Introduction to Nuclear Fusion as An Energy Source

Po-Yu Chang

Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 4

2025 spring semester

Tuesday 9:00-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

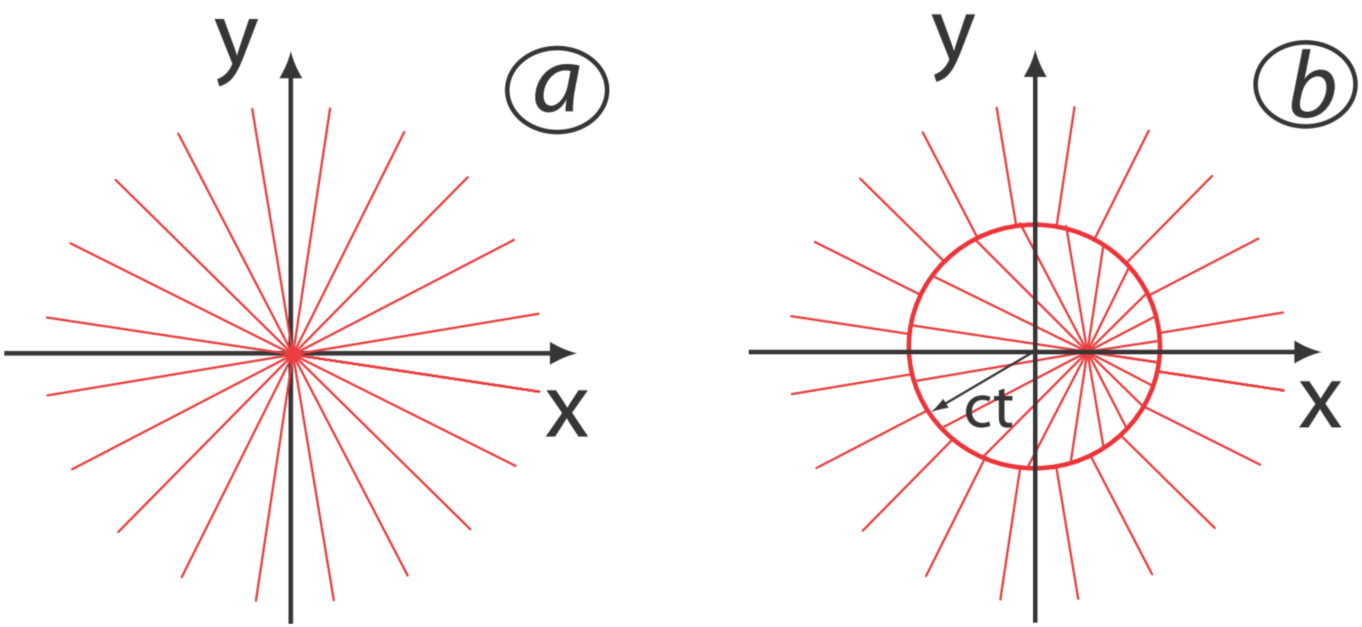
Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=mf1a33a5dab5eb71de9da4380ae888592

2025/3/11 updated

Electromagnetic wave is radiated when a charge particle is accelerated

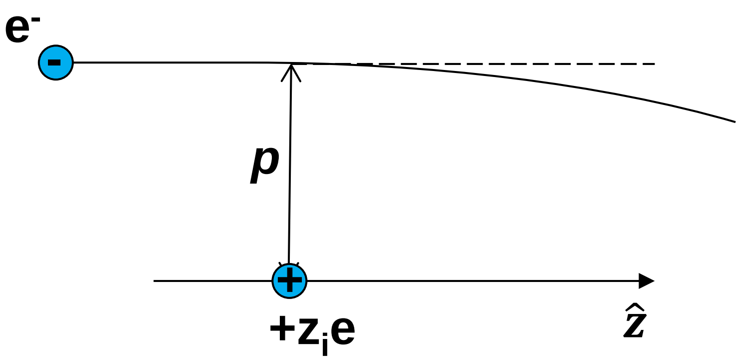
- **The retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.**



https://jseldredphysics.files.wordpress.com/2018/04/lecture18.pdf

Bremsstrahlung emission

- When an electron collides with a nucleus via coulomb interaction, the electron is accelerated and thus radiates, called Bremsstrahlung radiation.
- In the non-relativistic limit, the Bremsstrahlung radiation power is:



$$P_{\mathrm{B,e1,i1}} = \frac{e^2}{6\pi\epsilon_o}\frac{\dot{v}^2}{c^3}$$

p: Impact parameter

- The coulomb force experienced by the electron is:

$$\dot{v} = \frac{F}{m_e} = \frac{ze^2}{4\pi\epsilon_o m_e r^2} = \frac{ze^2}{4\pi\epsilon_o m_e[p^2+(vt)^2]} \approx \frac{ze^2}{4\pi\epsilon_o m_e p^2}$$

=>

$$P_{\mathrm{B,e1,i1}} = \frac{z^2 e^6}{96\pi^3 {\epsilon_o}^3 c^3 {m_e}^2}\frac{1}{p^4} \quad (\mathrm{W})$$

Bremsstrahlung emission

- For multiple ion species: n_j, z_j

$$\bar{P}_B = \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) n_e T_e{}^{1/2} \sum_j z_j^2 n_{i,j} \left(\frac{W}{m^3}\right)$$

$$= \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) Z_{eff} n_e^2 T_e{}^{1/2} \left(\frac{W}{m^3}\right)$$

where $Z_{eff} \equiv \frac{\sum_j z_j{}^2 n_j}{n_e} = \frac{\sum_j z_j{}^2 n_j}{\sum_j z_j n_j}$ $\qquad n_e = \sum_j z_j n_j$

$$\bar{P}_B = 5.35 \times 10^{-37} Z_{eff} n^2_{e\,(m^{-3})} T^{1/2}_{e\,(keV)} \left(\frac{W}{m^3}\right)$$

$$\bar{P}_B \equiv C_B Z_{eff} n^2_{e\,(m^{-3})} T^{1/2}_{e\,(keV)} \left(\frac{W}{m^3}\right)$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_h: external heating

S_α: α particle heating

$$D + T \longrightarrow He^4\ (3.5\ \mathrm{MeV}) + n\ (14.1\ \mathrm{MeV})$$

$$S_f = E_f n_1 n_2 \langle \sigma v \rangle (\mathrm{W/m^3}) \quad n_D = n_T = \frac{1}{2} n$$

$$S_\alpha = \frac{1}{4} E_\alpha n^2 \langle \sigma v \rangle = \frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2}$$

E_α = 3.5 MeV

$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

S_B: Bremsstrahlung radiation

$$S_B = C_B Z_{\mathrm{eff}} n^2_{\mathrm{e}\ (\mathrm{m}^{-3})} T^{1/2}_{\mathrm{e}\ (\mathrm{keV})} \left(\frac{W}{m^3} \right)$$

$$= \frac{1}{4} C_B \frac{p^2}{T^{3/2}}$$

S_k: heat conduction lost

$$S_\kappa = \frac{3}{2} \frac{p}{\tau}$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$$S_\alpha + S_h = S_B + S_k$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

$$p\tau \geqslant \frac{6}{\frac{1}{4} E_\alpha \frac{\langle \sigma v \rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

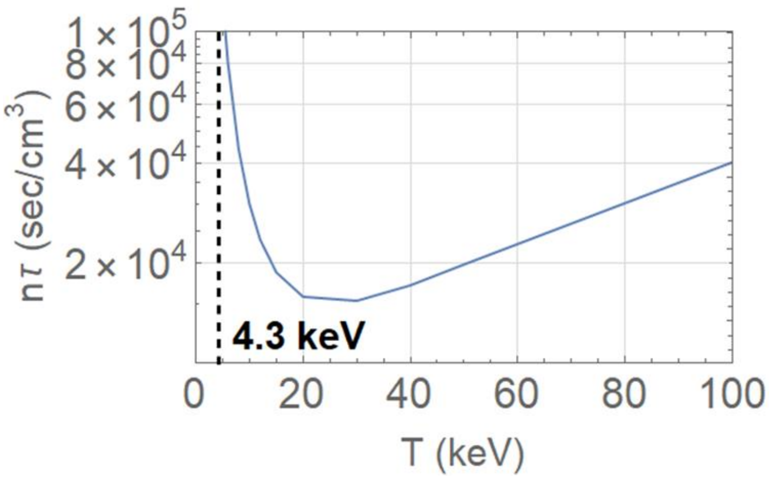
$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

