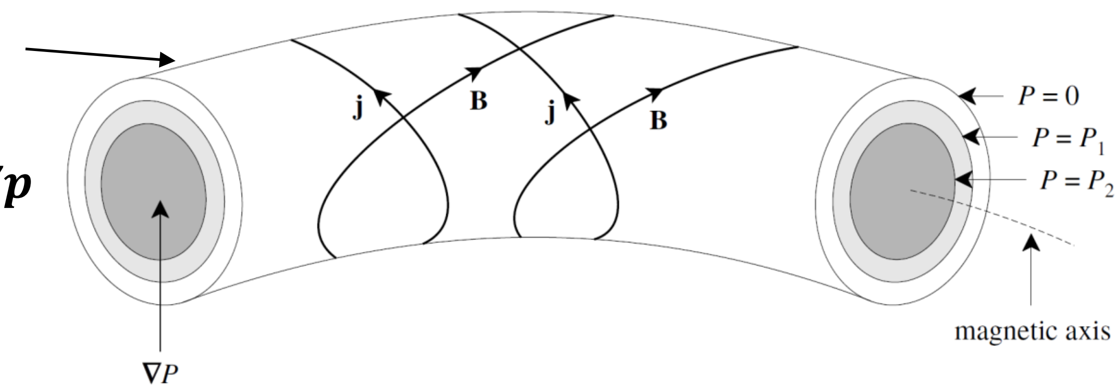




$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

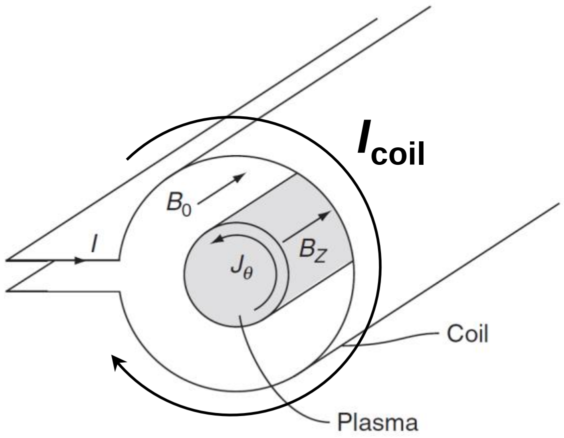
$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z\hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

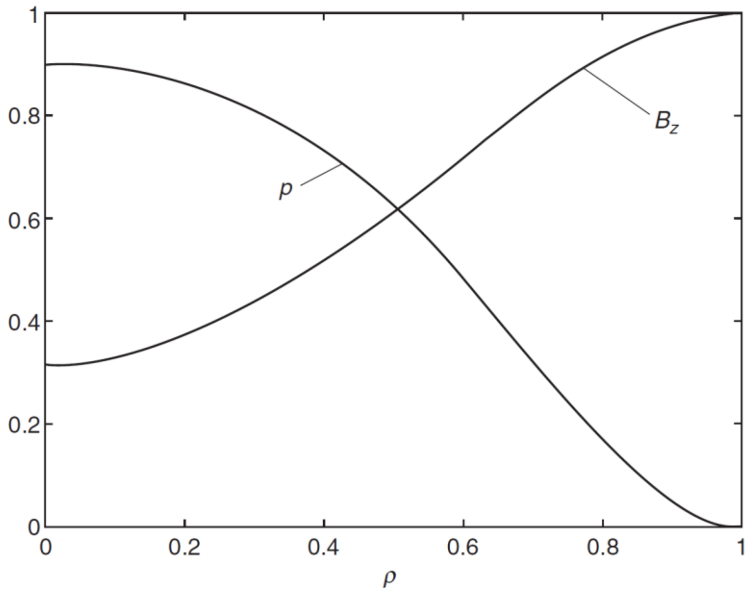
$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma



$$\frac{2\mu_o p(r)}{{B_o}^2} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$

$$\frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{{B_o}^2} \qquad \rho = \frac{r}{a}$$

