

Course Outline



- **Magnetic confinement fusion (MCF)**
 - Gyro motion, MHD
 - 1D equilibrium (z pinch, theta pinch)
 - Drift: ExB drift, grad B drift, and curvature B drift
 - Tokamak, Stellarator (toroidal field, poloidal field)
 - Magnetic flux surface
 - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
 - Stability (Kink instability, sausage instability, Safety factor Q)
 - Central-solenoid (CS) start-up (discharge) and current drive
 - CS-free current drive: electron cyclotron current drive, bootstrap current.
 - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Course Outline



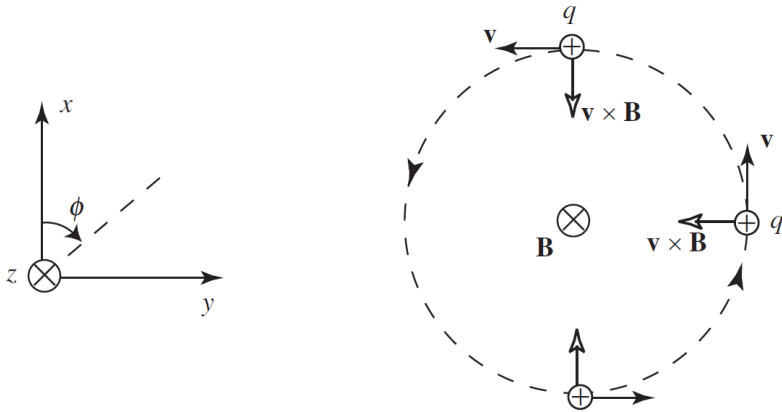
- **Magnetic confinement fusion (MCF)**
 - Gyro motion, MHD
 - 1D equilibrium (z pinch, theta pinch)
 - Drift: ExB drift, grad B drift, and curvature B drift
 - Tokamak, Stellarator (toroidal field, poloidal field)
 - Magnetic flux surface
 - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
 - Stability (Kink instability, sausage instability, Safety factor Q)
 - Central-solenoid (CS) start-up (discharge) and current drive
 - CS-free current drive: electron cyclotron current drive, bootstrap current.
 - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Course Outline



- **Magnetic confinement fusion (MCF)**
 - Gyro motion, MHD
 - 1D equilibrium (z pinch, theta pinch)
 - Drift: ExB drift, grad B drift, and curvature B drift
 - Tokamak, Stellarator (toroidal field, poloidal field)
 - Magnetic flux surface
 - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
 - Stability (Kink instability, sausage instability, Safety factor Q)
 - Central-solenoid (CS) start-up (discharge) and current drive
 - CS-free current drive: electron cyclotron current drive, bootstrap current.
 - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Charged particles gyro around the magnetic field line



$$m \frac{d\vec{v}}{dt} = q \vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \quad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m} \dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m} \dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

$$\omega_c \equiv \frac{|q|B}{m} \quad \text{Cyclotron frequency or gyrofrequency}$$

$$\ddot{v}_x + \omega_c^2 v_x = 0$$

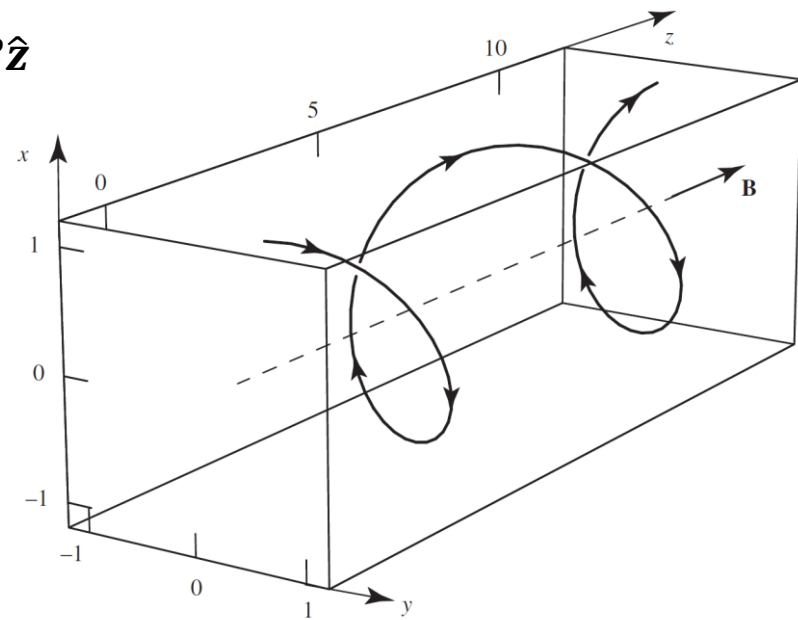
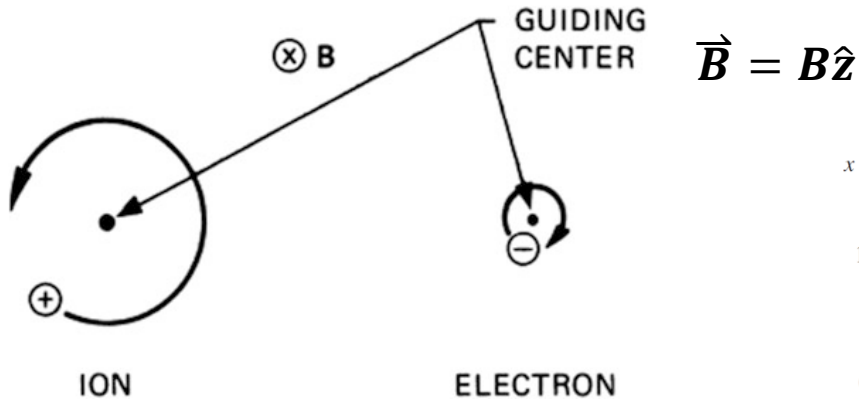
$$\ddot{v}_y + \omega_c^2 v_y = 0$$

$$v_x = v_{\perp} \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_{\perp} \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$v_x = v_{\perp} \cos(\pm\omega_c t + \psi)$$

$$v_y = v_{\perp} \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{\parallel}$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left| \frac{mv_{\perp}^2}{r} \right| = |q \vec{v} \times \vec{B}| = |qv_{\perp}B|$$

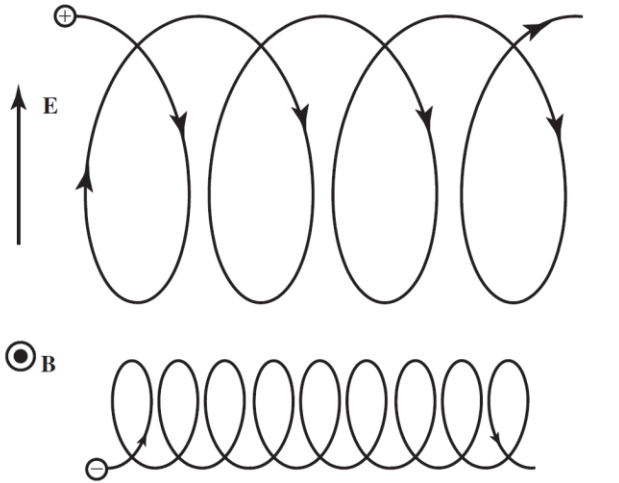
$$r_c = \frac{v_{\perp}}{\omega_c} = \frac{mv_{\perp}}{|q|B} \quad \text{Larmor radius or gyroradius}$$

$$x = \mp r_c \sin(\pm\omega_c t + \psi) + (x_0 - r_c \sin\psi)$$

$$y = \pm r_c \cos(\pm\omega_c t + \psi) + (y_0 + r_c \cos\psi)$$

$$z = z_0 + v_{\parallel} t$$

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_\perp + \hat{z}E_\parallel = \hat{x}E_\perp + \hat{z}E_\parallel$$

$$m \frac{dv_\parallel}{dt} = qE_\parallel$$

$$m \frac{d\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$v_\parallel(t) = \frac{qE_\parallel}{m}t + v_{\parallel,0}$$

$$\vec{v}_\perp(t) = \vec{v}_E + \vec{v}_{ac}(t)$$

$$m \frac{d}{dt} (\vec{v}_E + \vec{v}_{ac}(t)) = q[\hat{x}E_\perp + (\vec{v}_E + \vec{v}_{ac}(t)) \times \hat{z}B]$$

$$m \frac{d\vec{v}_{ac}(t)}{dt} = q[\hat{x}E_\perp + \vec{v}_E \times \hat{z}B + \vec{v}_{ac}(t) \times \hat{z}B]$$

No E field case: $m \frac{d\vec{v}}{dt} = q \vec{v} \times \vec{B}$



$$\hat{x}E_\perp + \vec{v}_E \times \hat{z}B = 0$$

$$\times \hat{z}B \quad (\vec{C} \times \vec{B}) \times \vec{A} = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$\vec{v}_E = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

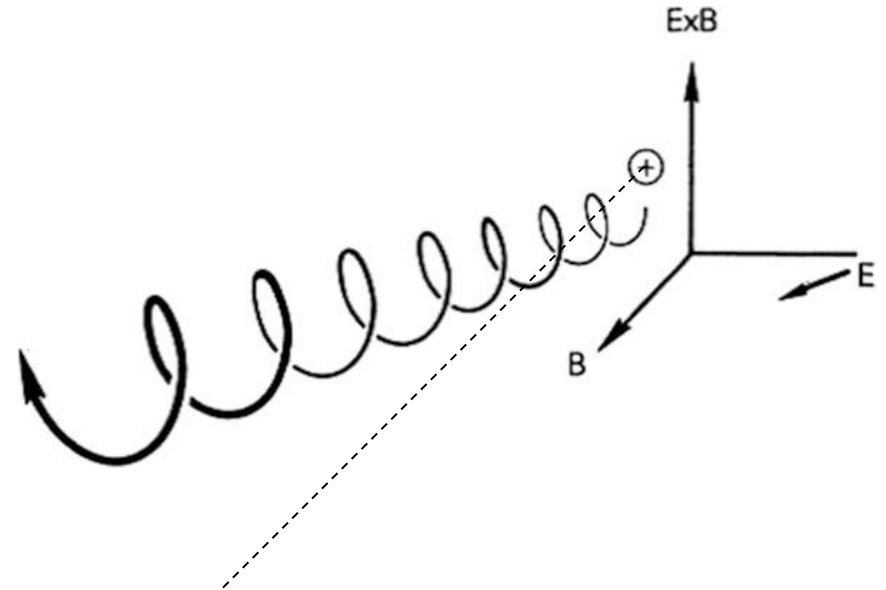
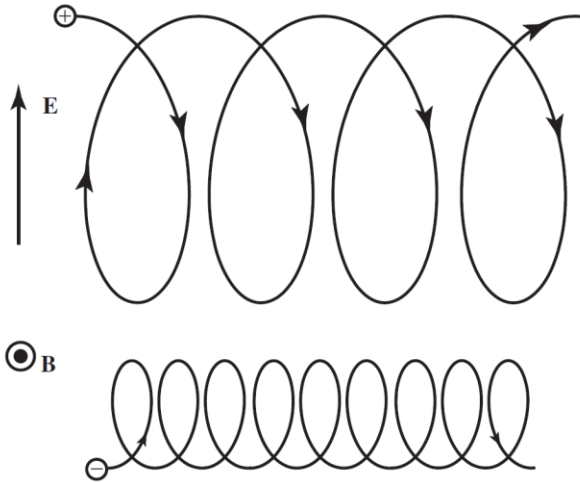
$$m \frac{d\vec{v}_{ac}(t)}{dt} = q \vec{v}_{ac}(t) \times \hat{z}B \quad \text{Gyro motion}$$

$$\vec{v}(t) = \hat{z}v_\parallel(t) + \vec{v}_E + \vec{v}_{ac}(t)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_\parallel(t) + \vec{v}_E$$

• Electrons and ions drift in the same direction.

No current is generated in ExB drift



$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2} \quad \text{ExB drift velocity}$$

- Electrons and ions drift in the same direction.

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{\parallel} = \hat{x}E_{\perp} + \hat{z}E_{\parallel}$$

$$m \frac{dv_{\parallel}}{dt} = qE_{\parallel}$$

$$m \frac{d\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_{\perp} + \hat{z}F_{\parallel} = \hat{x}F_{\perp} + \hat{z}F_{\parallel}$$

$$m \frac{dv_{\parallel}}{dt} = F_{\parallel}$$

$$m \frac{d\vec{v}_{\perp}}{dt} = q \left(\hat{x} \frac{F_{\perp}}{q} + \vec{v}_{\perp} \times \hat{z}B \right)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_F$$

$$\vec{v}_F = \frac{\hat{x}(F_{\perp}/q) \times \hat{z}B}{B^2} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

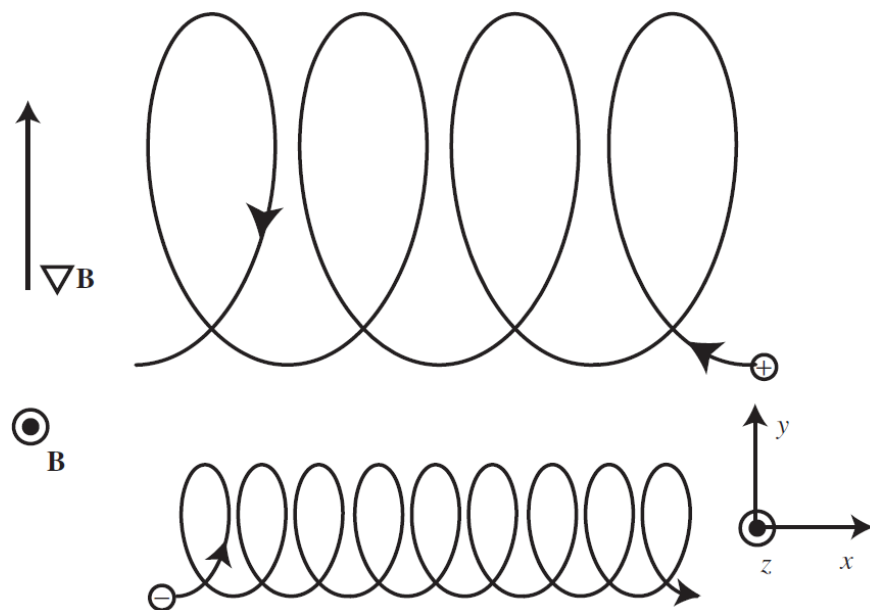
Gravitational drift velocity

- Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

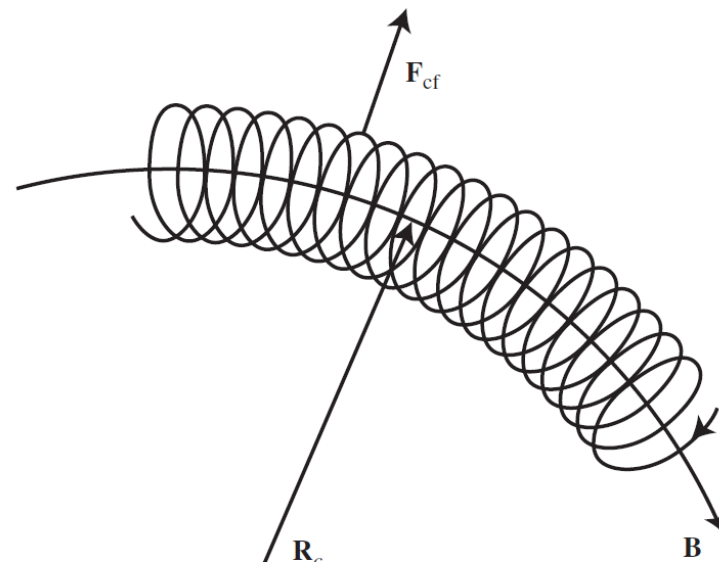


- Gradient-B drift



$$\vec{v}_{\nabla} = \frac{mv_{\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$$