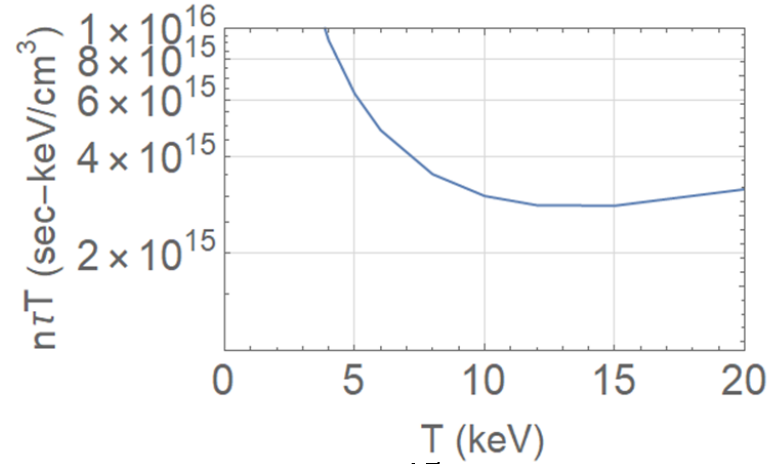
$$nT\tau > \frac{3T^2}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

$$n\tau > \frac{3T}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

Temperature needs to be greater than ~5 keV to ignite

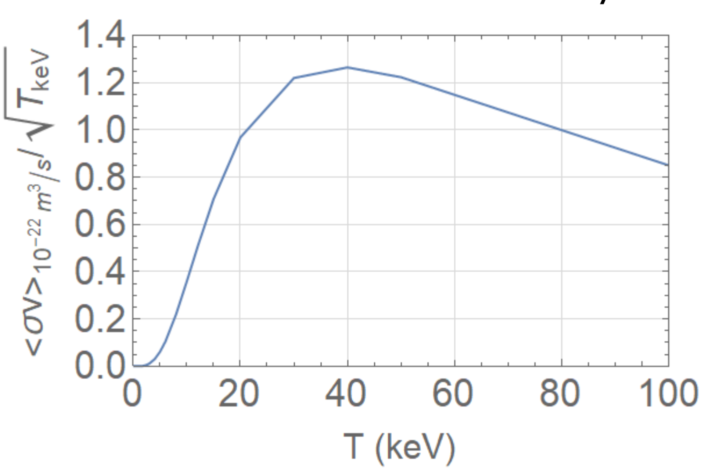
$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$





$$n\tau > 2\times10^4\ \mathrm{sec/cm^3}$$

$$nT\tau > 3.5\times10^{15}\ \mathrm{keV-sec/cm^3}$$



$S_\alpha > S_B$ $$\frac{1}{4}E_\alpha n^2\langle\sigma v\rangle > C_B n^2 T^{1/2}$$

$$\frac{\langle\sigma v\rangle}{T^{1/2}} > \frac{4C_B}{E_\alpha}$$ $$T > 4.3\ \mathrm{keV}$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$$S_\alpha + S_h = S_B + S_k$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

$$p\tau \geqslant \frac{6}{\frac{1}{4} E_\alpha \frac{\langle \sigma v \rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

$$nT\tau > \frac{3T^2}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

$$n\tau > \frac{3T}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

Under what conditions the plasma keeps itself hot?

- Steady state 0-D power balance:

$$S_\alpha + S_h = S_B + S_k$$

S_α: α particle heating

S_h: external heating

S_B: Bremsstrahlung radiation

S_k: heat conduction lost

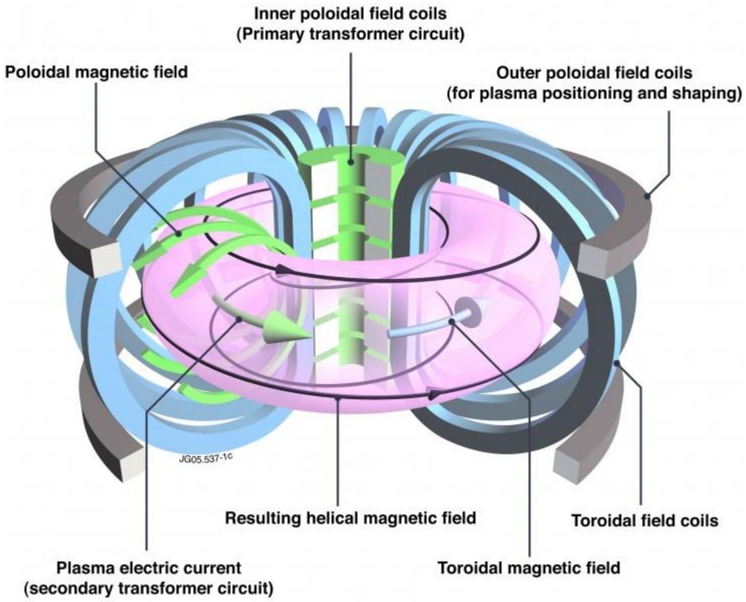
Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

- P: pressure, or called energy density
- τ is confinement time

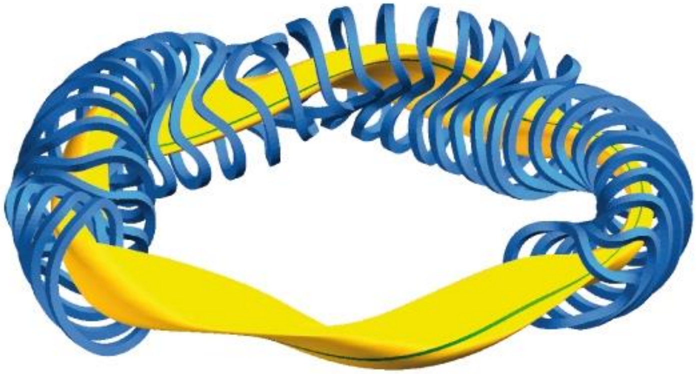
The plasma is too hot to be contained

- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. P~atm, τ~sec, T~10 keV (10^8 ⁰C)**

Tokamak



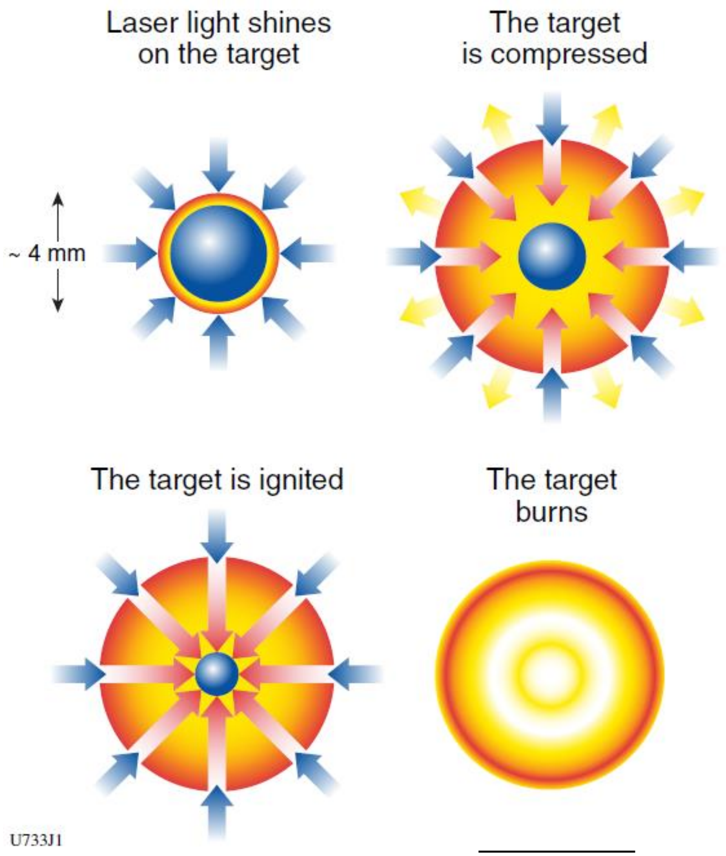
Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Don’t confine it!