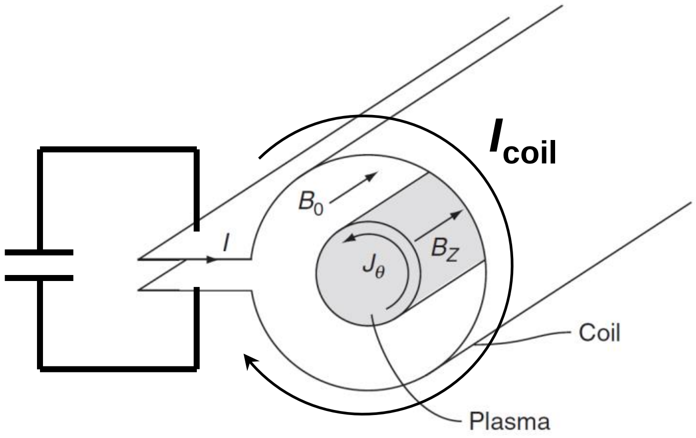
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{{B_o}^2} = \frac{4\mu_o}{a^2{B_o}^2}\int_0^a prdr = 2\int_0^1\left(1-\frac{{B_z}^2}{{B_o}^2}\right)\rho d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_o}{2}\ ,\beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad \Rightarrow \quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$0 < \beta < 1$

Theta pinches provide good radial confinement but NOT axially



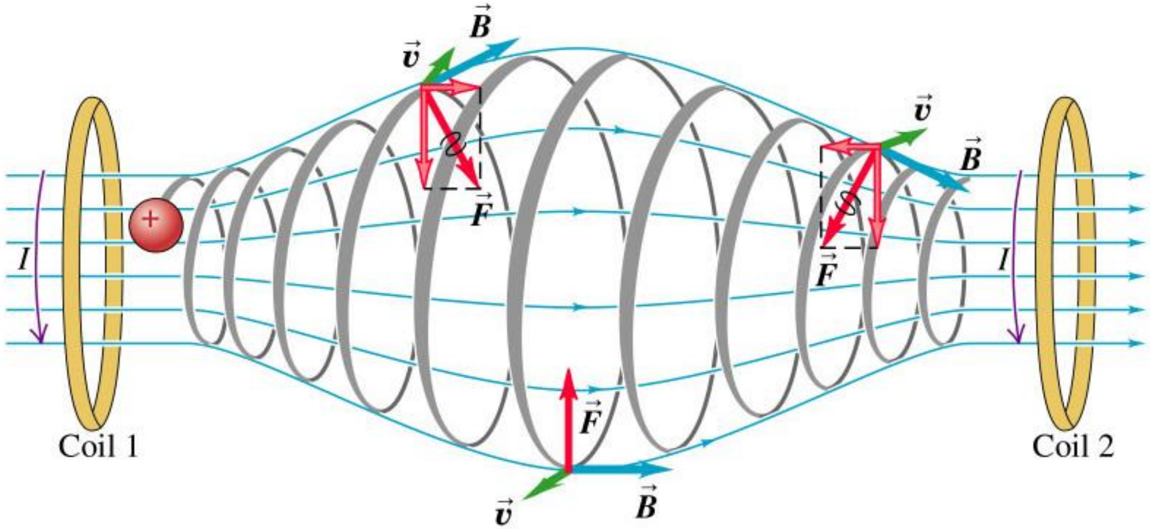
- **The gas is initially preionized.**
- **The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.**
- **The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.**

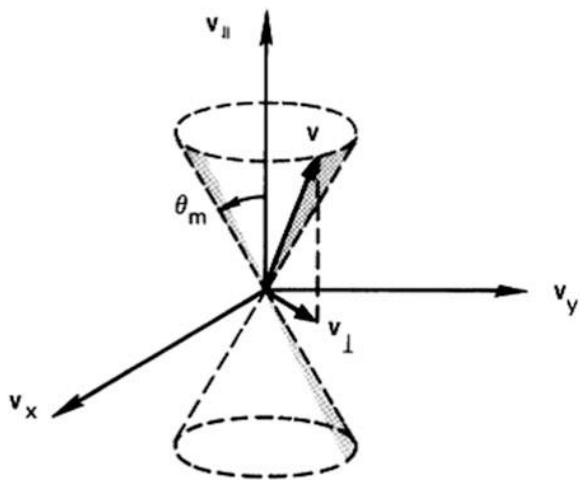
- **This energy is ultimately converted to heat after repeated reflections off the converging piston.**
- **T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.**
- **The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{Ti} \sim 10\mu s$ for L = 5 m.**

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.





$$v'_{\perp}{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_{\perp}{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv sin^2\theta$$