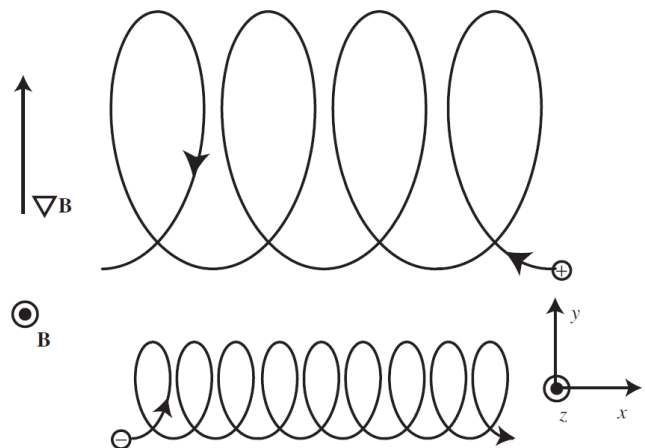
- Curvature drift



$$\vec{v}_R = \frac{mv_{\parallel}^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm \omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm \omega_c t + \psi)$$

$$v_x = v_{\perp} \cos(\pm \omega_c t + \psi)$$

$$v_y = -v_{\perp} \sin(\pm \omega_c t + \psi)$$

$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x}qv_y B_z - \hat{y}qv_x B_z$$

$$\approx \hat{x}qv_y \left(B_0 + y \frac{\partial B_z}{\partial y} \right) - \hat{y}qv_x \left(B_0 + y \frac{\partial B_z}{\partial y} \right)$$

$$B_z(y) = B_0 + y \frac{\partial B_z}{\partial y} + y^2 \frac{1}{2} \frac{\partial^2 B_z}{\partial y^2} + \dots$$

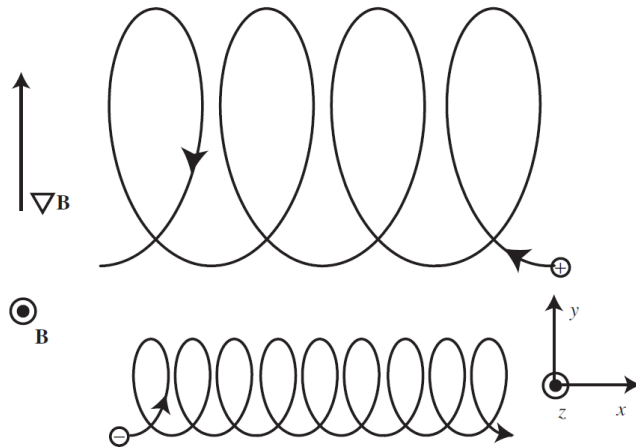
$$F_x = qv_y \left(B_0 + y \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -qv_x \left(B_0 + y \frac{\partial B_z}{\partial y} \right)$$

$$F_x \approx -qv_{\perp} \sin(\pm \omega_c t + \psi) \times \left(B_0 \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -qv_{\perp} \cos(\pm \omega_c t + \psi) \times \left(B_0 \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_{\perp} \sin(\pm\omega_c t + \psi) \left[B_0 \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y} \right]$$

$$F_y \simeq -qv_{\perp} \cos(\pm\omega_c t + \psi) \left[B_0 \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y} \right]$$

$$\langle F_x \rangle = 0$$

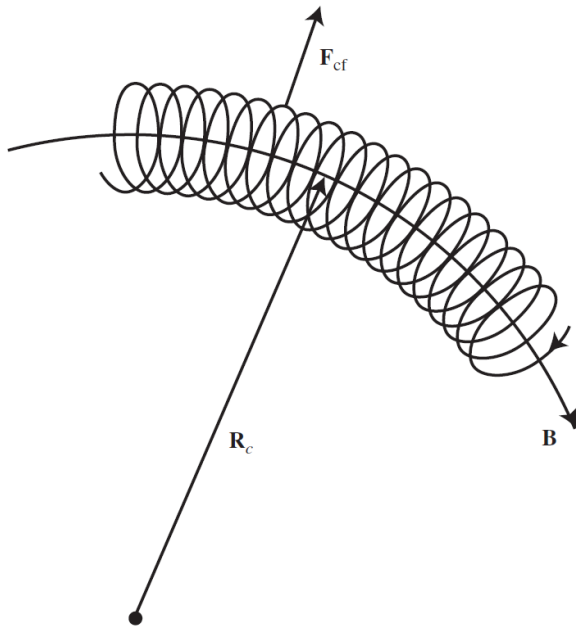
$$\langle F_y \rangle = \mp \frac{qv_{\perp} r_c}{2} \frac{\partial B_z}{\partial y} = -\frac{mv_{\perp}^2}{2B} \frac{\partial B_z}{\partial y}$$

$$r_c = \frac{v_{\perp}}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2} \quad \vec{v}_{\nabla} = \frac{1}{q} \frac{\langle F_y \rangle \hat{y} \times \hat{z} B_z}{B_z^2} = -\frac{mv_{\perp}^2}{2qB_z^2} \frac{\partial B_z}{\partial y} \hat{x}$$

- More general:
$$\vec{v}_{\nabla} = \frac{mv_{\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$$

Charge particles drift across magnetic field lines when the magnetic field line is curved

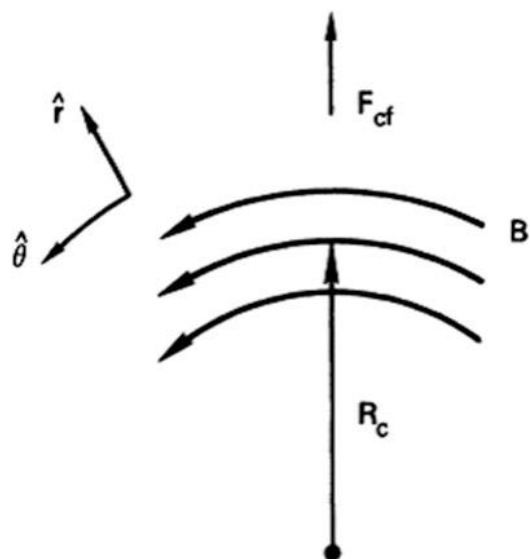


$$\vec{F}_{cf} = mv_{\parallel}^2 \frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{mv_{\parallel}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B\hat{r}$$

Cylindrical coordinate

$$\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3} \quad \vec{v}_R = \frac{mv_\parallel^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

$$\nabla \times \vec{B} = 0$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0 \quad (\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (rB_\theta) - \frac{1}{r} \frac{\partial B_r}{\partial \theta}$$

$$\nabla \times \vec{B} = (\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (rB_\theta) = 0 \quad B_\theta \propto \frac{1}{r}$$

$$\frac{\nabla |B|}{|B|} = -\frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)$$

- Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.

Quick summary of different drifts



- **ExB drift:** $\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$ Independent to charge
- **Gravitational drift:** $\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$ Depended on charge
- **Grad-B drift:** $\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ Depended on charge
- **Curvature drift:** $\vec{v}_R = \frac{mv_{\parallel}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$ Depended on charge
- **Non-uniform B drift:**

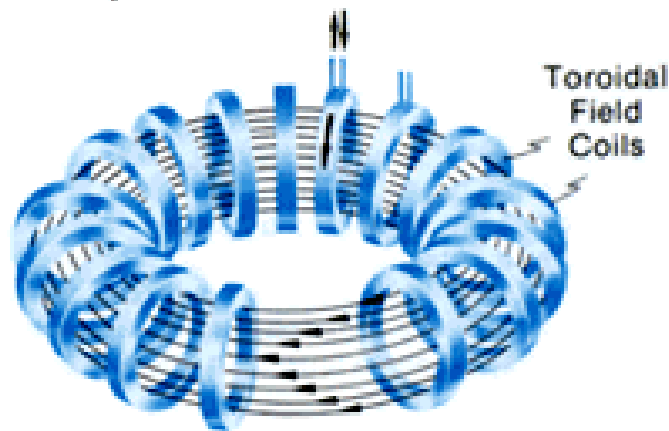
$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)$$

Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field



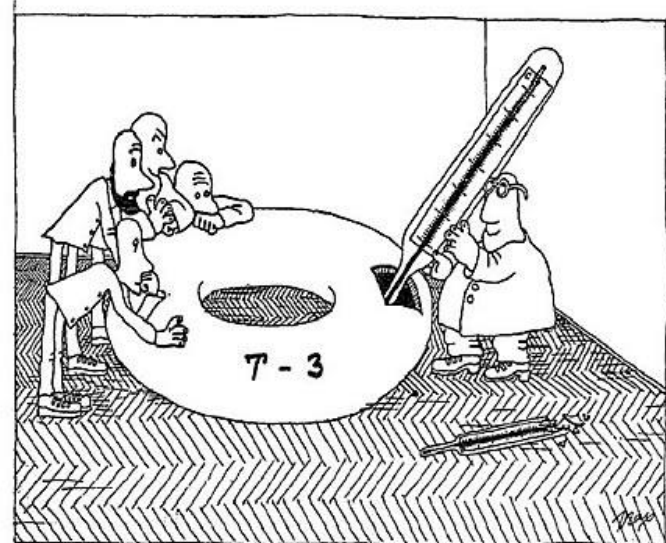
- Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)

Relatively Constant Electric Current



Nature

Constant Toroidal Field



Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

N. J. PEACOCK
D. C. ROBINSON
M. J. FORREST
P. D. WILCOCK
UKAEA Research Group,
Culham Laboratory,
Abingdon, Berkshire

V. V. SANNIKOV
I. V. Kurchatov Institute,
Moscow

$$T_e = 100 \sim 1 \text{ keV}$$

$$n_e = 1\text{-}3 \times 10^{13} \text{ cm}^{-3}$$

Electron temperatures of 100 eV up to 1 keV and densities in the range $1\text{-}3 \times 10^{13} \text{ cm}^{-3}$ have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

<https://www.iter.org/mach/tokamak>

https://en.wikipedia.org/wiki/Tokamak#cite_ref-4

Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.

N. J. Peacock, et al., Nature **224**, 488 (1969)

Quick summary of different drifts



- ExB drift: $\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$

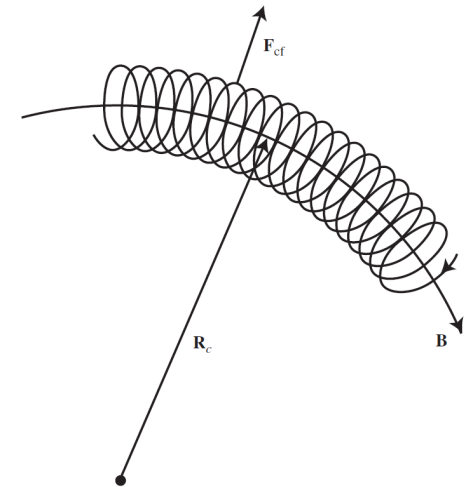
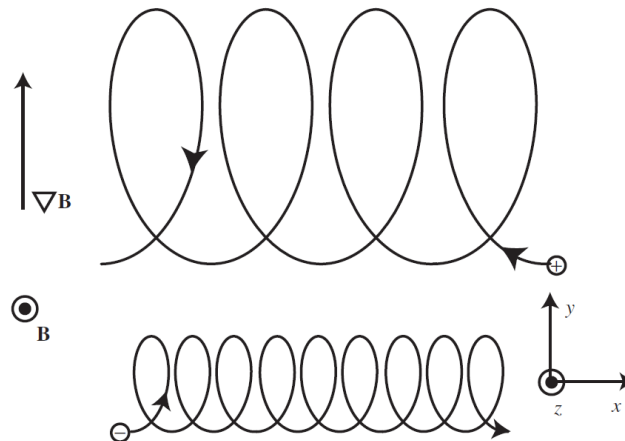
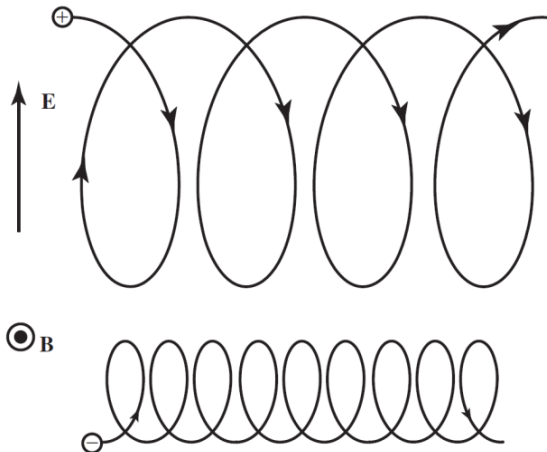
Independent to charge

- Grad-B drift: $\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$

Depended on charge

- Curvature drift: $\vec{v}_R = \frac{mv_{||}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$