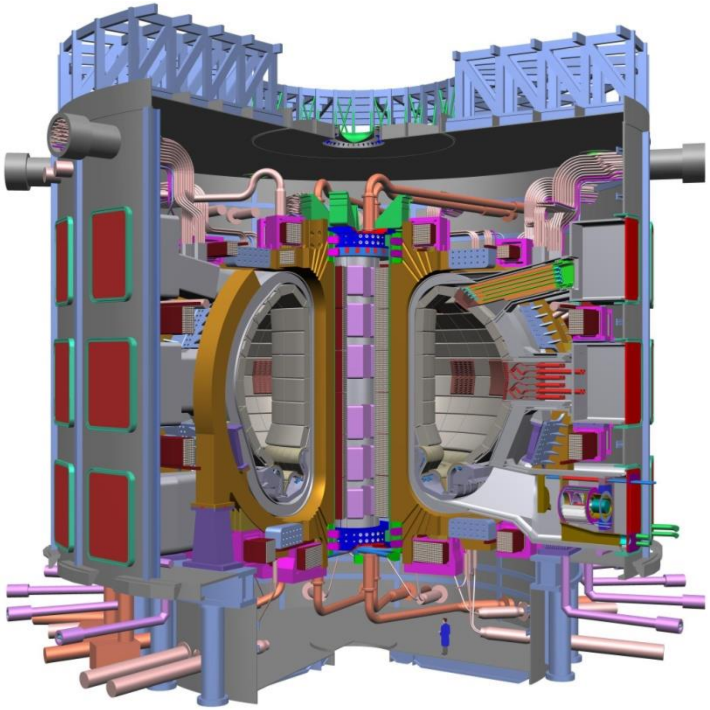
- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ~nsec, T~10 keV (10^8 ^{0}C)**



Inertial confinement fusion: an introduction, Laboratory for Laser Energetics, University of Rochester

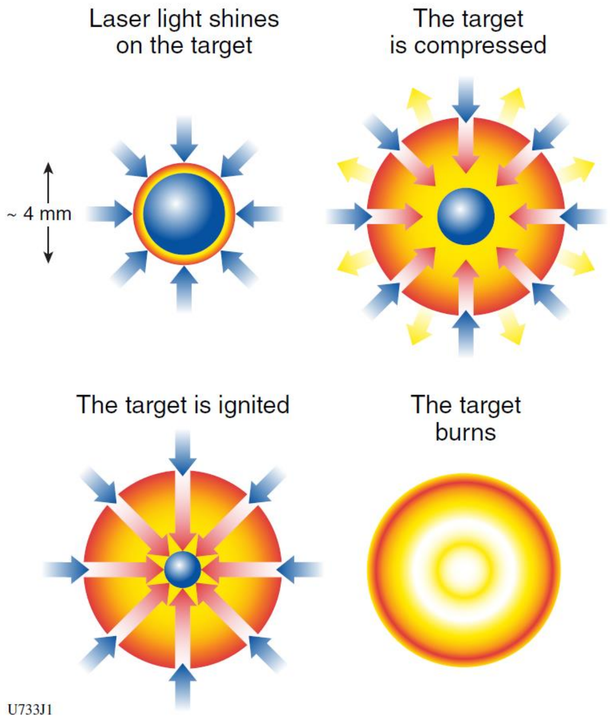
To control? Or not to control?

- **Magnetic confinement fusion (MCF)**



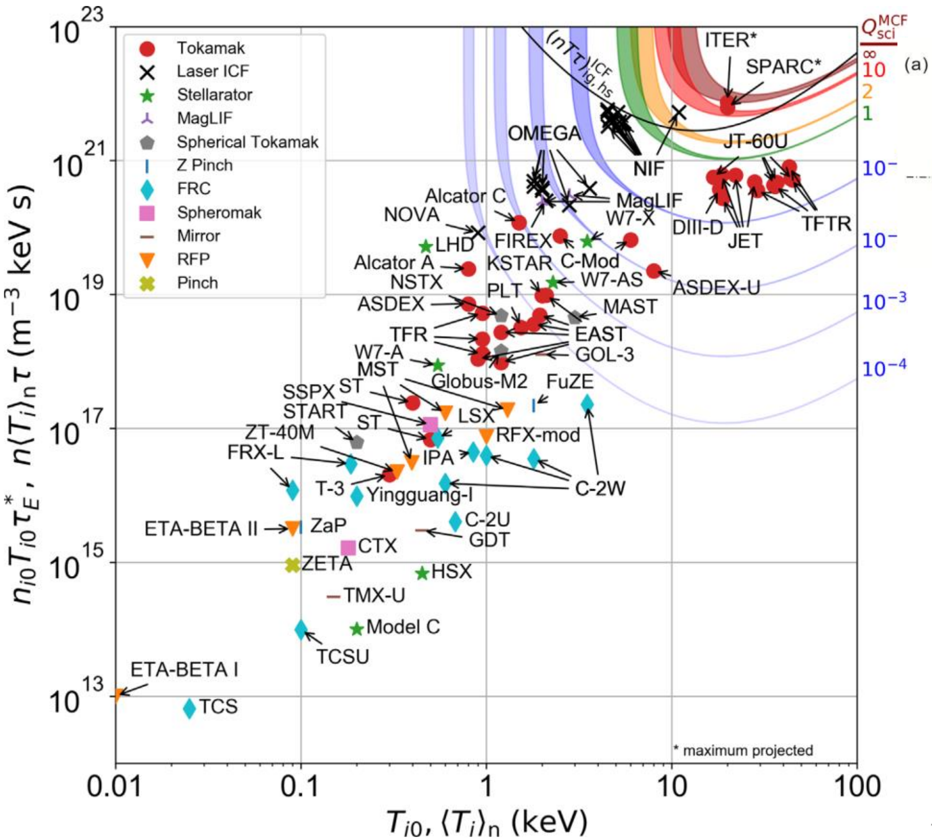
  - **Plasma is confined by toroidal magnetic field.**

- **Inertial confinement fusion (ICF)**

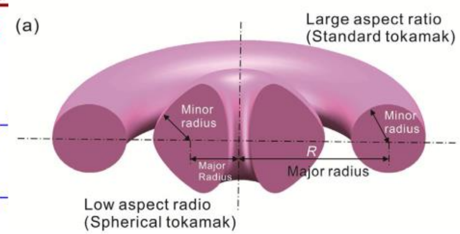


  - **A DT ice capsule filled with DT gas is imploded by laser.**

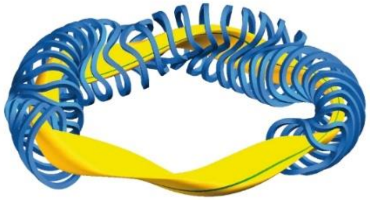
We are closed to ignition!