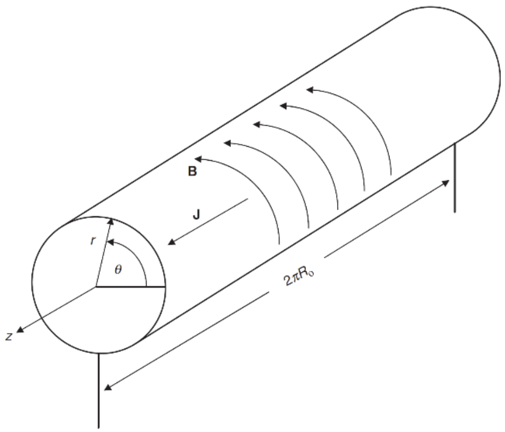
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_{\perp}{}^2 \quad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_{\perp}{}^2}{B}$$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = sin^2\theta_m$$

- Large $v_{||}$ may occur from collisions between particles.
- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$(\nabla \times \vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

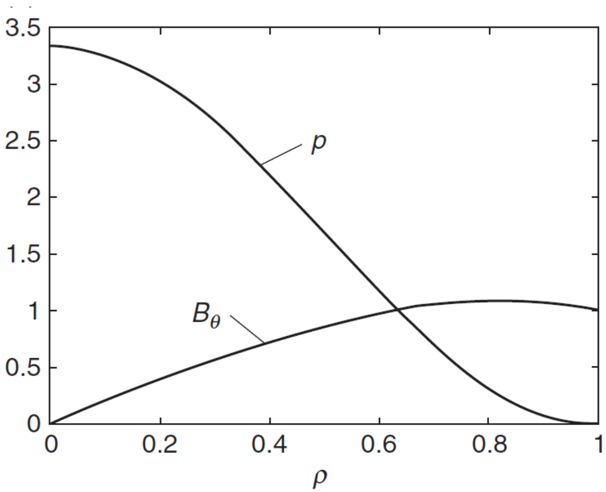
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

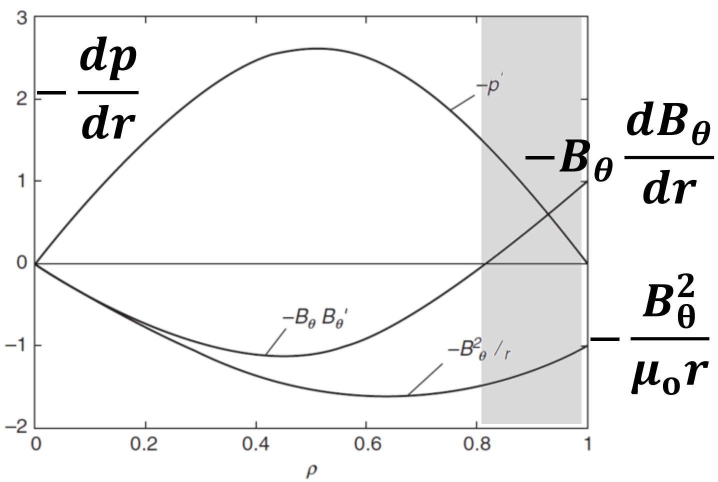
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β





$$\frac{d}{dr}\left(p+\frac{B_\theta^2}{2\mu_o}\right)+\frac{B_\theta^2}{\mu_o r}=0$$

$$\frac{2\mu_o p(r)}{B_{\theta a}{}^2}=\frac{2}{3}(5-2\rho^2)(1-\rho^2)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}}=2\rho\left(1-\frac{\rho^2}{2}\right)$$

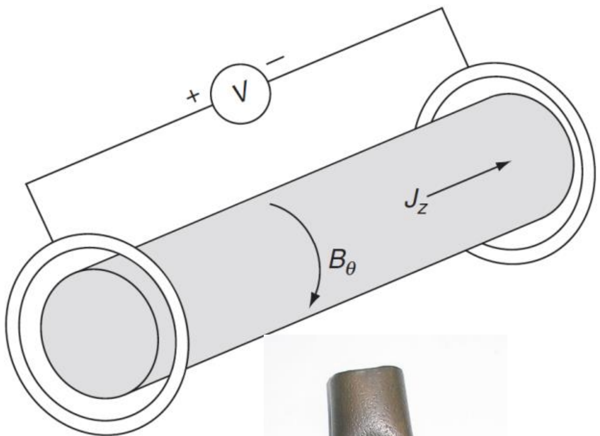
$$\frac{a\mu_o j_z(r)}{B_{\theta a}}=4(1-\rho^2)$$

$$B_{\theta a}\equiv B_\theta(a)=\frac{\mu_o I}{2\pi a}$$

$$\beta\equiv\beta_p=\frac{2\mu_o\langle p\rangle}{B_{\theta a}{}^2}=\frac{4\mu_o}{a^2 B_{\theta a}{}^2}\int_0^a p r dr=1$$