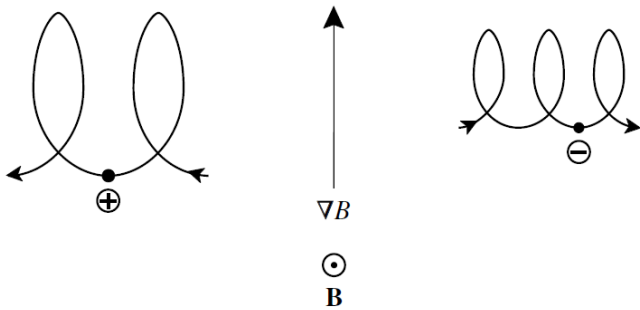
Depended on charge



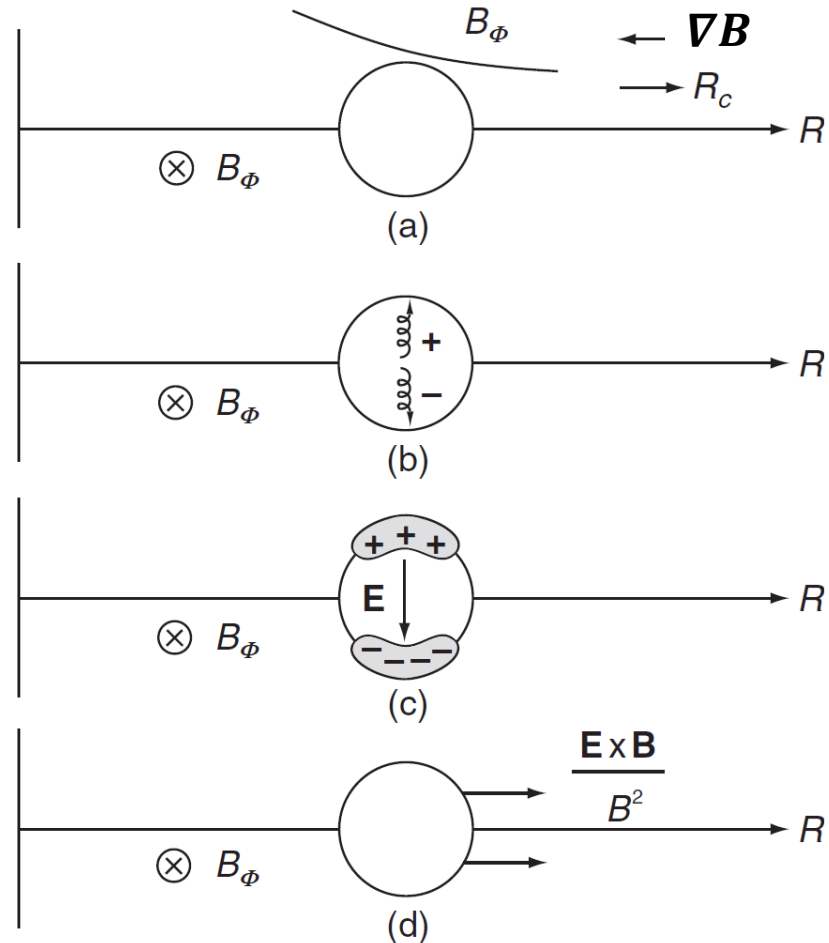
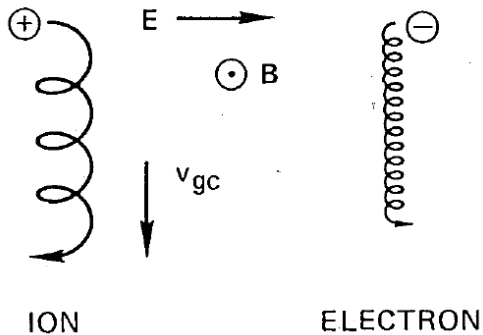
Charged particles drift across field lines



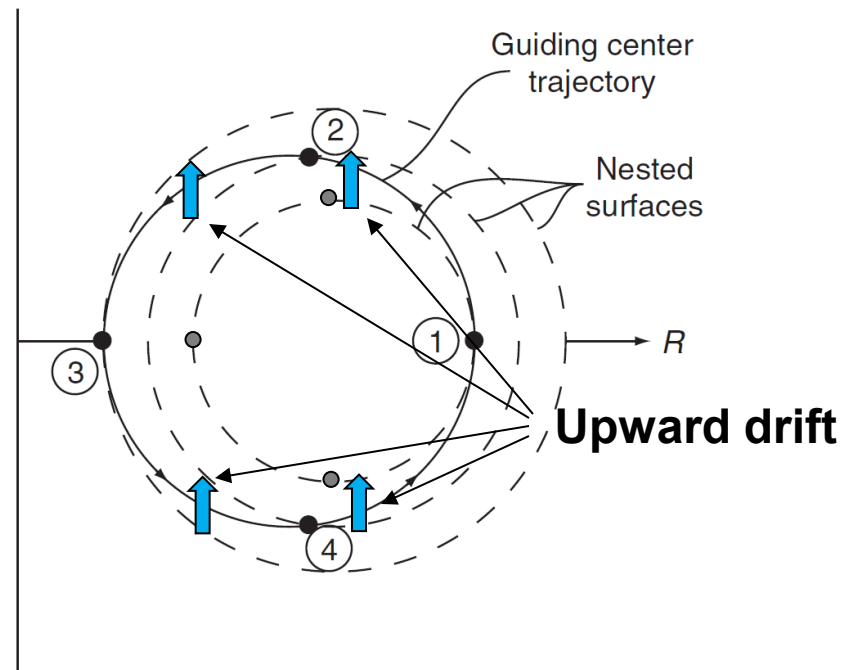
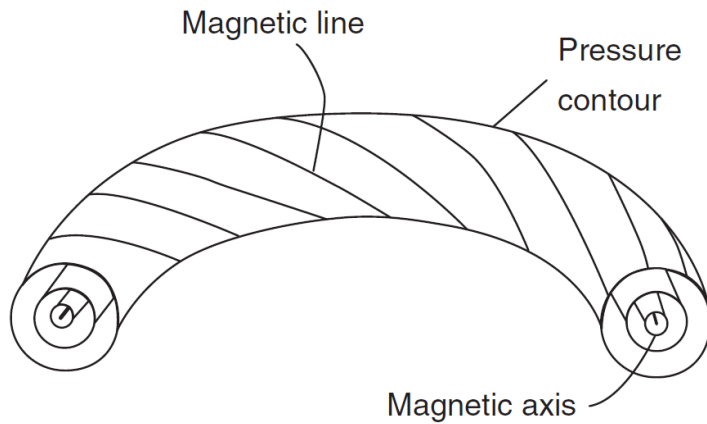
- Grad-B drift**



- ExB drift**

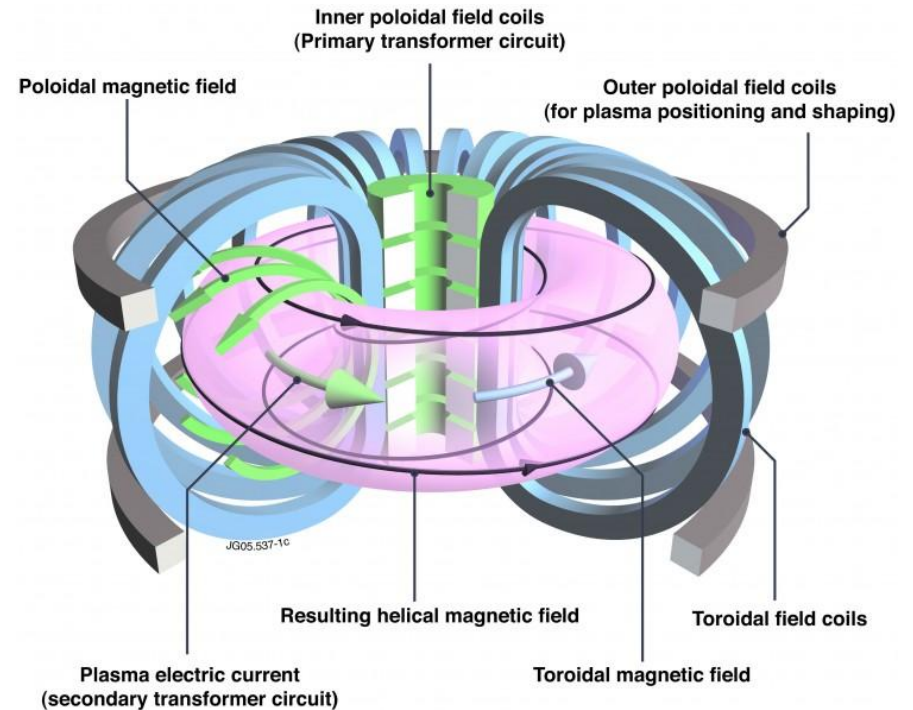
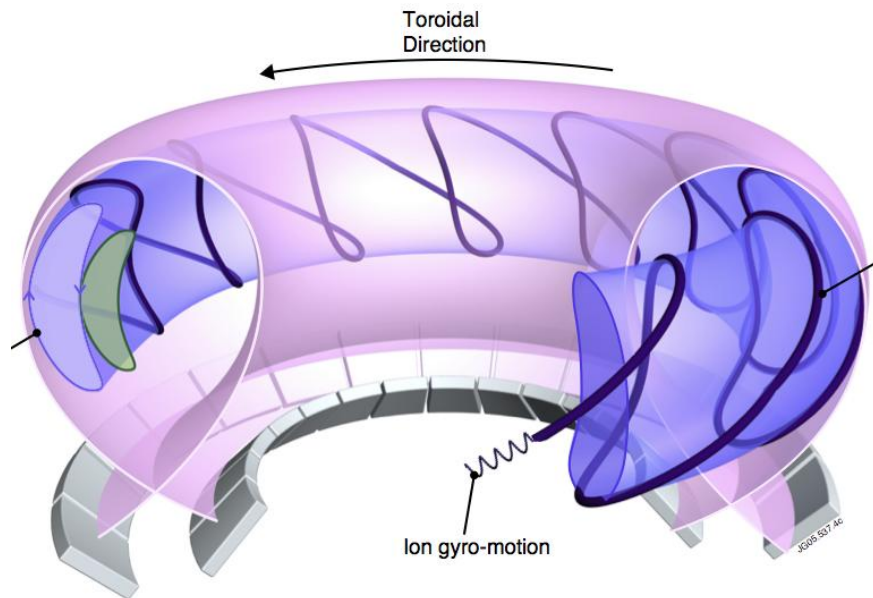


The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field



• Points with no drift

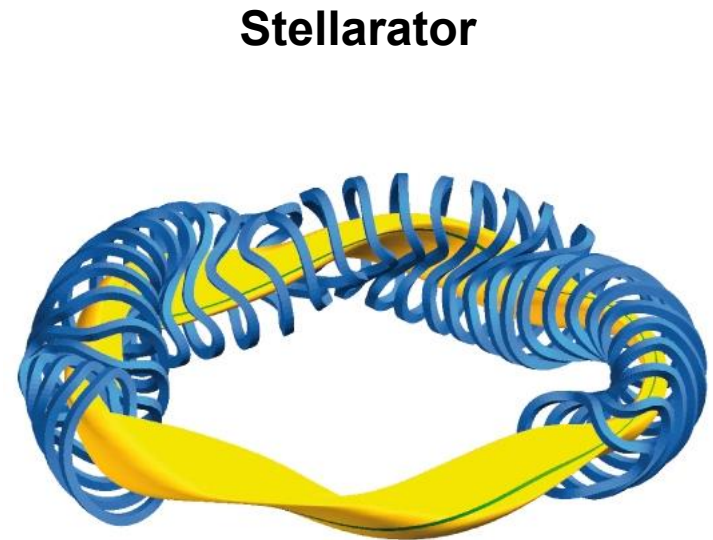
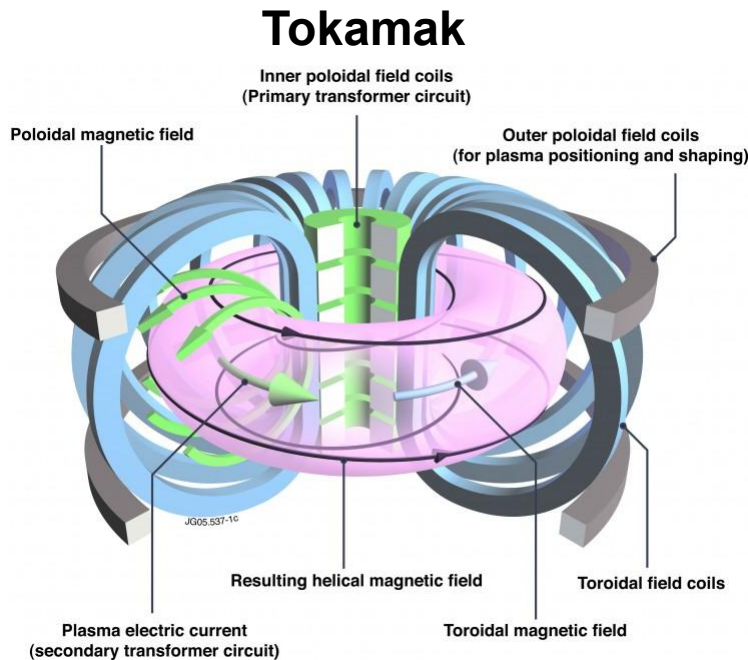
A poloidal magnetic field is required to reduce the drift across field lines



A poloidal magnetic field is required to reduce the drift across field lines



Stellarator uses twisted coil to generate poloidal magnetic field



Magnetohydrodynamics description of plasma



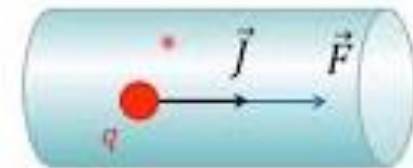
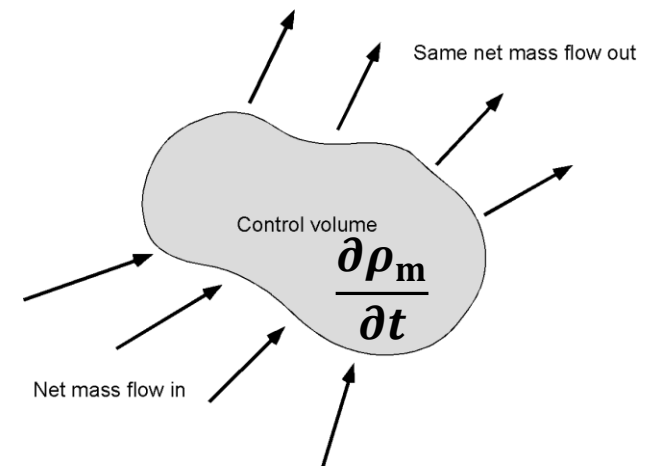
- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho_q \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$
- **Ohm's law:** $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt} \left(\frac{P}{\rho_m^\gamma} \right) = 0$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$



Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation



- Continuity eq: $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ w/ long wavelength ($\lambda \gg \lambda_d$)
- Momentum eq: $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \cancel{\rho_q \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$
- Ohm's law: $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- Equation of state: $\frac{d}{dt} \left(\frac{P}{\rho_m^\gamma} \right) = 0$

- Maxwell's eqs:

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0} \approx 0 \quad \text{w/ long wavelength (} \lambda \gg \lambda_d \text{)} \Rightarrow \text{quasi neutral}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \cancel{\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega \ll \omega_{pe}$)

Ideal MHD



- Continuity eq: $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
- Momentum eq: $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$
- Ohm's law: $\vec{E} + \vec{v} \times \vec{B} \approx 0$
- Equation of state: $\frac{d}{dt} \left(\frac{P}{\rho_m^\gamma} \right) = 0$
- Maxwell's eqs:

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- Requirement:

- High collisionality – fluid model
- Small gyro radius – low frequency
- Small resistivity – a perfect conductor

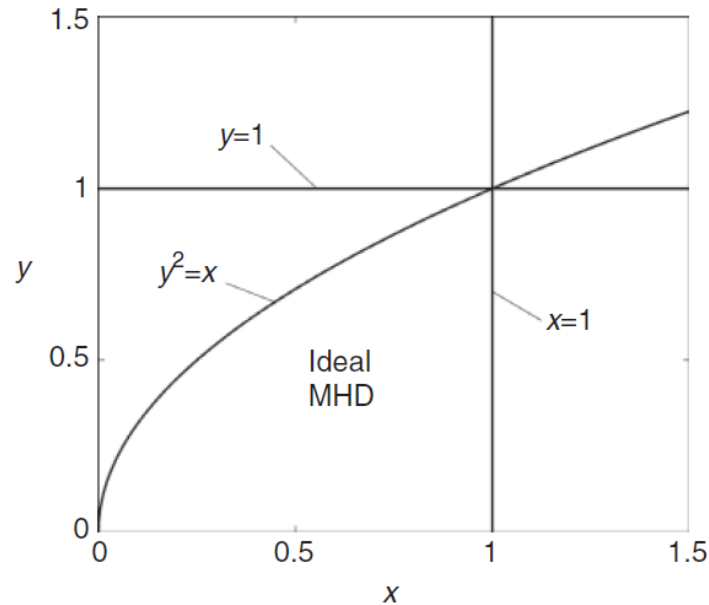
Conflict!



$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \quad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \quad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a} \frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} \ll 1$$

Scale length of nonuniformity.

Region of validity for ideal MHD



Mean free path: $\lambda_i \sim v_{Ti} \tau_{ii}$

$$x = \left(\frac{m_i}{m_e} \right)^{1/2} \frac{v_{Ti} \tau_{ii}}{a}$$

$$y = \frac{r_{Li}}{a}$$

- **Requirement:**

- **High collisionality** $x \ll 1$

- **Small gyro radius** $y \ll 1$

- **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement (small η)



$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \quad \eta \vec{j} = \vec{E} + \vec{v} \times \vec{B} \quad \frac{|\eta j|}{|\vec{v} \times \vec{B}|} \sim ?$$

$$|j \times B| \sim |\nabla p| \quad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a} \frac{nT}{B} \sim \frac{1}{a} \frac{nm_i v_{Ti}^2}{B} \quad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \quad \omega_{ci} = \frac{v_{Ti}}{r_{Li}}$$

$$\eta \sim \frac{m_e}{ne^2 \tau_{ei}} \quad \tau_{ei} \sim \tau_{ee} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \tau_{ii} \quad k \sim \nabla \sim \frac{1}{a} \quad \omega_{ci} = \frac{eB}{m_i}$$

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti} \tau_{ii}}{a} \quad y = \frac{r_{Li}}{a}$$

$$\frac{|\eta j|}{|\vec{v} \times \vec{B}|} \sim \frac{\eta j}{v_{Ti} B} \sim \frac{m_e}{ne^2 \tau_{ei}} \frac{1}{a} \frac{nm_i v_{Ti}^2}{B} \frac{1}{v_{Ti} B} = \frac{m_e v_{Ti}}{\tau_{ei} a} \frac{m_i}{e^2 B^2} = \frac{m_e v_{Ti}}{m_i \tau_{ei} a} \frac{m_i^2}{e^2 B^2} = \frac{m_e v_{Ti}}{m_i \tau_{ei} a \omega_{ci}^2}$$

$$\sim \frac{m_e}{m_i \tau_{ii}} \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}}{a \omega_{ci}^2} = \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{Ti} r_{Li}^2}{\tau_{ii} a v_{Ti}^2} = \left(\frac{m_e}{m_i}\right)^{1/2} \frac{1}{\tau_{ii} a} \frac{r_{Li}^2}{v_{Ti}} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{1}{\omega \tau_{ii}} \left(\frac{r_{Li}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

Fusion plasma is not in the ideal MHD region!

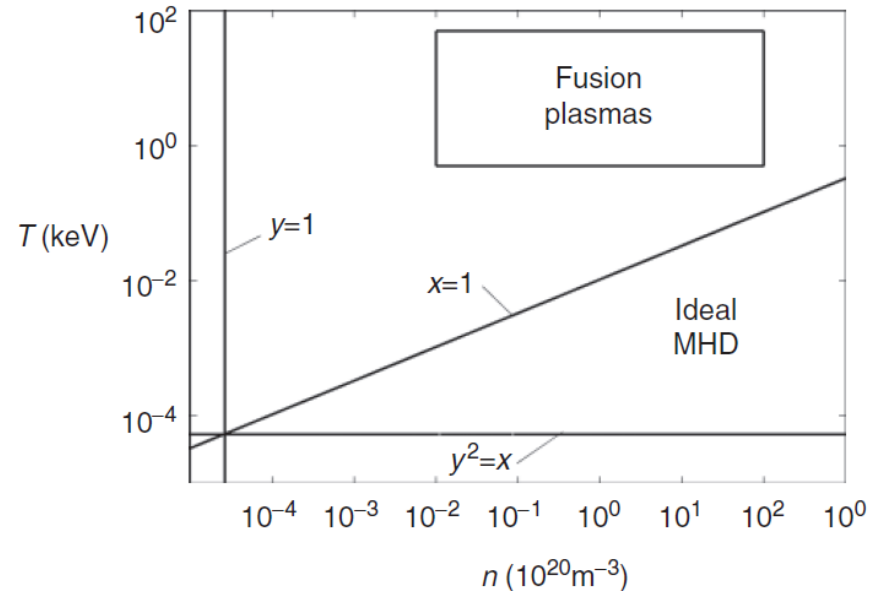


$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti} \tau_{ii}}{a} \quad y = \frac{r_{Li}}{a}$$

$$10^{18} \text{ m}^{-3} < n < 10^{22} \text{ m}^{-3}$$

$$0.5 \text{ keV} < T < 50 \text{ keV}$$

$$\beta \equiv \frac{2\mu_0 n T}{B^2}$$



- Requirement:

- High collisionality $x = 3 \times 10^3 \frac{T^2}{an} \ll 1$

- Small gyro radius $y = 2.3 \times 10^{-2} \left(\frac{\beta}{na^2}\right)^{1/2} \ll 1$

- Small resistivity $\frac{y^2}{x} = 1.8 \times 10^{-7} \frac{\beta}{aT^2} \ll 1$

• With strong B, the gyromotion mimic the collisional characteristics.

Ideal MHD



- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$
- **Ohm's law:** $\vec{E} + \vec{v} \times \vec{B} \approx 0$
- **Equation of state:** $\frac{d}{dt} \left(\frac{P}{\rho_m^\gamma} \right) = 0$

- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**

- **High collisionality – fluid model**
- **Small gyro radius – low frequency**
- **Small resistivity – a perfect conductor**

Additional simplification of the momentum equation



- Momentum eq: $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$

$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} \end{pmatrix}$$

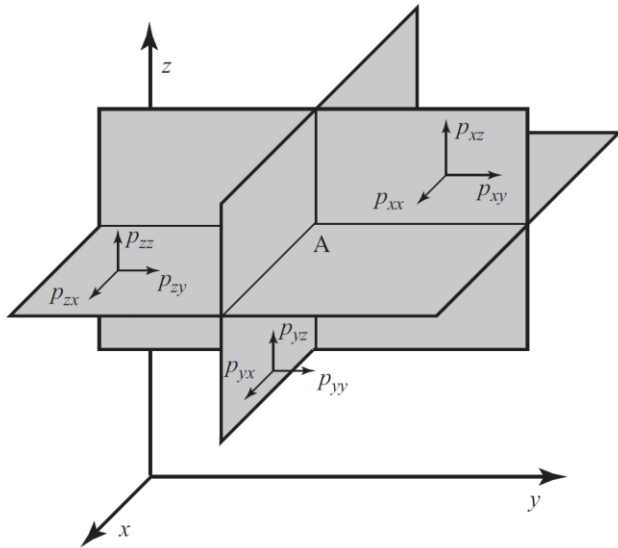
$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial p_{xx}}{\partial x} \\ \frac{\partial p_{yy}}{\partial y} \\ \frac{\partial p_{zz}}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{pmatrix}$$

Additional simplification of the momentum equation



$$\nabla \cdot \overleftrightarrow{\mathbf{P}} = \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right) \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right) \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$



$$= \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right) \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \underbrace{\left(\begin{array}{c} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{array} \right)}_{\text{Viscosity } \nabla \cdot \overleftrightarrow{\mathbf{\Pi}}}$$

- **Isotropic plasma:** $p_{xx} = p_{yy} = p_{zz} \equiv p$ $\begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} \equiv p \overleftrightarrow{\mathbf{1}}$

$$\nabla \cdot \overleftrightarrow{\mathbf{P}} = \nabla p + \nabla \cdot \overleftrightarrow{\mathbf{\Pi}}$$

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\mathbf{\Pi}}$$

Viscosity is negligible in a collision-dominated plasma



- Y component of momentum transfer through the surface A.

$$\pi_{xy}^{ii} \sim (mv_y n) \frac{dv_x}{dx} dl \sim \mu \frac{dv_x}{dx}$$

$$\mu \sim mnvdl \sim mnv(v\tau_{ii}) \sim mn \left(\sqrt{\frac{T_i}{m}} \right)^2 \tau_{ii} \sim nT_i \tau_{ii}$$

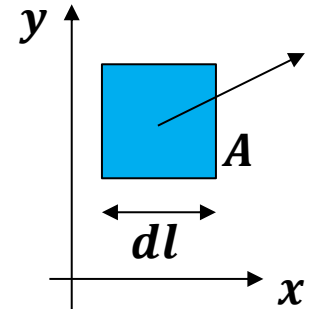
$$\vec{\Pi} \sim \mu \left(2\nabla_{\parallel} \cdot \vec{v}_{\parallel} - \frac{2}{3} \nabla \cdot \vec{v} \right) \sim \mu \frac{v_{Ti}}{a}$$

$$\left| \frac{\nabla \cdot \vec{\Pi}}{\nabla p} \right| \sim \frac{\vec{\Pi} a}{ap} \sim \frac{nT_i \tau_{ii} v_{Ti}}{ap} \sim \frac{\tau_{ii} v_{Ti}}{a} \sim \frac{\lambda_i}{a} \ll 1 \quad \lambda_i \sim v_{Ti} \tau_{ii}$$

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \vec{\Pi}$$



$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$



Ideal MHD



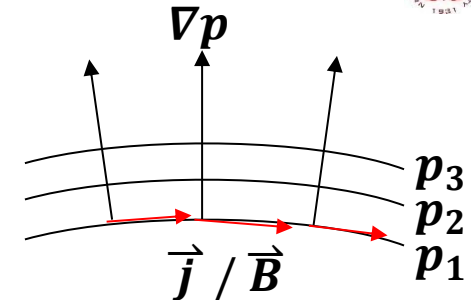
- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$
- **Ohm's law:** $\vec{E} + \vec{v} \times \vec{B} \approx 0$
- **Equation of state:** $\frac{d}{dt} \left(\frac{P}{\rho_m^\gamma} \right) = 0$
- **Maxwell's eqs:**
 - $\nabla \cdot \vec{E} \approx 0$
 - $\nabla \cdot \vec{B} = 0$
 - $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
 - $\nabla \times \vec{B} = \mu_0 \vec{j}$
 - $\nabla \cdot \vec{j} = 0$
- **Requirement:**
 - High collisionality – fluid model
 - Small gyro radius – low frequency
 - Small resistivity – a perfect conductor

When forces are balanced, the system is in the equilibrium state, or called “Magnetohydrostatics”



- Equilibrium state:

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p \equiv 0$$



$$\vec{j} \times \vec{B} = \nabla p$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = \frac{1}{\mu_0} \left[(\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla B^2 \right] = \nabla p$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}$$

Magnetic pressure

Magnetic tension

← Forces caused by curvature of the field lines

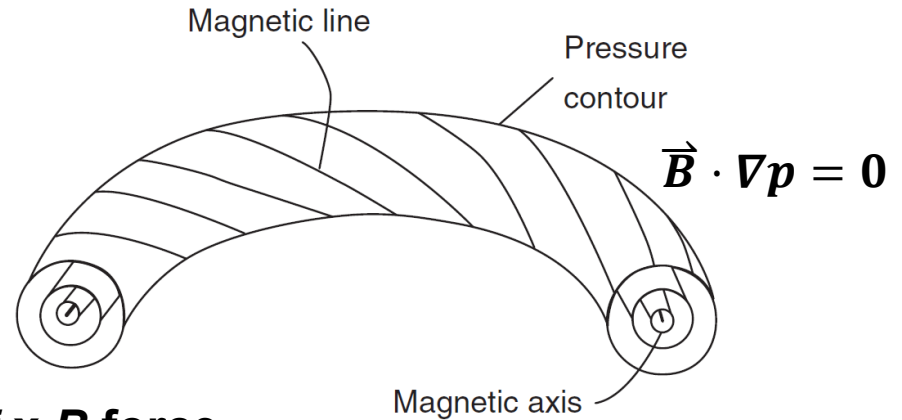
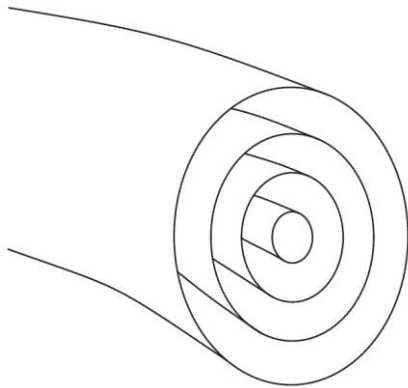
$$\vec{j} \perp \nabla p \quad \vec{B} \perp \nabla p \quad \Rightarrow \quad \vec{j} \cdot \nabla p = 0 \quad \vec{B} \cdot \nabla p = 0$$

- The surfaces with $p = \text{constant}$ are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).