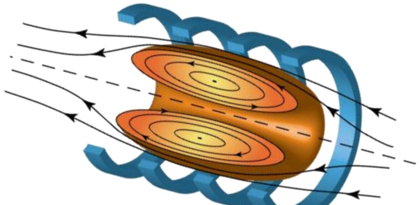
MCF | **ICF**



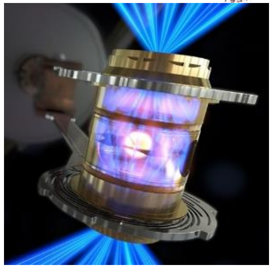
(Spherical) tokamak



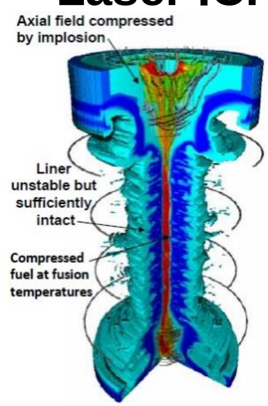
Stellerator



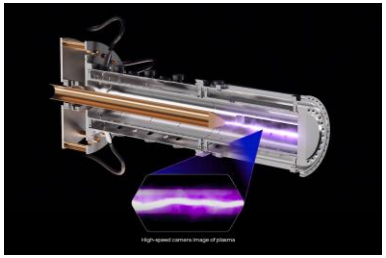
FRC



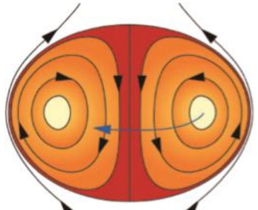
Laser ICF



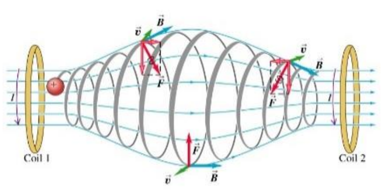
MagLIF



Sheared-flow Z pinch



Spheromak

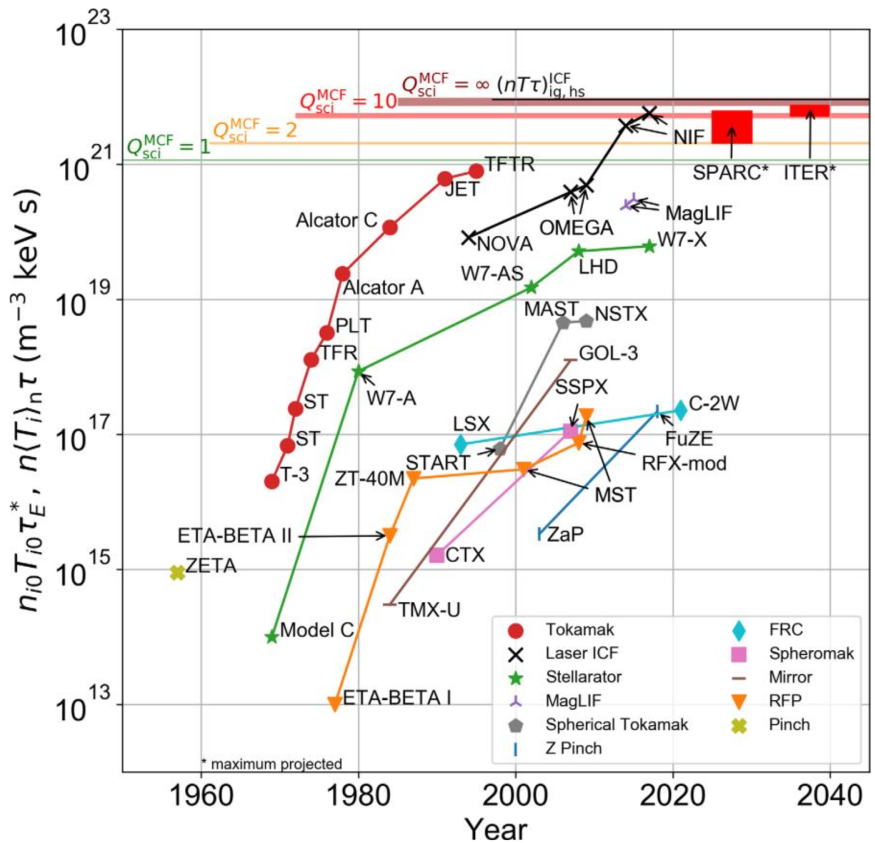


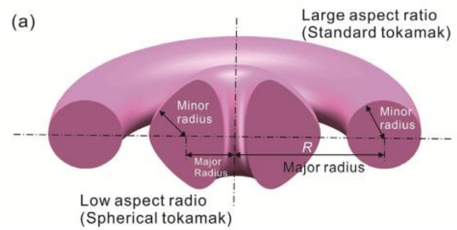
Mirror

Samuel E. Wurzel and Scott C. Hsu, Phys. Plasmas, **29**, 062103 (2022)

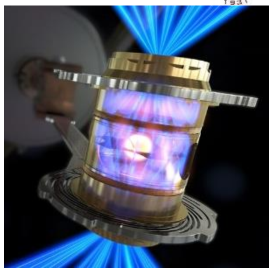
We are closed to ignition!

MCF | **ICF**

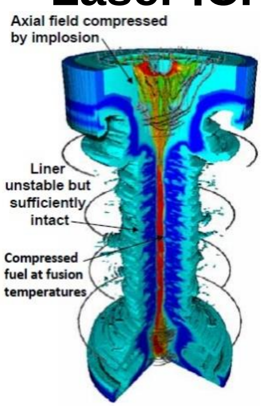




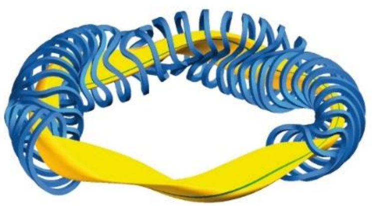
(Spherical) tokamak



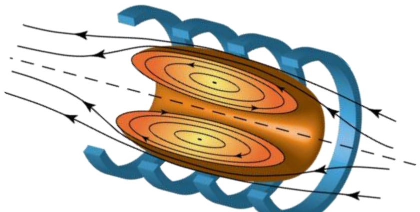
Laser ICF



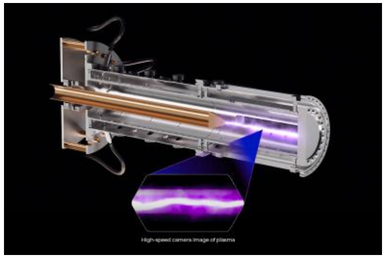
MagLIF



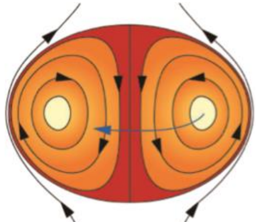
Stellerator



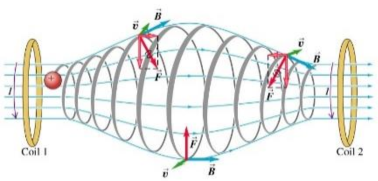
FRC



Sheared-flow Z pinch



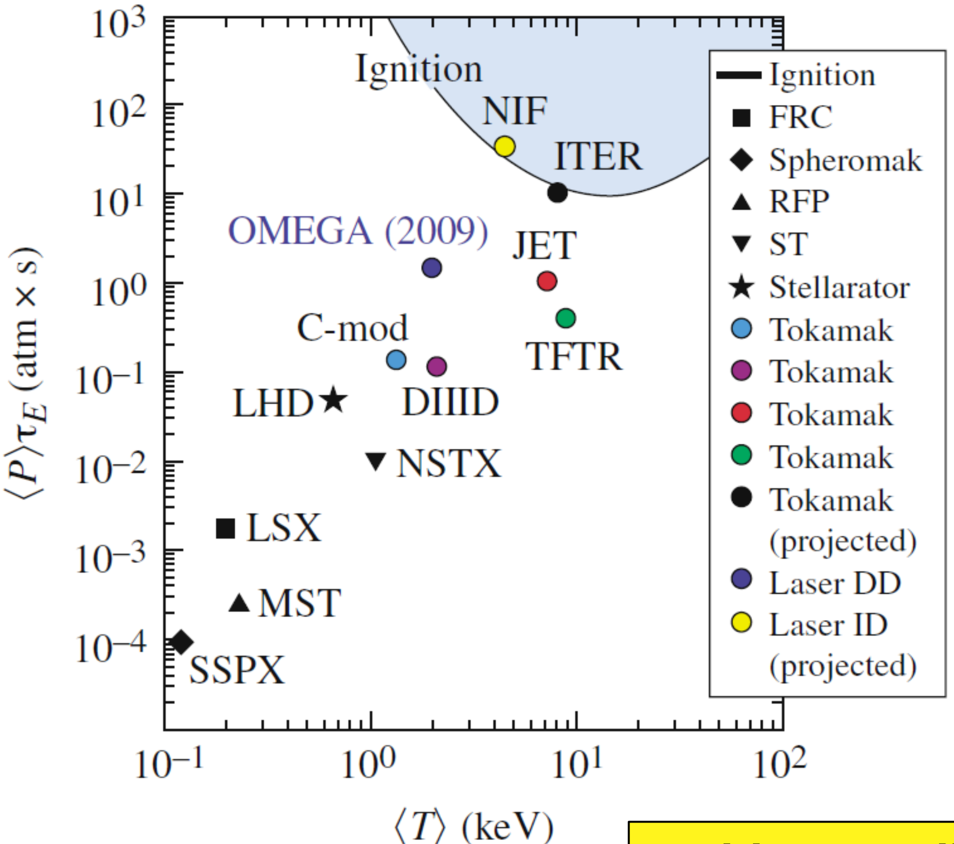
Spheromak

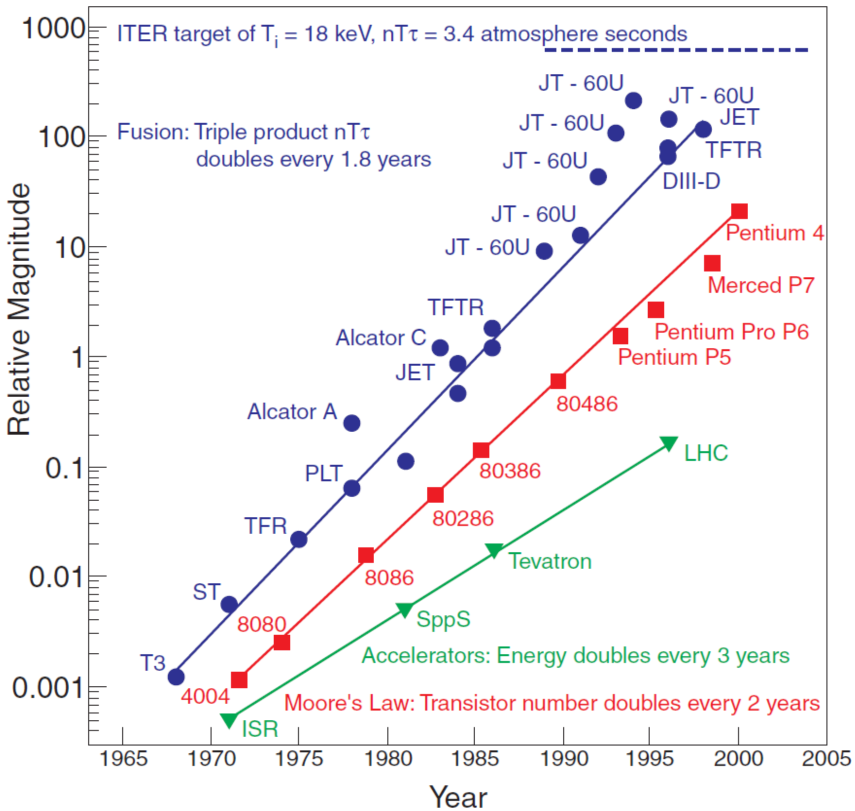


Mirror

Samuel E. Wurzel and Scott C. Hsu, Phys. Plasmas, **29**, 062103 (2022)

We are closed to ignition!