Bennett pinch relation: $\beta = 1$

Huge instabilities occur in a z pinch

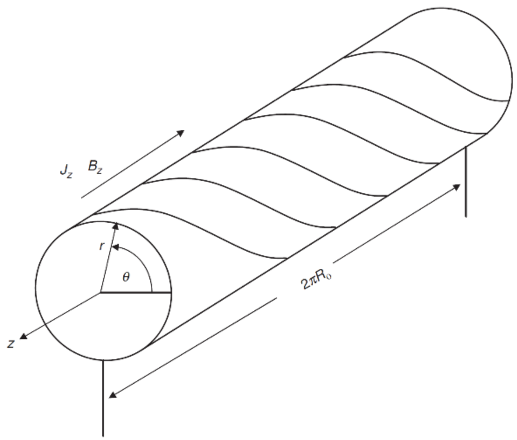


- A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.
- The gas is ionized by the high voltage and produces a z current flowing along the plasma.
- Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.

Main issue: unstable.

https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

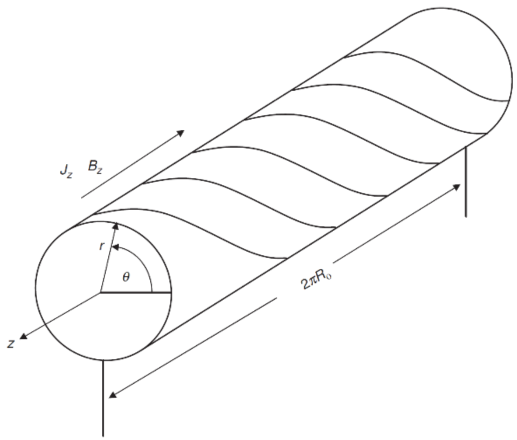
$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

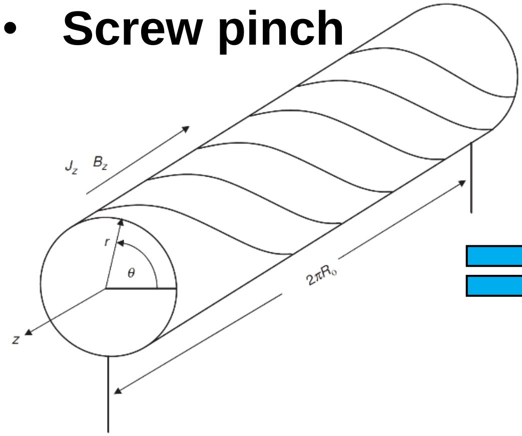
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

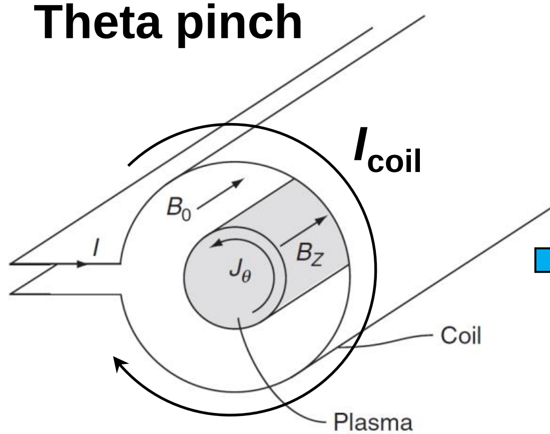
General screw pinch is flexible with varies range of β