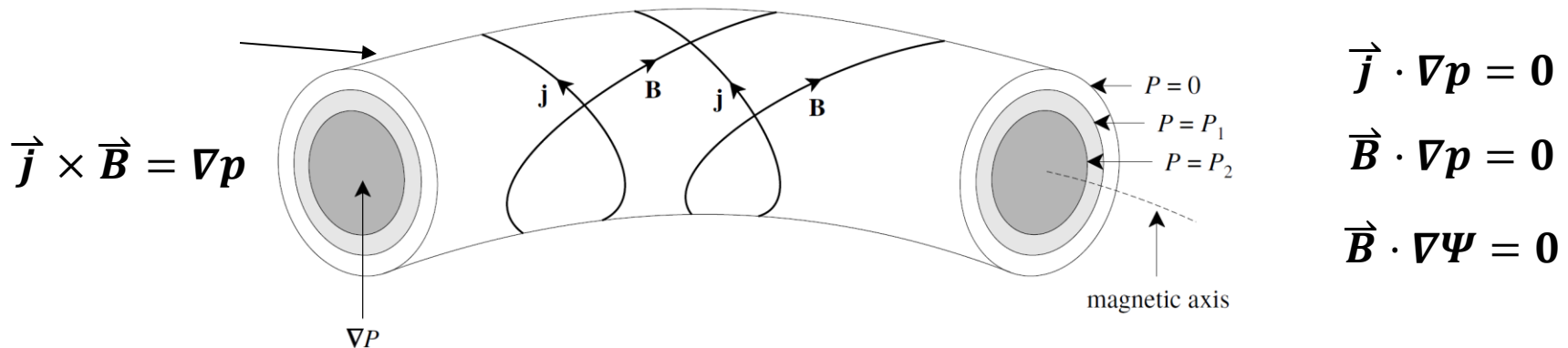
Magnetic lines lying on pressure contour



- Contours of constant pressure
- Magnetic lines lying on pressure contour



- Pressure gradient is balanced by the $\vec{j} \times \vec{B}$ force



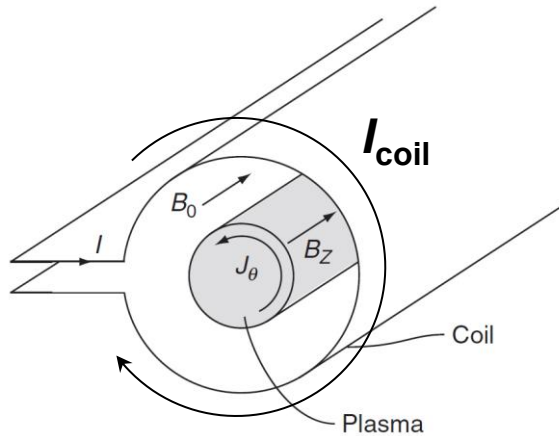
- A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to \vec{B} .

Course Outline



- **Magnetic confinement fusion (MCF)**
 - Gyro motion, MHD
 - 1D equilibrium (z pinch, theta pinch)
 - Drift: ExB drift, grad B drift, and curvature B drift
 - Tokamak, Stellarator (toroidal field, poloidal field)
 - Magnetic flux surface
 - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
 - Stability (Kink instability, sausage instability, Safety factor Q)
 - Central-solenoid (CS) start-up (discharge) and current drive
 - CS-free current drive: electron cyclotron current drive, bootstrap current.
 - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$
 $\vec{B} = B_z \hat{z}$
- **All quantities are only functions of the radius r .**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{r} \frac{\partial B_r}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r}$$

$$\nabla \left(P + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B} = 0$$

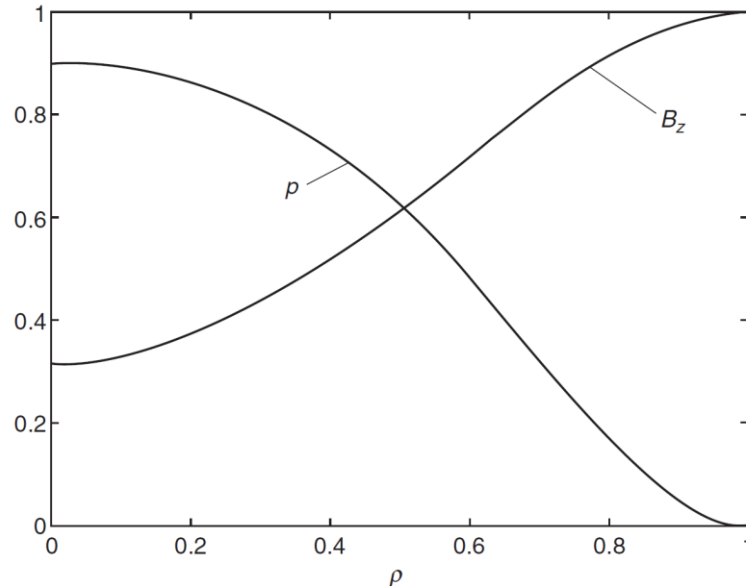
$$P + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$\vec{j} \times \vec{B} = \nabla p \quad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma



- Example:



$$\frac{2\mu_0 p(r)}{B_0^2} = 1 - \left[1 - \hat{\beta}(1 - \rho^2)\right]^2$$

$$\frac{B_z(r)}{B_0} = 1 - \hat{\beta}(1 - \rho^2)$$

$$j_\theta B_z = \frac{dp}{dr} \quad \rightarrow \quad \frac{a\mu_0 j_\theta(r)}{B_0} = -4\hat{\beta}\rho(1 - \rho^2)$$

$$\hat{\beta} = \frac{\beta_0}{1 + \sqrt{(1 - \beta_0)}} \quad \beta_0 = \frac{2\mu_0 p_0}{B_0^2} \quad \rho = \frac{r}{a}$$

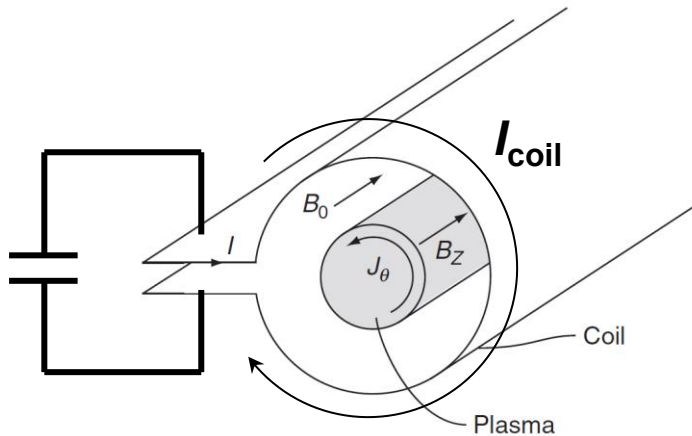
$$\beta \equiv \beta_t = \frac{2\mu_0 \langle p \rangle}{B_0^2} = \frac{4\mu_0}{a^2 B_0^2} \int_0^a p r dr = 2 \int_0^1 \left(1 - \frac{B_z^2}{B_0^2}\right) \rho d\rho = \hat{\beta} \left(\frac{2}{3} - \frac{\hat{\beta}}{5}\right)$$

$$\beta_0 \rightarrow 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_0}{2}, \quad \beta \approx \frac{\beta_0}{3}$$

$$\beta_0 \rightarrow 1 \quad \Rightarrow \quad \hat{\beta} \rightarrow 1, \quad \beta \approx \frac{7}{15}$$

$$0 < \beta < 1$$

Theta pinches provide good radial confinement but NOT axially



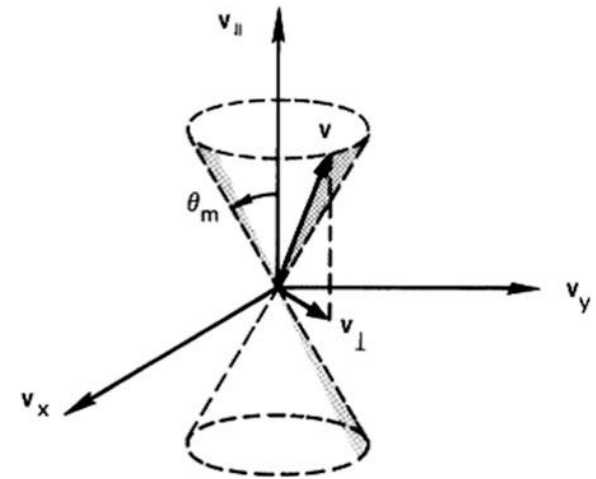
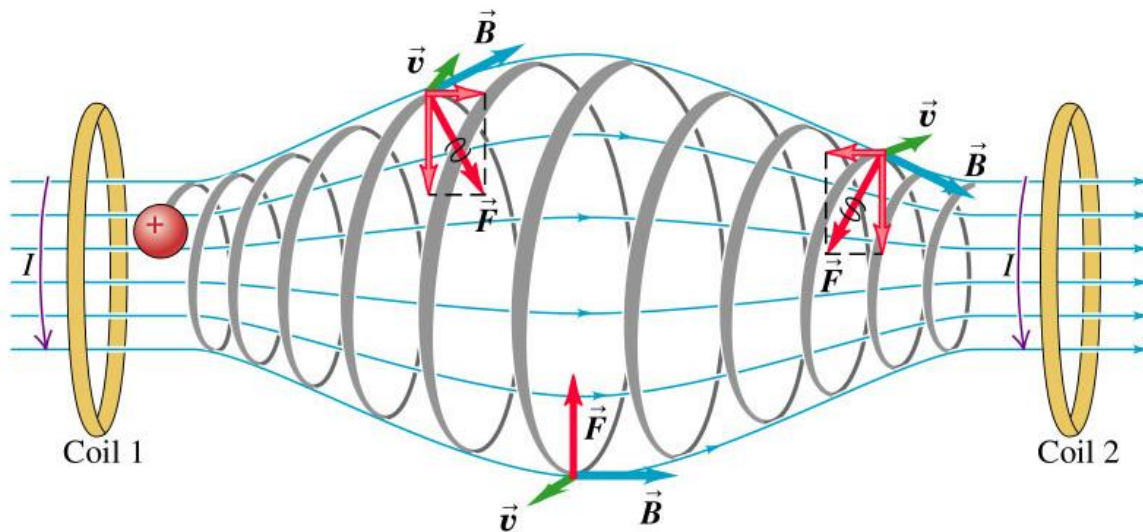
- The gas is initially preionized.
- The coil current is provided by a capacitor bank. The typical pulse length is 10-50 μs .
- The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.
- This energy is ultimately converted to heat after repeated reflections off the converging piston.
- $T_i \sim 1\text{-}4 \text{ keV}$, $n \sim 1\text{-}2 \times 10^{22} \text{ m}^{-3}$, $\beta_0 \sim 0.7\text{-}0.9$, $\beta \sim 0.05$.
- The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{Ti} \sim 10 \mu\text{s}$ for $L = 5 \text{ m}$.

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine



- Charged particles with small v_{\parallel} eventually stop and are reflected while those with large v_{\parallel} escape.



$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2 \quad \text{Invariant: } \mu \equiv \frac{1}{2} \frac{mv_{\perp}^2}{B}$$

$$v_{\perp}^{\prime 2} = v_{\perp 0}^2 + v_{\parallel 0}^2 \equiv v_0^2$$

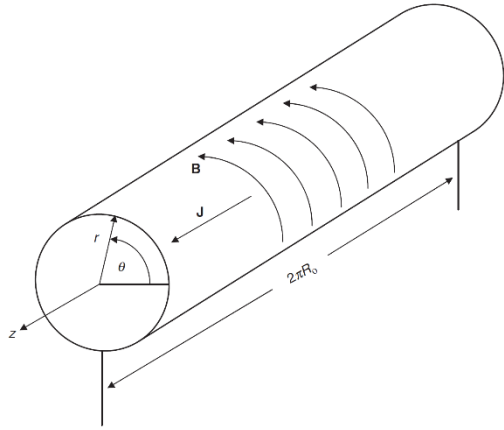
$$\frac{B_0}{B'} = \frac{v_{\perp 0}^2}{v_{\perp}^{\prime 2}} = \frac{v_{\perp 0}^2}{v_0^2} \equiv \sin^2 \theta$$

$$\frac{B_0}{B_m} \equiv \frac{1}{R_m} = \sin^2 \theta_m$$

- Large v_{\parallel} may occur from collisions between particles.

Those confined charged particle are eventually lost due to collisions.