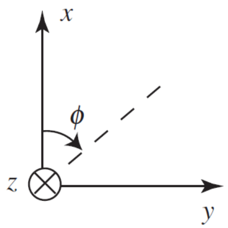
Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

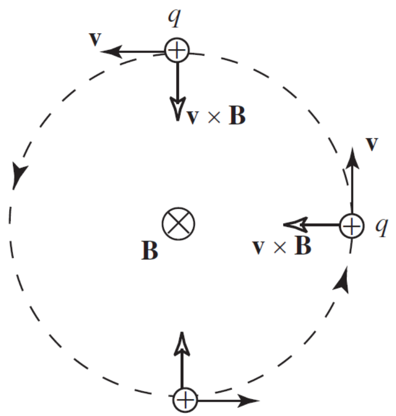
A. J. Webster, Phys. Educ. **38**, 135 (2003)
R. Betti, etc., Phys. Plasmas, **17**, 058102 (2010)

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Charged particles gyro around the magnetic field line





$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \qquad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

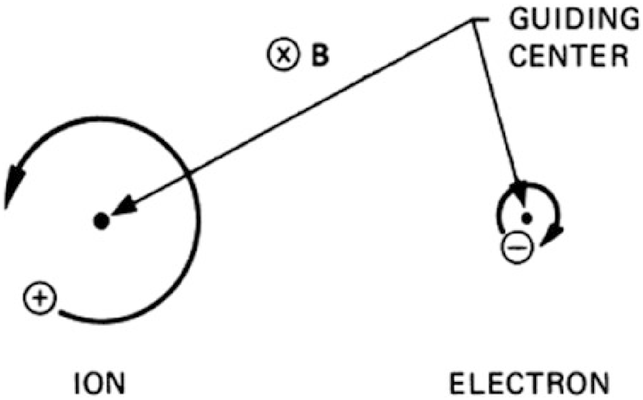
$$\omega_c \equiv \frac{|q|B}{m}$$ Cyclotron frequency or gyrofrequency

$$\ddot{v}_x + {\omega_c}^2 v_x = 0$$

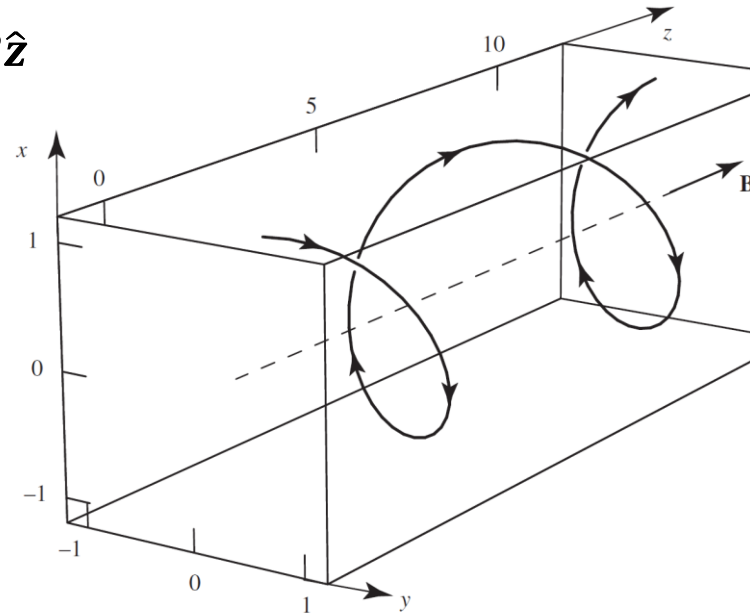
$$\ddot{v}_y + {\omega_c}^2 v_y = 0$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$\vec{B} = B\hat{z}$$



$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = v_\perp \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

$$x = \mp r_c sin(\pm\omega_c t + \psi) + (x_o - r_c sin\psi)$$

$$y = \pm r_c cos(\pm\omega_c t + \psi) + (y_o + r_c cos\psi)$$

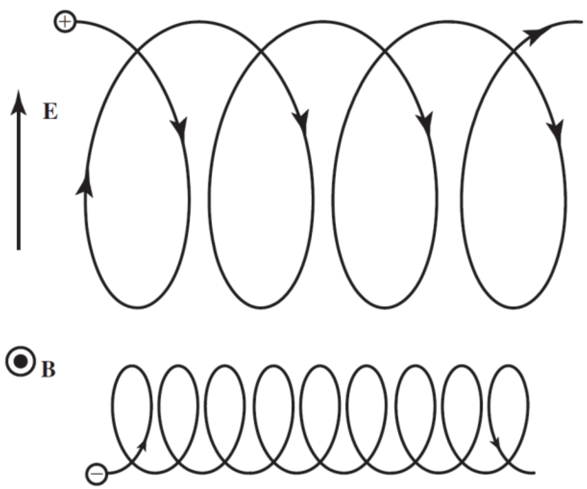
$$z = z_o + v_{||} t$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left|\frac{m v_\perp{}^2}{r}\right| = \left|q\,\vec{v} \times \vec{B}\right| = |q v_\perp B|$$

$$r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$$ **Larmor radius or gyroradius**

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{||} = \hat{x}E_{\perp} + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$v_{||}(t) = \frac{qE_{||}}{m}t + v_{||,o}$$

$$\vec{v}_{\perp}(t) = \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$m\frac{d}{dt}\left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) = q\left[\hat{x}E_{\perp} + \left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) \times \hat{z}B\right]$$

$$m\frac{d\vec{v}_{\mathrm{ac}}(t)}{dt} = q[\hat{x}E_{\perp} + \vec{v}_{\mathrm{E}} \times \hat{z}B + \vec{v}_{\mathrm{ac}}(t) \times \hat{z}B]$$

No E field case: $m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$

$$\hat{x}E_{\perp} + \vec{v}_{\mathrm{E}} \times \hat{z}B = 0$$

$\times\,\hat{z}B$

$$\left(\vec{C} \times \vec{B}\right) \times \vec{A} = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C}$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$m\frac{d\vec{v}_{\mathrm{ac}}(t)}{dt} = q\,\vec{v}_{\mathrm{ac}}(t) \times \hat{z}B$$

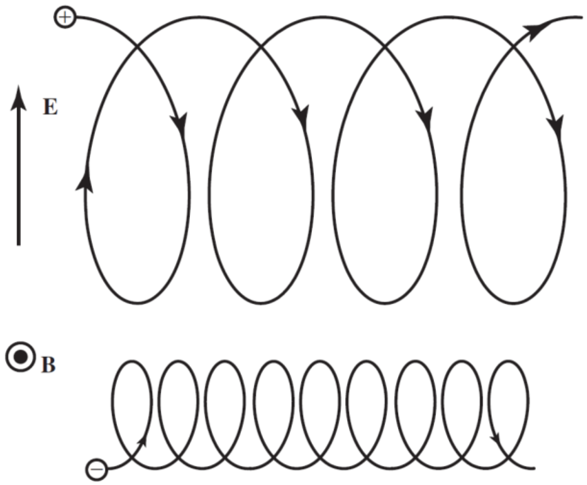
Gyro motion

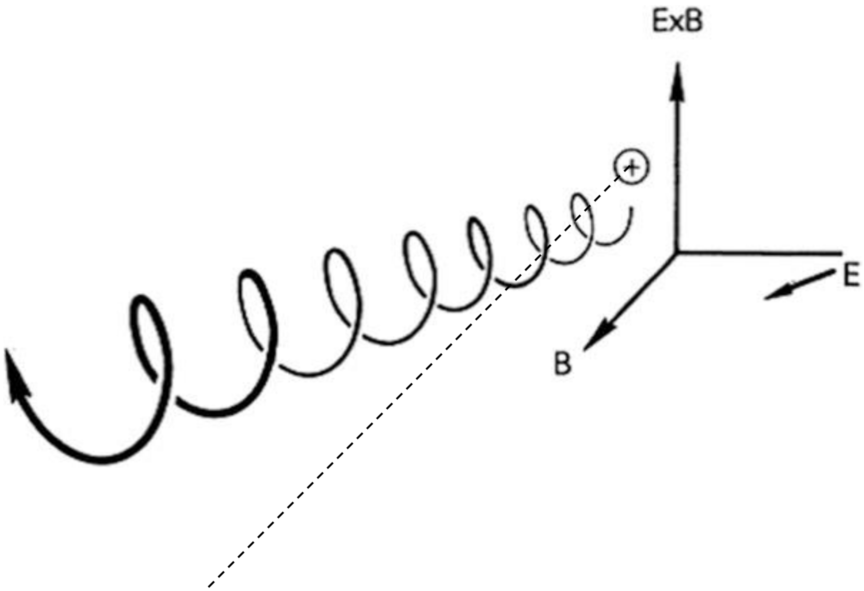
$$\vec{v}(t) = \hat{z}v_{||}(t) + \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$\langle\vec{v}(t)\rangle = \frac{1}{T_c}\int_0^{T_c}\vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