- **Screw pinch**



=

- **Theta pinch**



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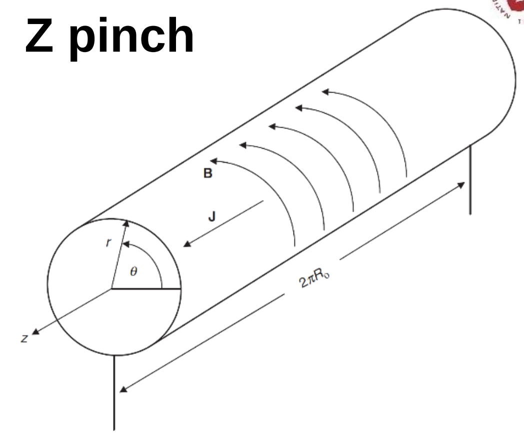
- **Z pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\vec{B} = B_z \hat{z}$$
$$\vec{j} = j_\theta \hat{\theta}$$

$$\vec{B} = B_\theta \hat{\theta}$$
$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

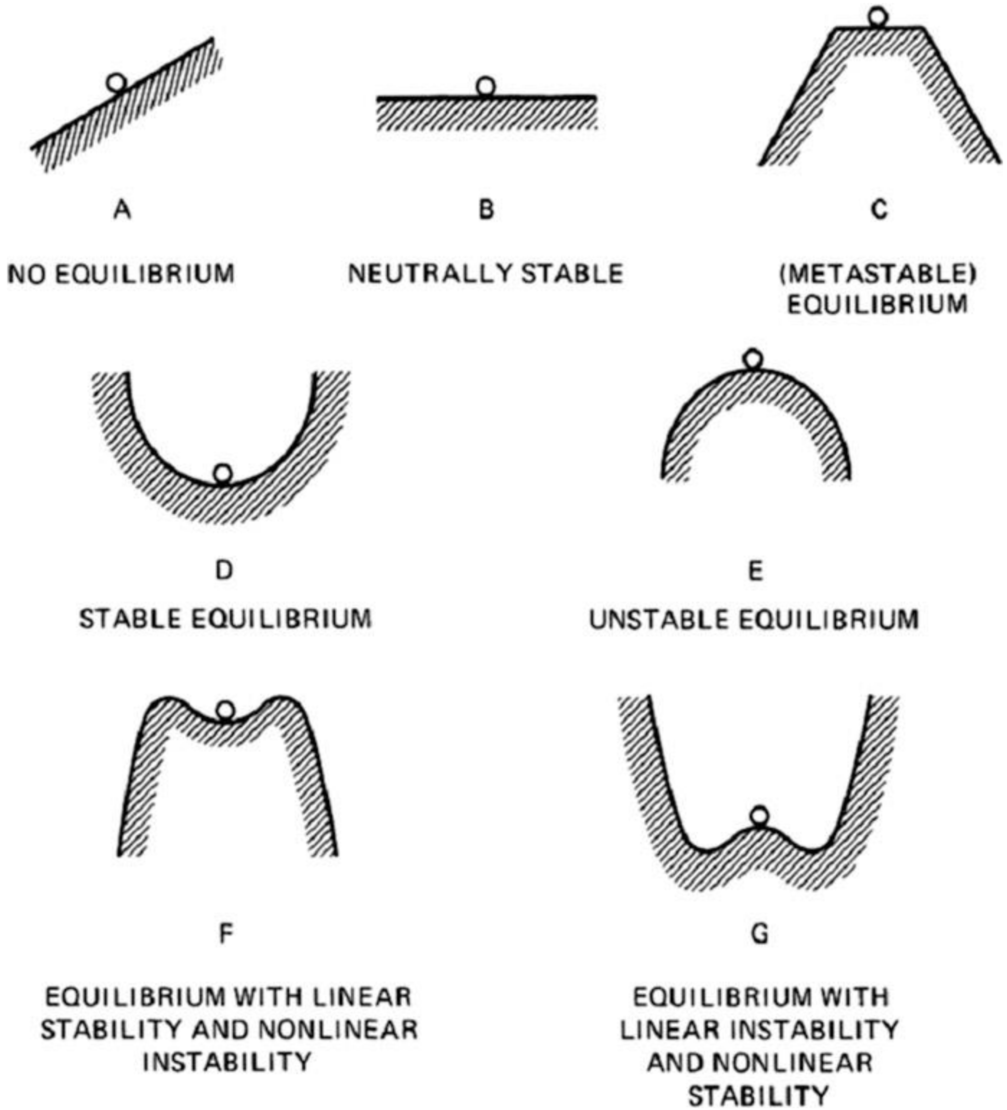
$$\int_0^a \pi r^2 dr \left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2 \rangle\right)$$

$$\beta_t = \frac{2\mu_o \langle p \rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o \langle p \rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_o \langle p \rangle}{B_o^2 + {B_{\theta a}}^2}$$