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius r .**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r} \frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{r} \frac{\partial B_r}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta)$$

$$j_z = \frac{1}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \quad j_z B_\theta = - \frac{dp}{dr}$$

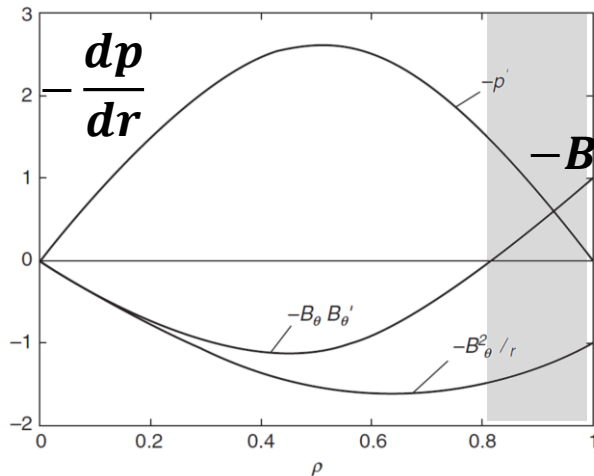
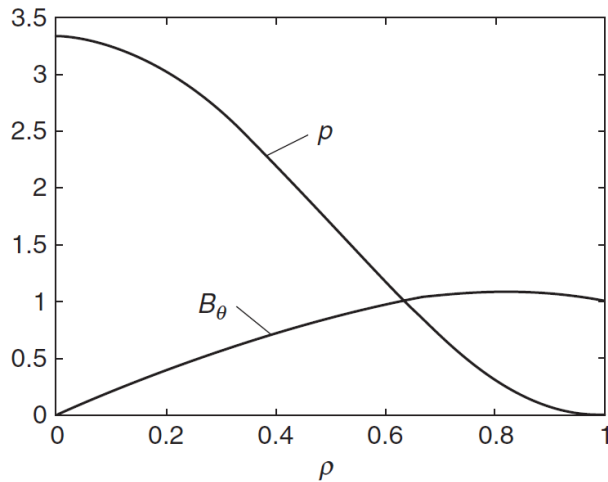
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta) = 0$$

$$\frac{d}{dr} \left(p + \frac{B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

Magnetic pressure

Magnetic tension

Z pinch – there is no flexibility in achieving small to moderate β



$$\frac{d}{dr} \left(p + \frac{B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

- Example:

$$\frac{2\mu_0 p(r)}{B_{\theta a}^2} = \frac{2}{3} (5 - 2\rho^2)(1 - \rho^2)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}} = 2\rho \left(1 - \frac{\rho^2}{2} \right)$$

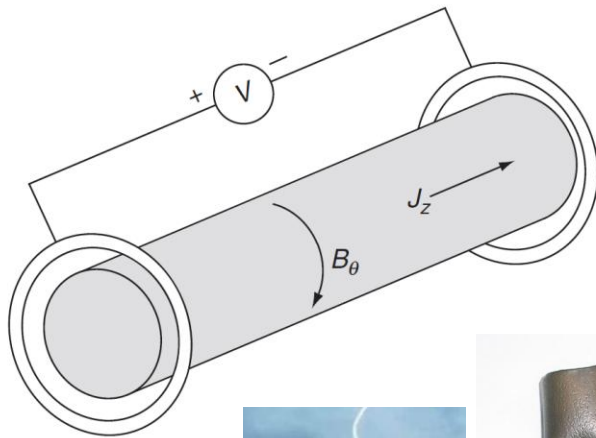
$$j_z = \frac{1}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta) \rightarrow \frac{a\mu_0 j_z(r)}{B_{\theta a}} = 4(1 - \rho^2)$$

$$B_{\theta a} \equiv B_\theta(a) = \frac{\mu_0 I}{2\pi a}$$

$$\beta \equiv \beta_p = \frac{2\mu_0 \langle p \rangle}{B_{\theta a}^2} = \frac{4\mu_0}{a^2 B_{\theta a}^2} \int_0^a p r dr = 1$$

Bennett pinch relation: $\beta = 1$

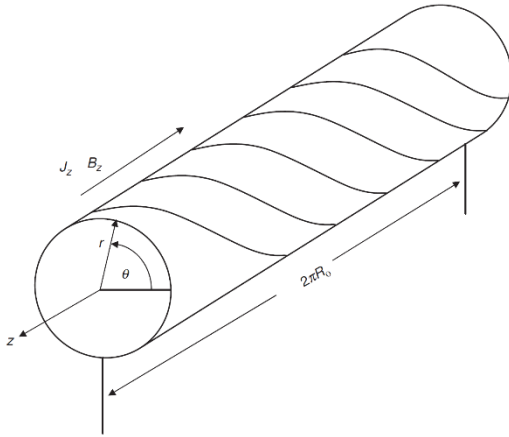
Huge instabilities occur in a z pinch



- A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.
- The gas is ionized by the high voltage and produces a z current flowing along the plasma.
- Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.

Main issue: unstable.

General screw pinch – linear superposition of the theta pinch and the z pinch



- Nonzero field: $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{r} \frac{\partial B_r}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta)$$

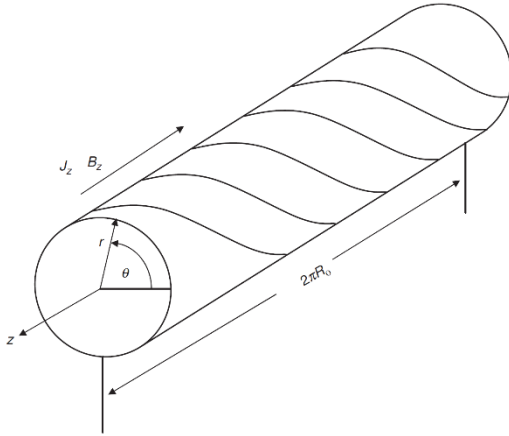
$$j_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r} \quad j_z = \frac{1}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \quad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_0} \frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr} \left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

General screw pinch – linear superposition of the theta pinch and the z pinch



- Nonzero field: $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) - \frac{1}{r} \frac{\partial B_r}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta)$$

$$j_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r} \quad j_z = \frac{1}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \quad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

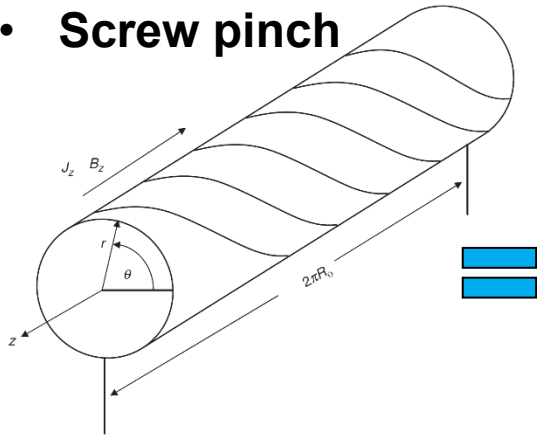
$$-\frac{B_z}{\mu_0} \frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr} \left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

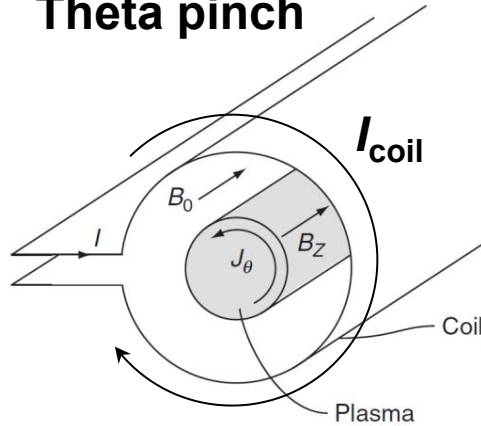
General screw pinch is flexible with varies range of β



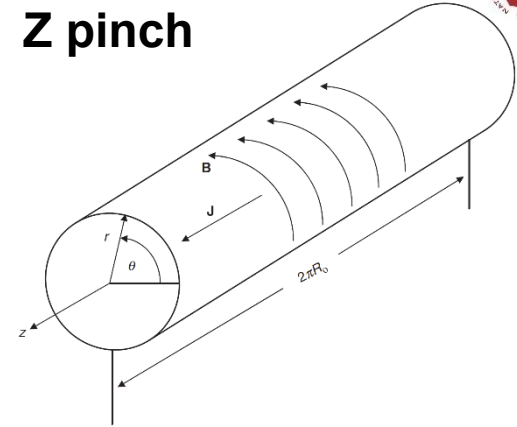
• Screw pinch



• Theta pinch



• Z pinch



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\frac{d}{dr} \left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

$$\vec{B} = B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta}$$

$$p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$\vec{B} = B_\theta \hat{\theta}$$

$$\vec{j} = j_z \hat{z}$$

$$\frac{d}{dr} \left(p + \frac{B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

$$\int_0^a \pi r^2 dr \left[\frac{d}{dr} \left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} \right] = 0$$

$$\langle p \rangle = \frac{B_{\theta a}^2}{2\mu_0} + \frac{1}{2\mu_0} (B_0^2 - \langle B_z^2 \rangle)$$

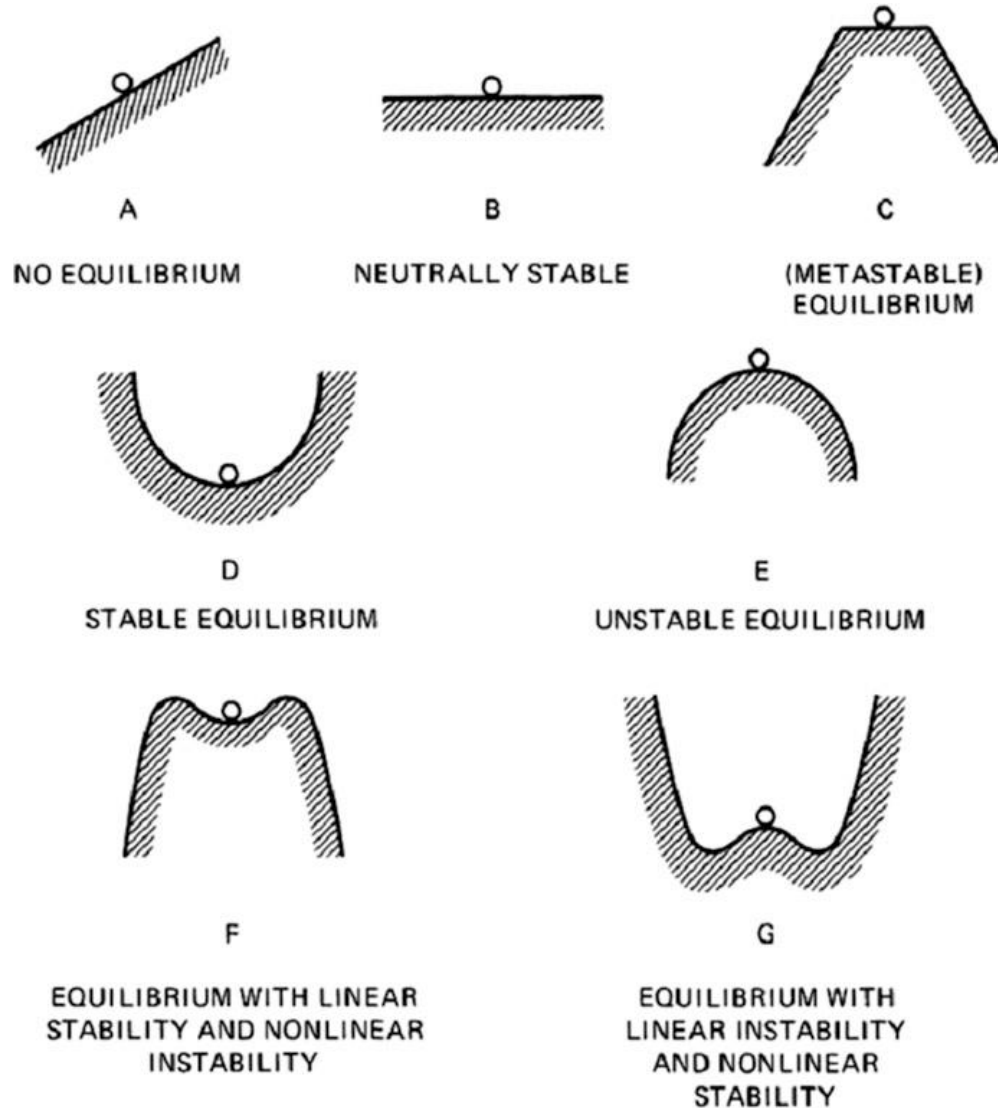
$$\beta_t = \frac{2\mu_0 \langle p \rangle}{B_0^2}$$

$$\beta_p = \frac{2\mu_0 \langle p \rangle}{B_{\theta a}^2}$$

$$\beta = \frac{\beta_t \beta_p}{\beta_t + \beta_p} = \frac{2\mu_0 \langle p \rangle}{B_0^2 + B_{\theta a}^2}$$

$$0 \leq \langle \beta \rangle \leq 1$$

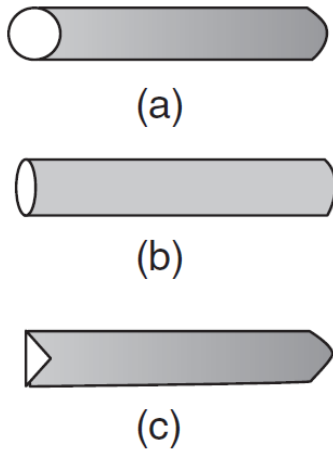
An equilibrium state may not be stable



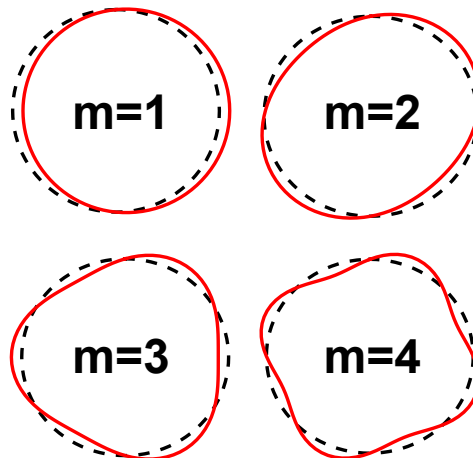
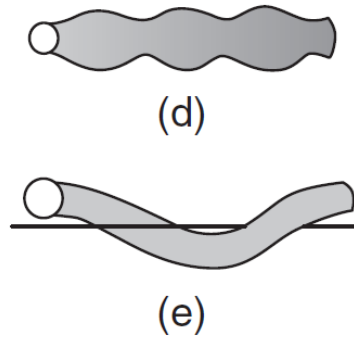
A cylindrical plasma column may not be stable