- **Electrons and ions drift in the same direction.**

No current is generated in ExB drift





$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_\mathrm{E} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E}\times\vec{B}}{B^2}$$ ExB drift velocity

- **Electrons and ions drift in the same direction.**

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs

$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_\mathrm{E} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_\perp + \hat{z}F_{||} = \hat{x}F_\perp + \hat{z}F_{||}$$

$$m\frac{dv_{||}}{dt} = F_{||}$$

$$m\frac{d\vec{v}_\perp}{dt} = q\left(\hat{x}\frac{F_\perp}{q} + \vec{v}_\perp \times \hat{z}B\right)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_F$$

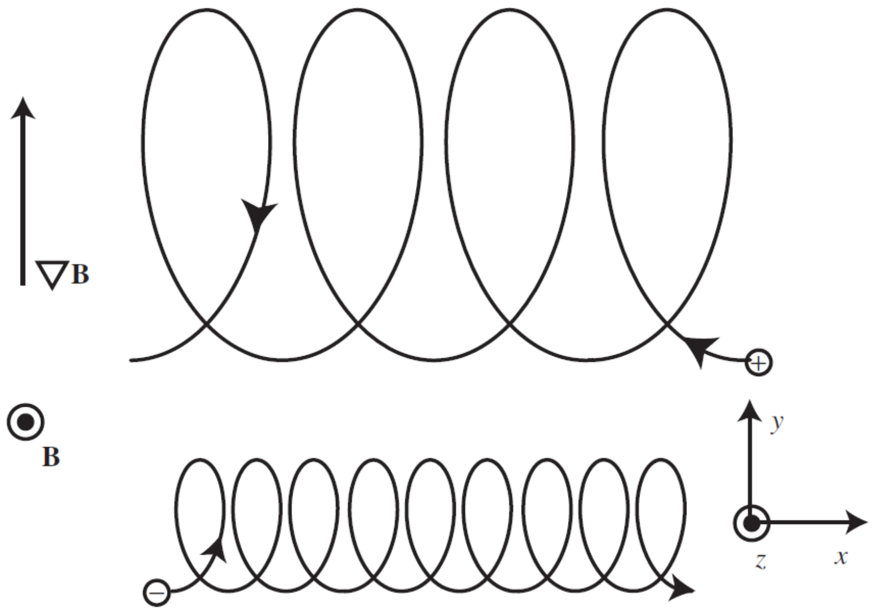
$$\vec{v}_\mathrm{F} = \frac{\hat{x}(F_\perp/q) \times \hat{z}B}{B^2} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

Gravitational drift velocity

- **Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.**

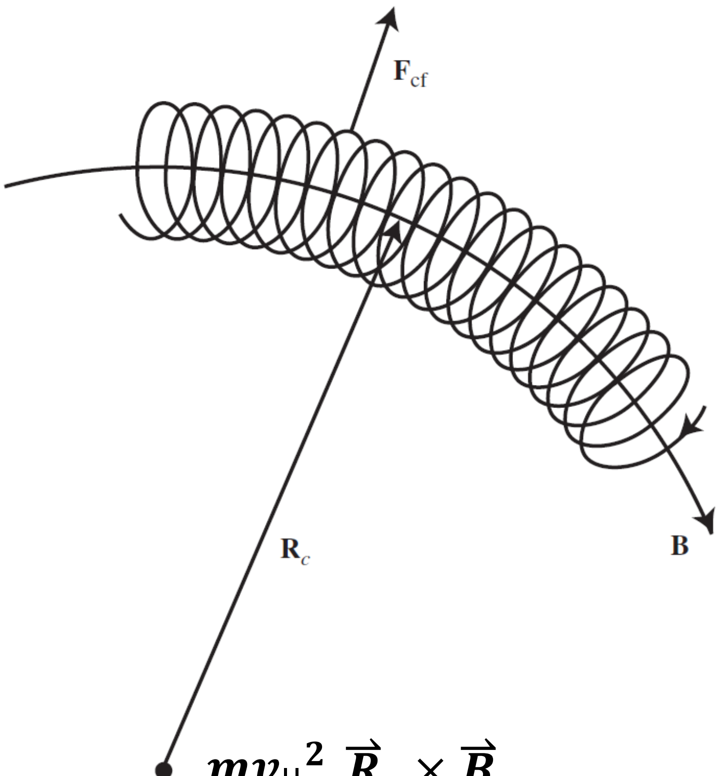
Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**



$$\vec{v}_{\nabla} = \frac{m{v_{\perp}}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$$

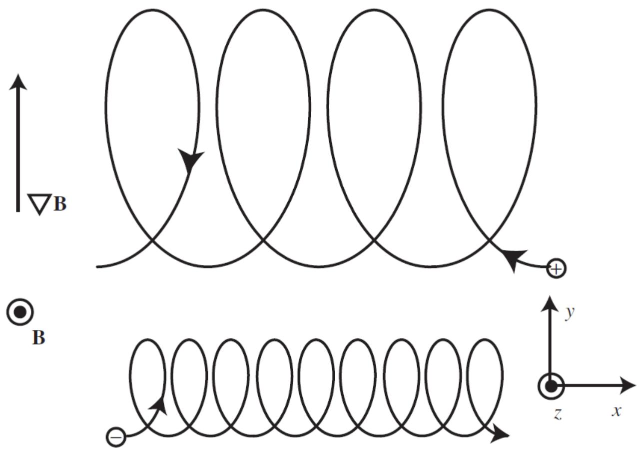
- **Curvature drift**



$$\vec{v}_{R} = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\vec{v}_{total} = \vec{v}_R + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x} q v_y B_z - \hat{y} q v_x B_z$$

$$\simeq \hat{x} q v_y \left(B_o + y \frac{\partial B_z}{\partial y} \right) - \hat{y} q v_x \left(B_o + y \frac{\partial B_z}{\partial y} \right)$$

$$B_z(y) = B_o + y \frac{\partial B_z}{\partial y} + y^2 \frac{1}{2} \frac{\partial^2 B_z}{\partial y^2} + \ldots$$

$$F_x = q v_y \left(B_o + y \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -q v_x \left(B_o + y \frac{\partial B_z}{\partial y} \right)$$

- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm \omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm \omega_c t + \psi)$$

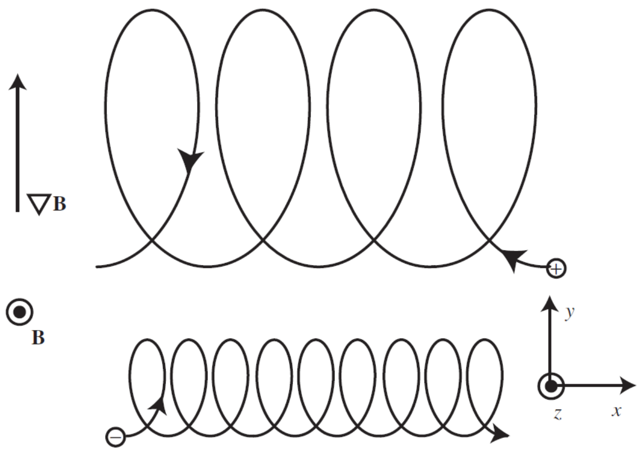
$$v_x = v_\perp \cos(\pm \omega_c t + \psi)$$

$$v_y = -v_\perp \sin(\pm \omega_c t + \psi)$$

$$F_x \simeq -q v_\perp \sin(\pm \omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -q v_\perp \cos(\pm \omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_\perp \sin(\pm\omega_c t + \psi)\left[B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right]$$

$$F_y \simeq -qv_\perp \cos(\pm\omega_c t + \psi)\left[B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right]$$