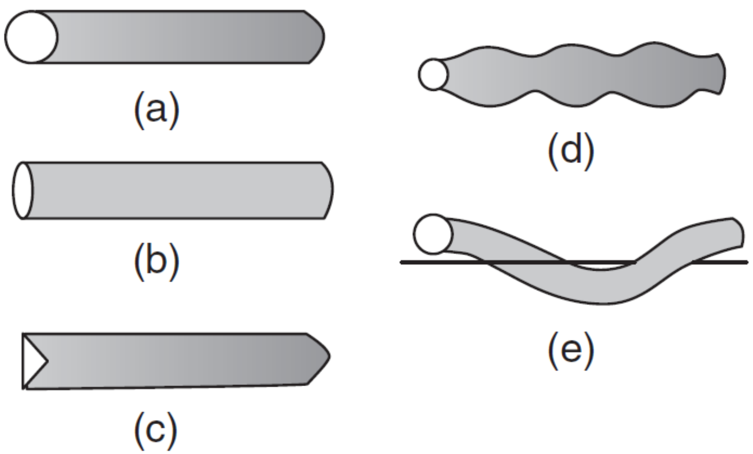
$$0 \le \langle \beta \rangle \le 1$$

An equilibrium state may not be stable



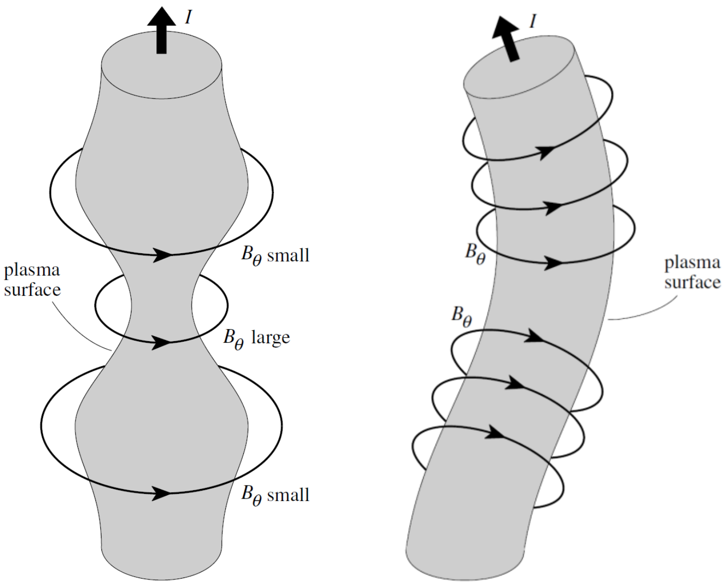
A cylindrical plasma column may not be stable

- **Instabilities of theta pinch**



(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0

- **Instabilities of z pinch**



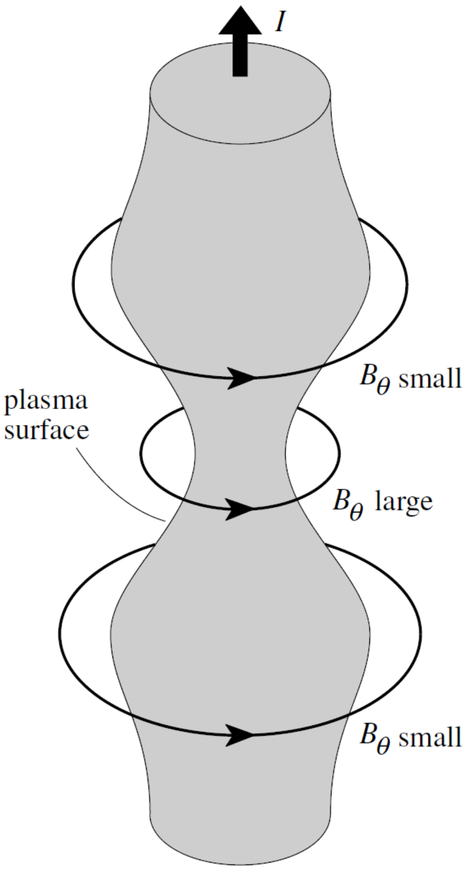
Sausage instability (m=0)

Kink instability (m=1)