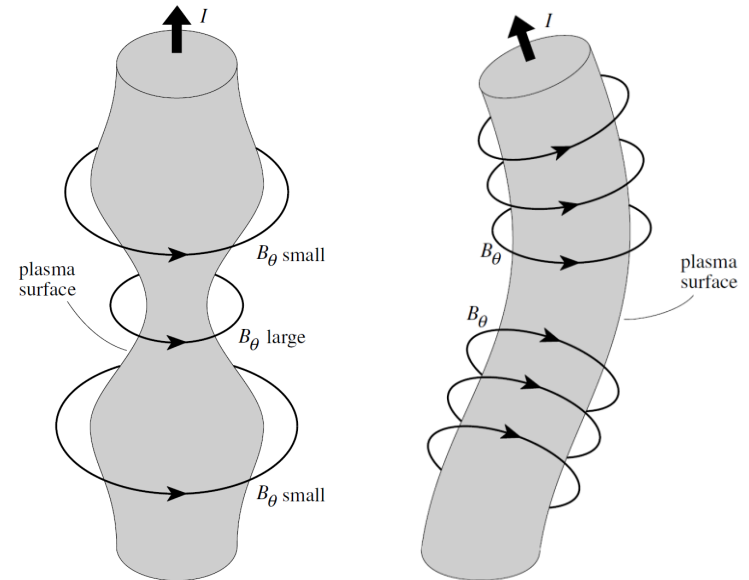
- Instabilities of theta pinch



- (a) Unperturbed
- (b) $m=2, k=0$
- (c) $m=3, k=0$
- (d) $m=0, k \neq 0$
- (e) $m=1, k \neq 0$



- Instabilities of z pinch



Sausage instability
($m=0$)

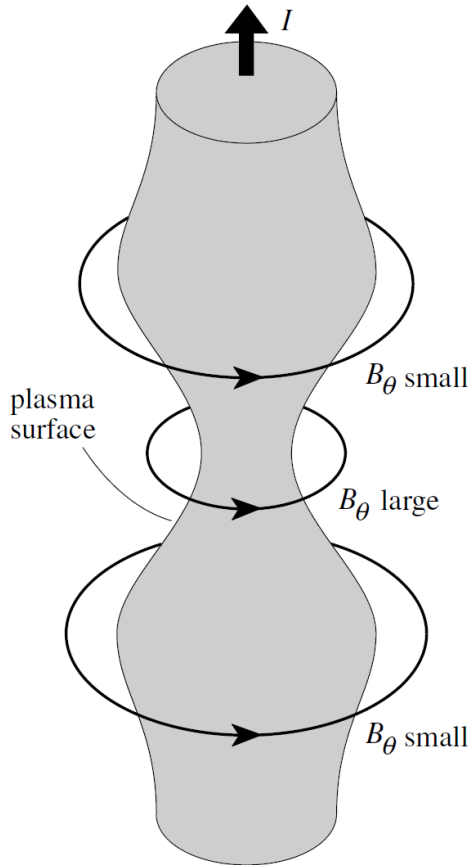
Kink instability
($m=1$)

$$\zeta(\vec{r}) = \zeta(r) \exp(im\theta + ikz)$$

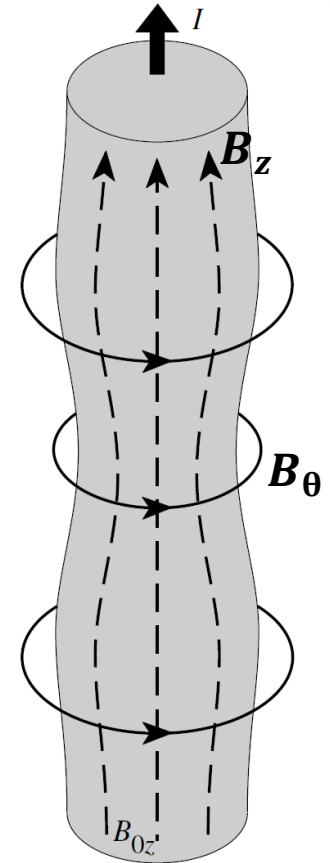
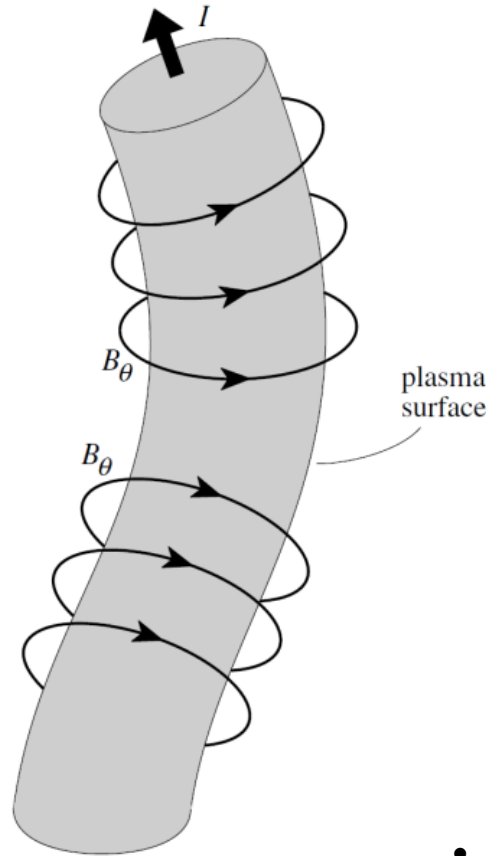
A cylindrical plasma column is stable when the safety factor is greater than unity (Kruskal–Shafranov limit)



- Sausage instability ($m=0$)



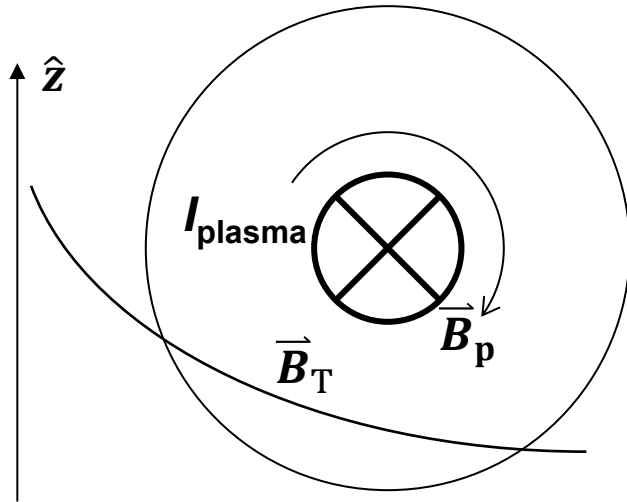
- Kink instability



- MHD Safety factor: $q(r) = \frac{rB_z(r)}{R_0B_\theta(r)}$
- Kruskal–Shafranov limit: $q(r) > 0$: stable

- The tension of B_z provides the stabilizing force and suppresses the instabilities.

Sawtooth oscillation is initiated when the core is heated



$$\eta \propto T_e^{-3/2} \quad E_{\parallel} = \eta J$$

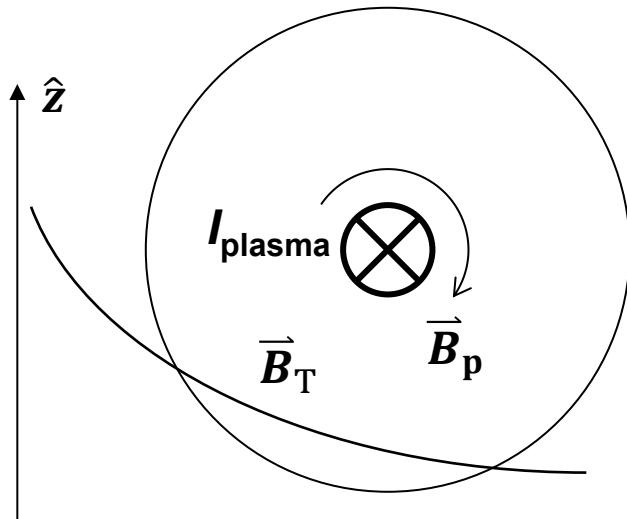
$$T_e \uparrow \Rightarrow \eta \downarrow \Rightarrow J \uparrow$$

- Current is less diffused.
- Current density with a higher peak is formed.

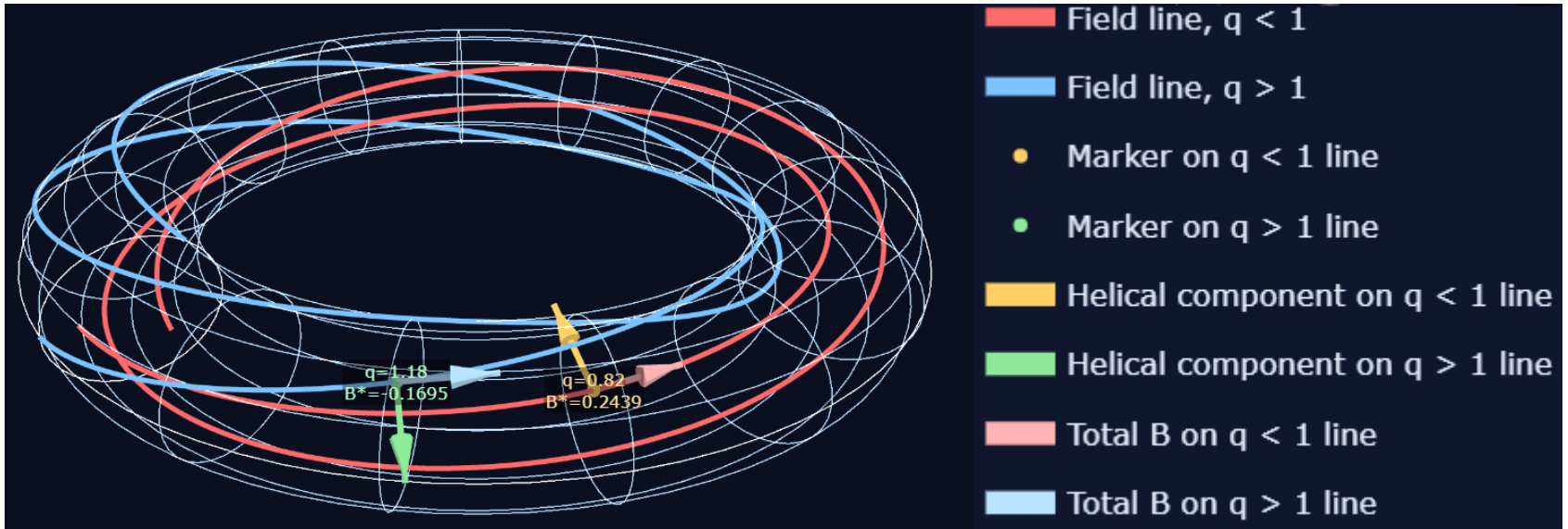
$$J \uparrow \Rightarrow \vec{B}_p \uparrow \Rightarrow q(r) = \frac{r B_T(r)}{R_0 B_p(r)} \downarrow$$

- For a certain radius a ,

$$q(a) = \frac{r B_T(a)}{R_0 B_p(a)} < 0 \Rightarrow \text{kink unstable}$$



The helical field component B^* is opposite across the surface of $q=1$



$$B_* = \vec{B} \cdot \nabla(m\theta - n\phi)$$

$$B_* = \vec{B} \cdot \nabla(\theta - \phi) = \frac{B_p}{r} - \frac{B_T}{R} = \frac{B_T}{R} \left(\frac{1}{q} - 1 \right)$$

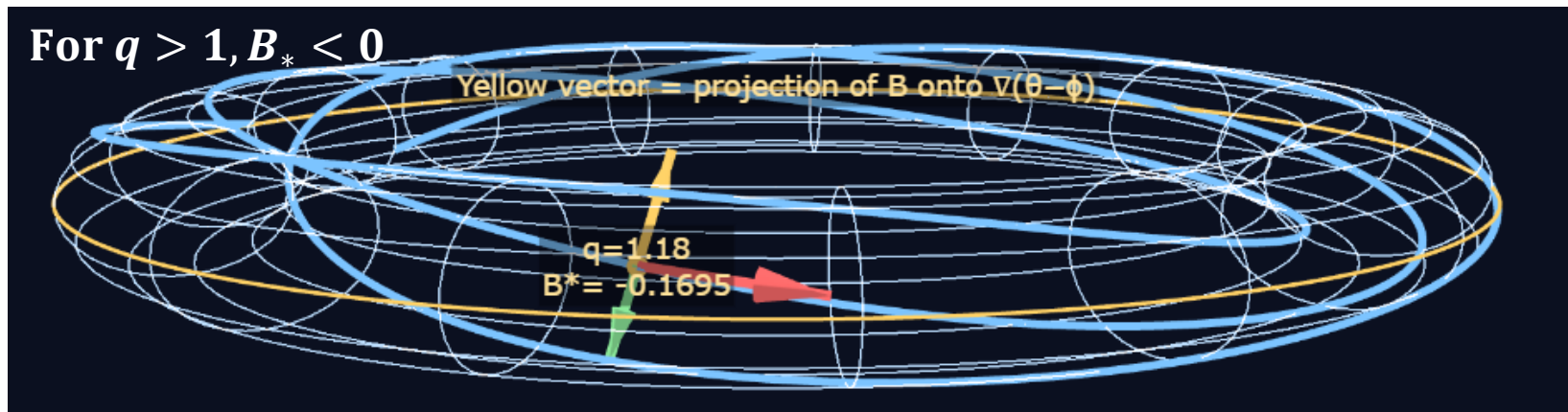
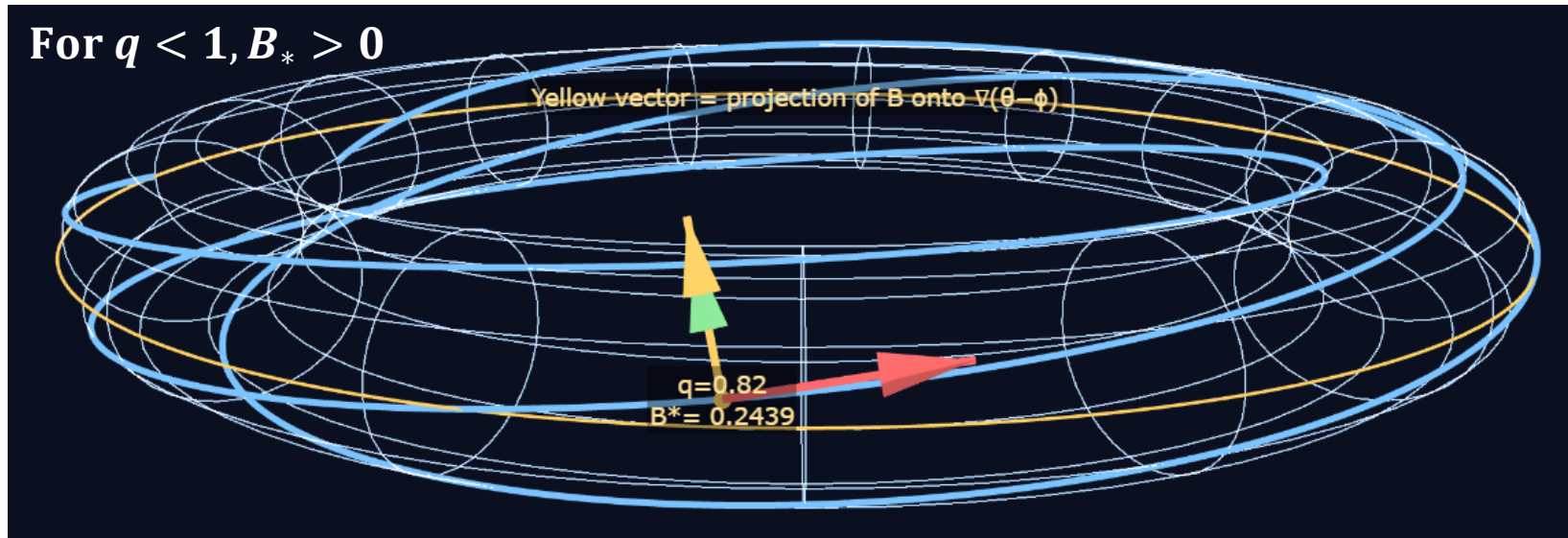
$$q(r) = \frac{rB_T(a)}{R_0B_p(a)}$$