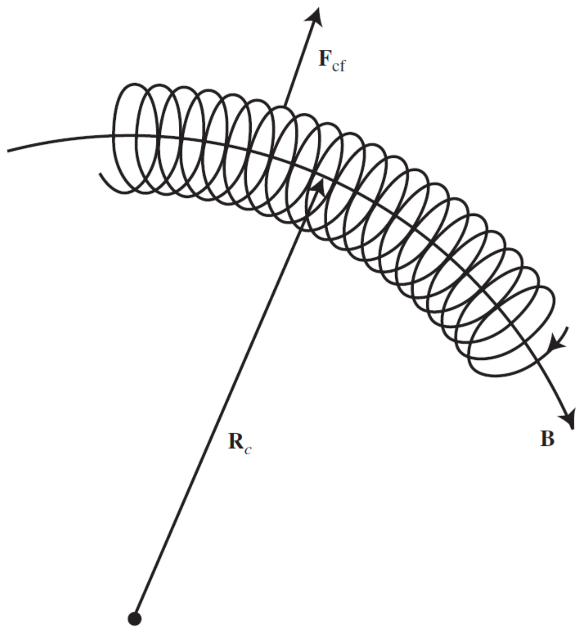
$$\langle F_x \rangle = 0$$

$$\langle F_y \rangle = \mp \frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} = -\frac{m{v_\perp}^2}{2B}\frac{\partial B_z}{\partial y} \qquad r_c = \frac{v_\perp}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \qquad \vec{v}_\nabla = \frac{1}{q}\frac{\langle F_y\rangle\hat{y}\times\hat{z}B_z}{{B_z}^2} = -\frac{m{v_\perp}^2}{2q{B_z}^2}\frac{\partial B_z}{\partial y}\hat{x}$$

- More general: $\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$

Charge particles drift across magnetic field lines when the magnetic field line is curved

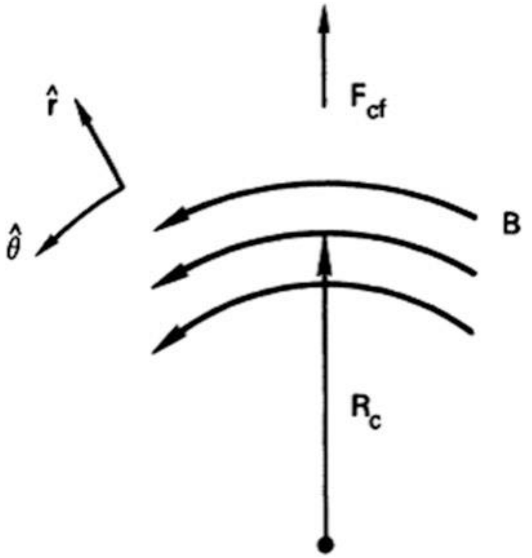


$$\vec{F}_{cf} = m v_{||}^2 \frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q}\frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m v_{||}^2}{q}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3} \qquad \vec{v}_R = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\nabla\times\vec{B} = 0$$

$$\left(\nabla\times\vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z} = 0 \qquad \left(\nabla\times\vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial\theta}$$

$$\vec{B} = B\hat{\theta}$$

$$\nabla\times\vec{B} = \left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \qquad B_\theta \propto \frac{1}{r}$$

$$\nabla B = \nabla B\hat{r}$$

$$\frac{\nabla|B|}{|B|} = -\frac{\vec{R}_c}{{R_c}^2}$$

Cylindrical coordinate

$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right)$$

- **Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.**

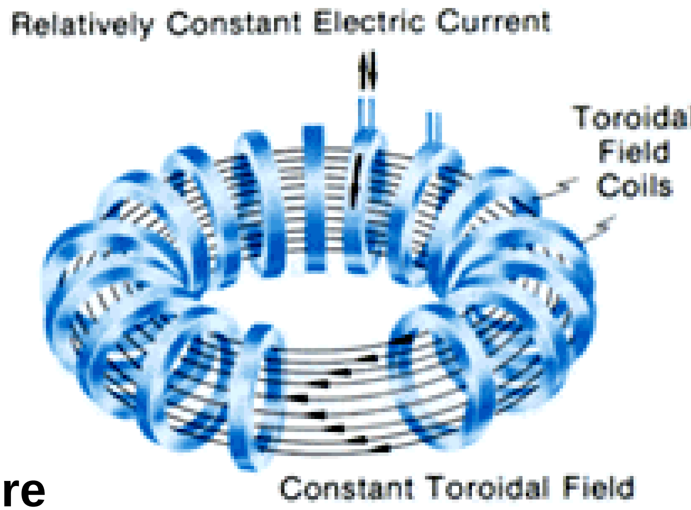
Quick summary of different drifts

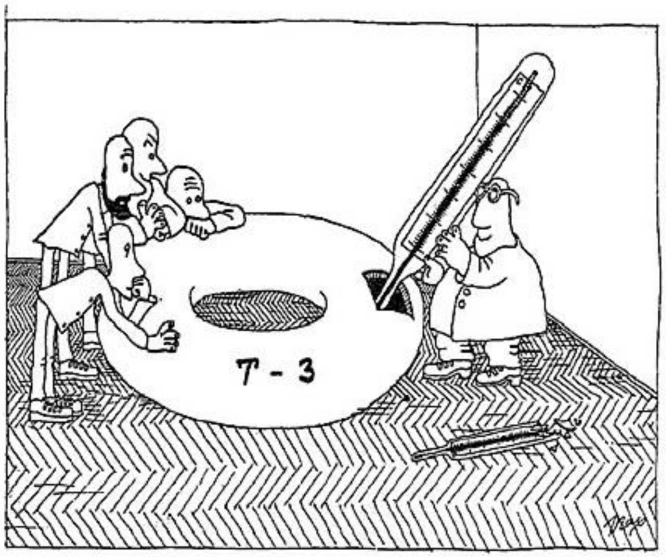
- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m{v_\perp}^2}{2q}\frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \frac{m{v_{||}}^2}{q}\frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_c}^2 B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}} B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right)$$

Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field

- **Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)**





Nature

Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

N. J. PEACOCK
D. C. ROBINSON
M. J. FORREST
P. D. WILCOCK
UKAEA Research Group, Culham Laboratory, Abingdon, Berkshire

V. V. SANNIKOV
I. V. Kurchatov Institute, Moscow

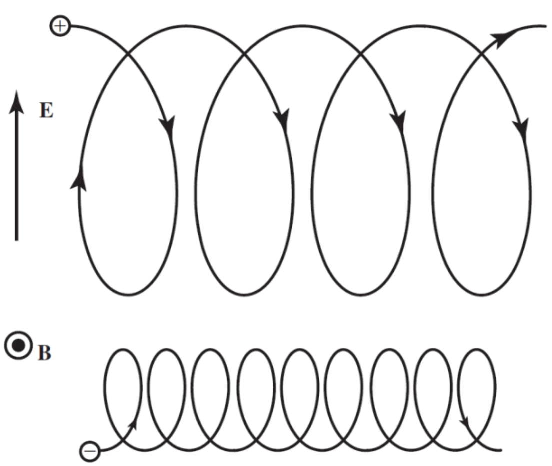
Electron temperatures of 100 eV up to 1 keV and densities in the range $1–3 \times 10^{13}$ cm^{-3} have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

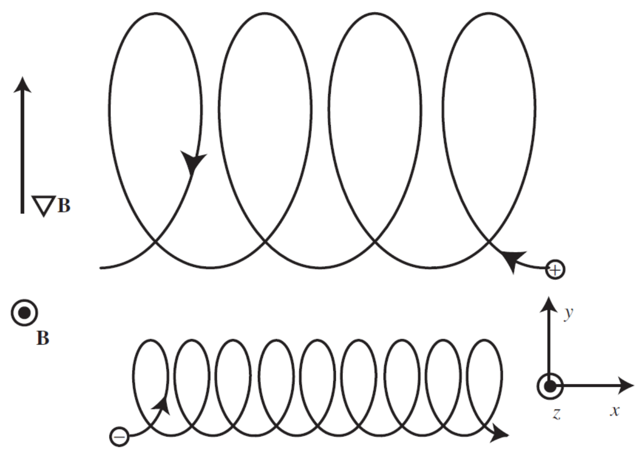
$\boldsymbol{T_e}$ **= 100 ~ 1 keV**
$\boldsymbol{n_e}$ **= 1-3 x 10^{13} cm^{-3}**

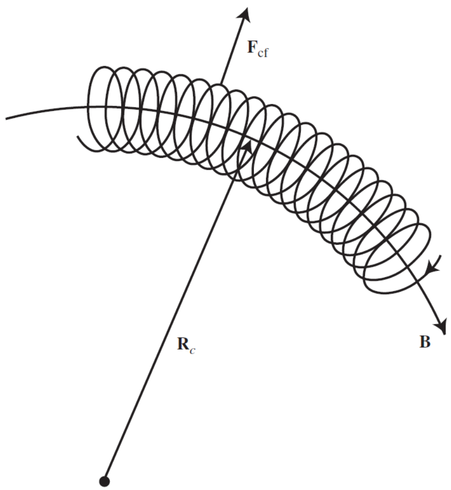
https://www.iter.org/mach/tokamak
https://en.wikipedia.org/wiki/Tokamak#cite_ref-4
Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.
N. J. Peacock,et al., Nature **224**, 488 (1969)