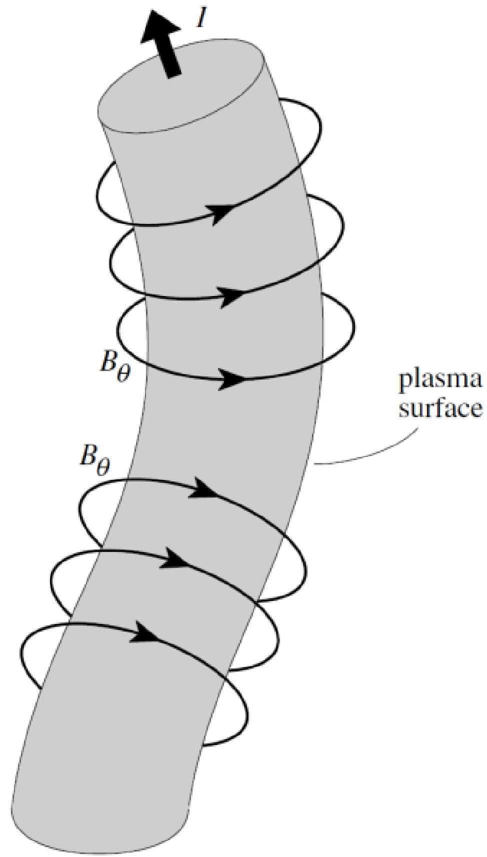
$$\zeta(\vec{r}) = \zeta(r)exp(im\theta + ikz)$$

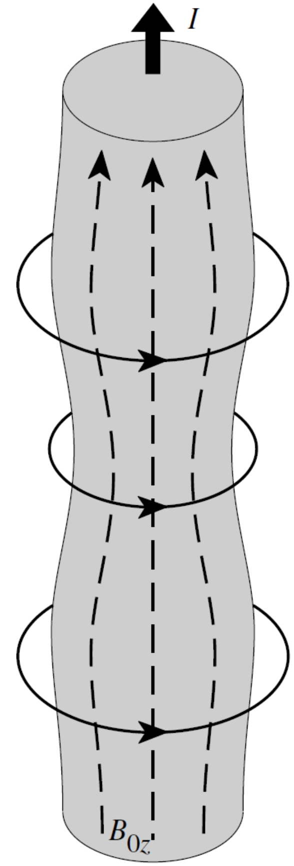
A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**



- **Kink instability**

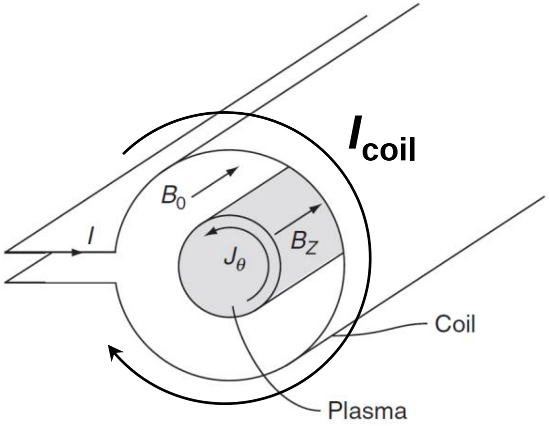


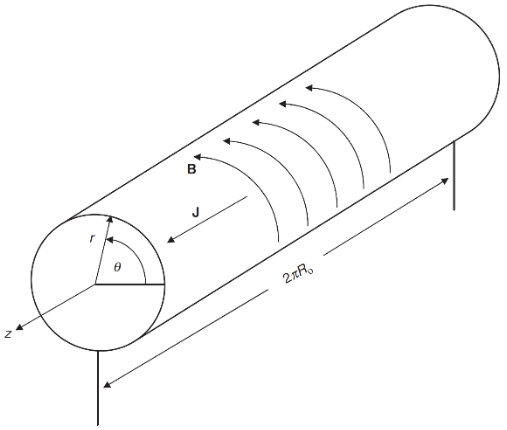


- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_o B_\theta(r)}$ **Kruskal–Shafranov limit**

Theta pinch is stable while z pinch is unstable

- **Theta pinch**
- **Z pinch**





$$\vec{B} = B_z\hat{z}$$

$$\vec{B} = B_\theta\hat{\theta}$$

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$

$$q_\theta = \infty$$

$$q_z = 0$$

Stable

Unstable