For $q < 1$, $B_* > 0$

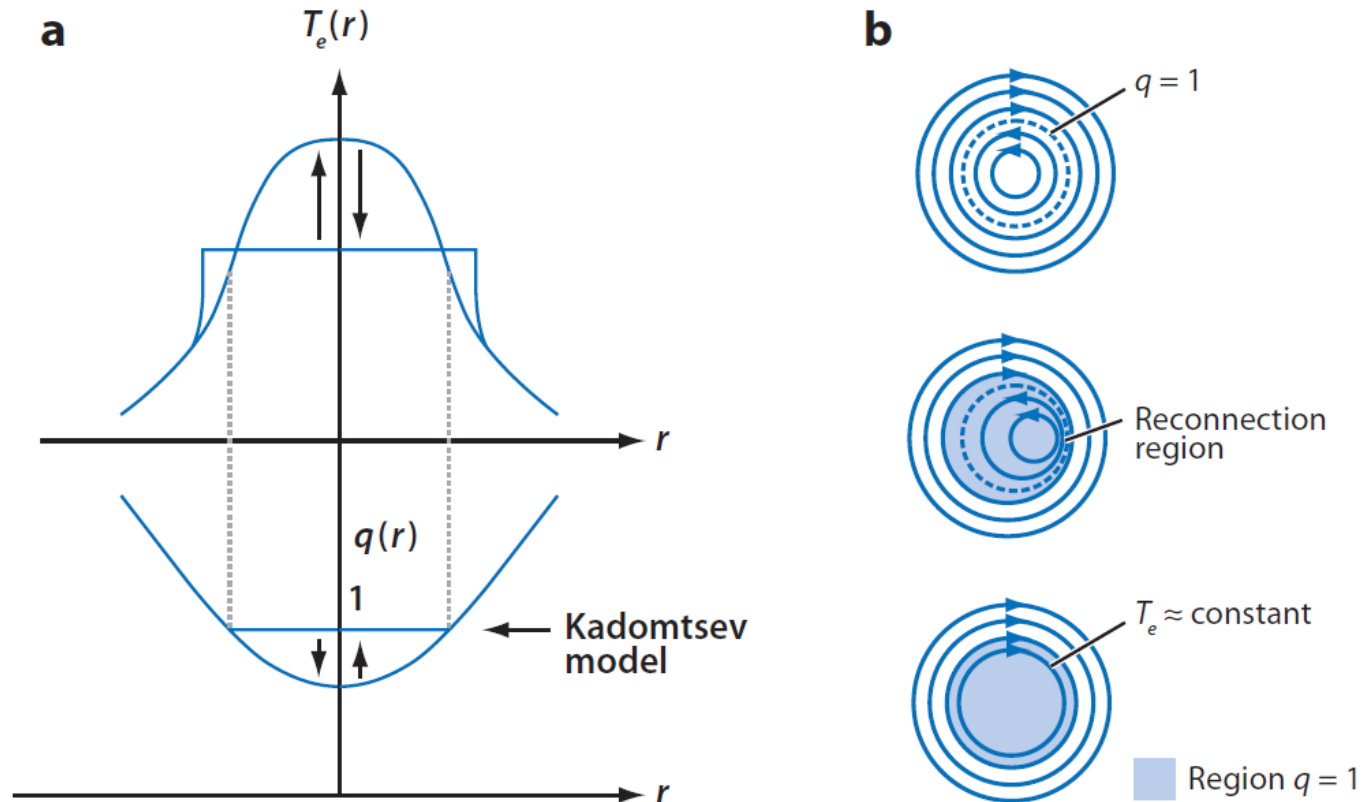
For $q = 1$, $B_* = 0$

For $q > 1$, $B_* < 0$

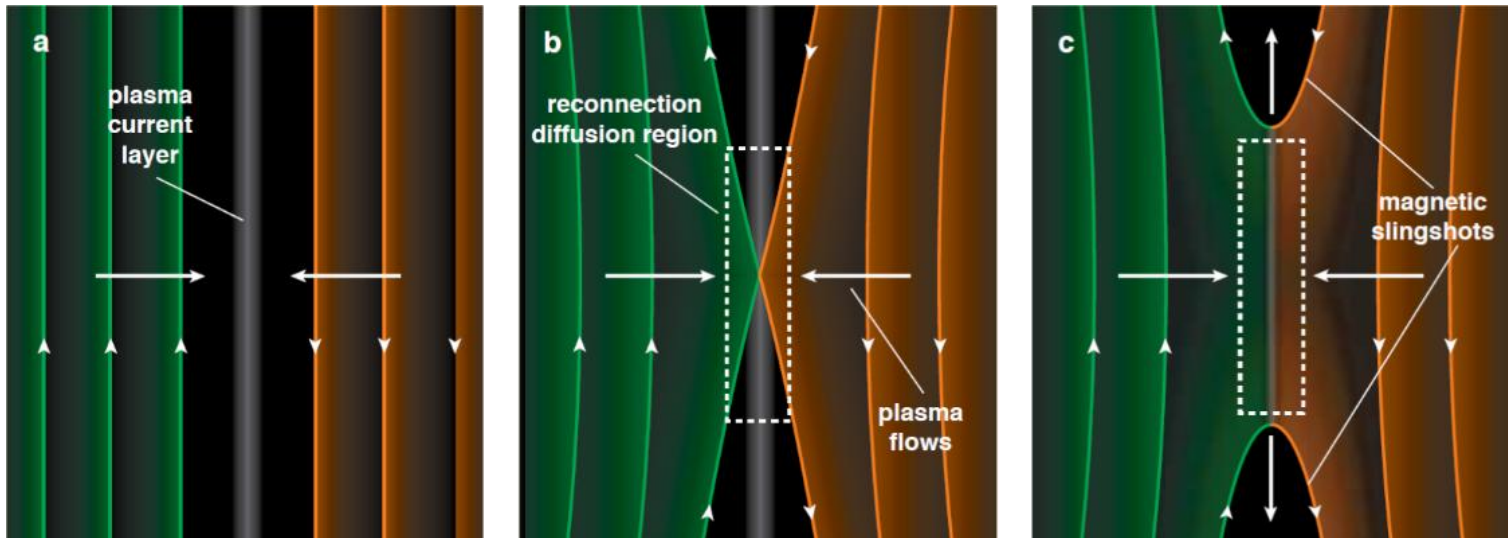
The helical field component B^* is opposite across the surface of $q=1$



Kink instabilities occur at $q < 0$ leading to reconnection events



Reconnection converts the magnetic field energy to kinetic energy of particles



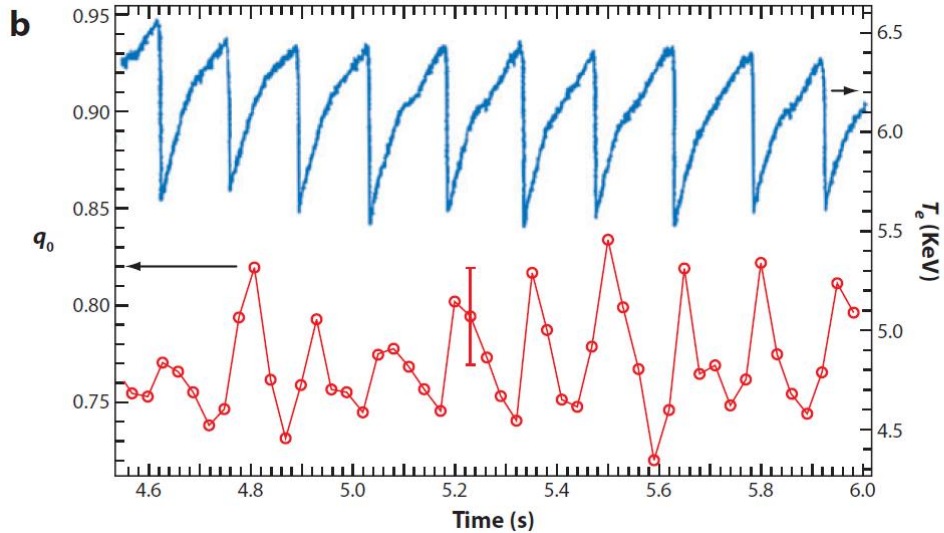
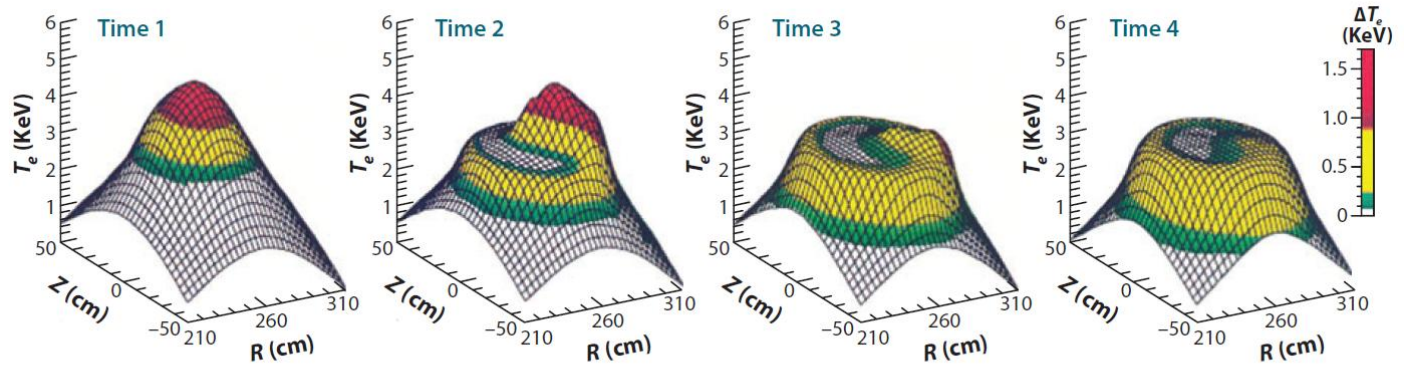
<https://www.youtube.com/watch?v=7sS3Lpzh0Zw>

- Energetic particles are thermalized when they collide with the surrounding particles.

Temperature and safety factor oscillates during the sawtooth crash



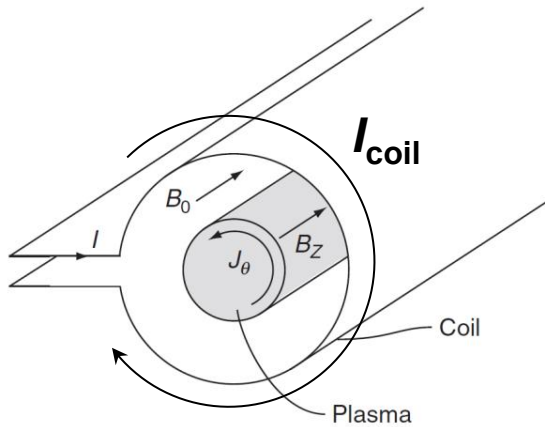
a



Theta pinch is stable while z pinch is unstable



- **Theta pinch**

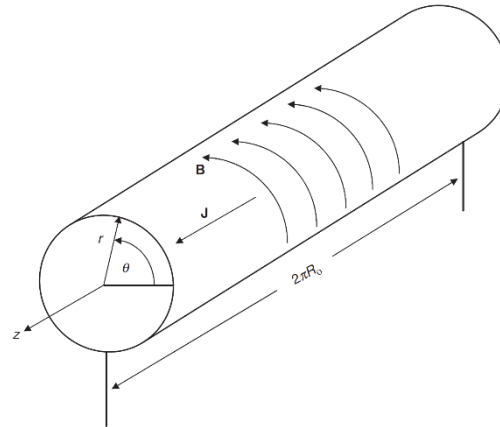


$$\vec{B} = B_z \hat{z}$$

$$q_\theta = \infty$$

Stable

- **Z pinch**



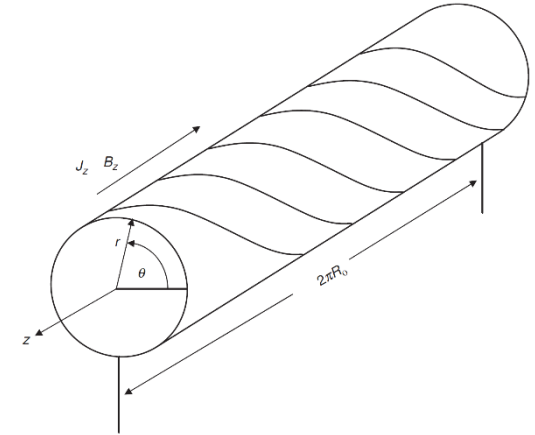
$$\vec{B} = B_\theta \hat{\theta}$$

$$q_z = 0$$

Unstable

$$q(r) = \frac{r B_z(r)}{R_0 B_\theta(r)}$$

- **Screw pinch**



$$\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

q can be controlled.

Stable/Unstable