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m{v_\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_R = \frac{m{v_{||}}^2}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_c}^2 B^2}$ **Depended on charge**

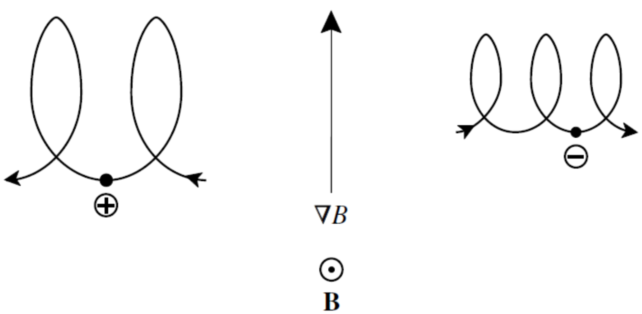




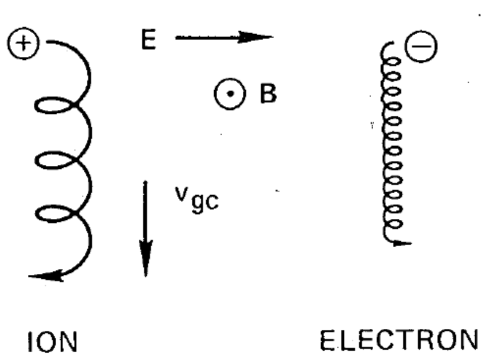


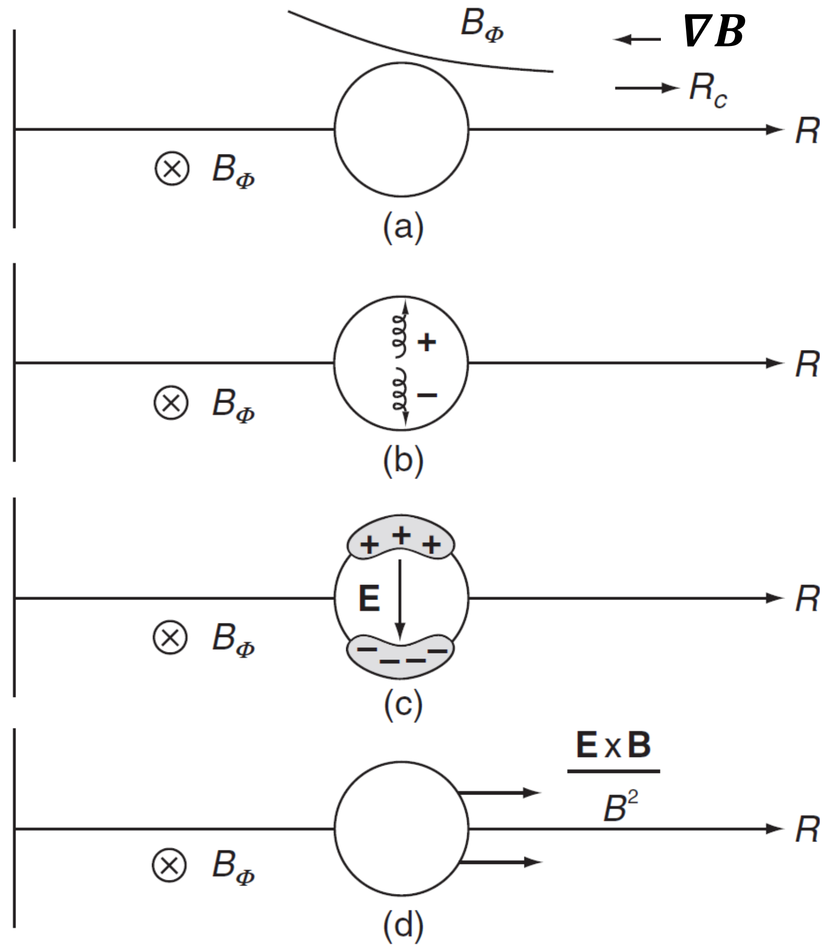
Charged particles drift across field lines

- **Grad-B drift**

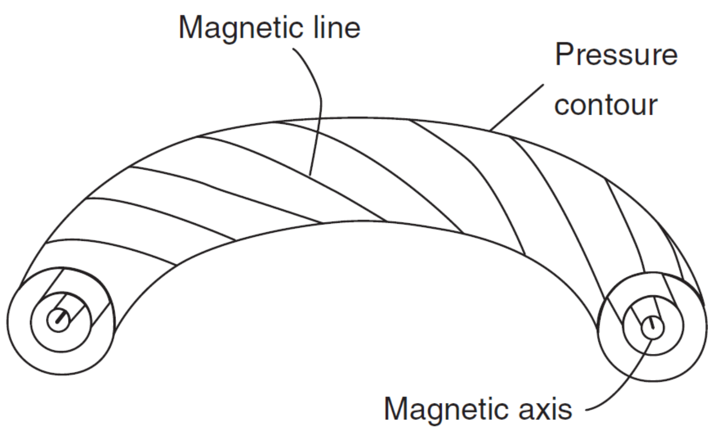


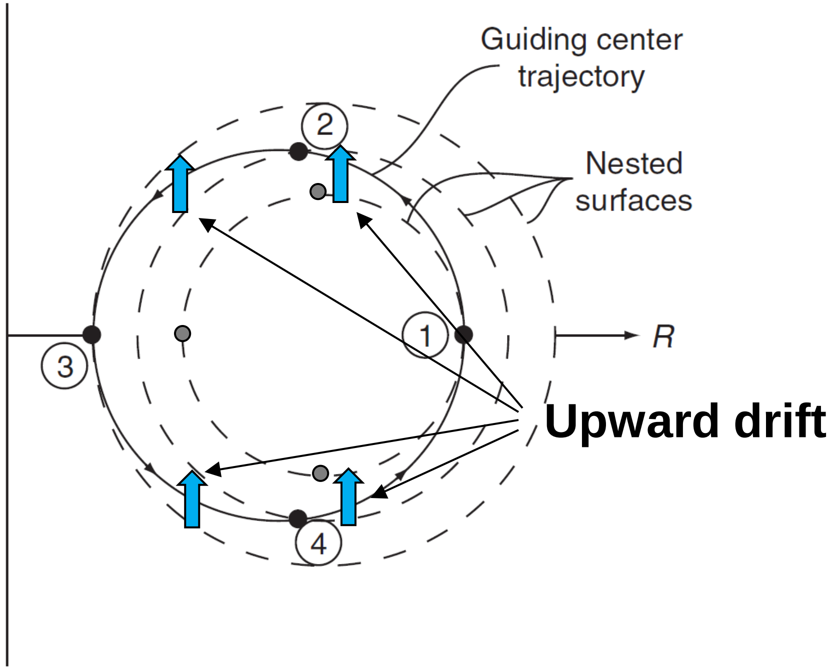
- **ExB drift**





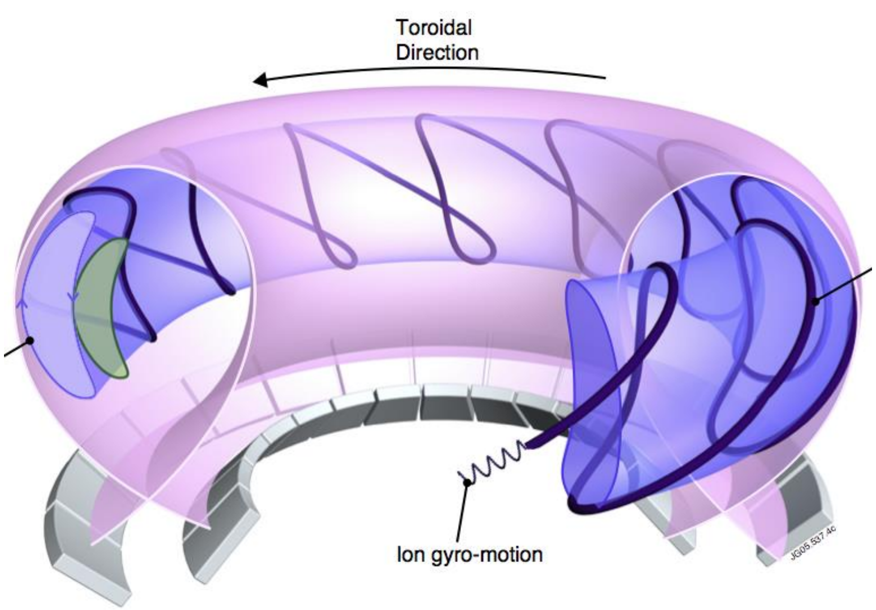
The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

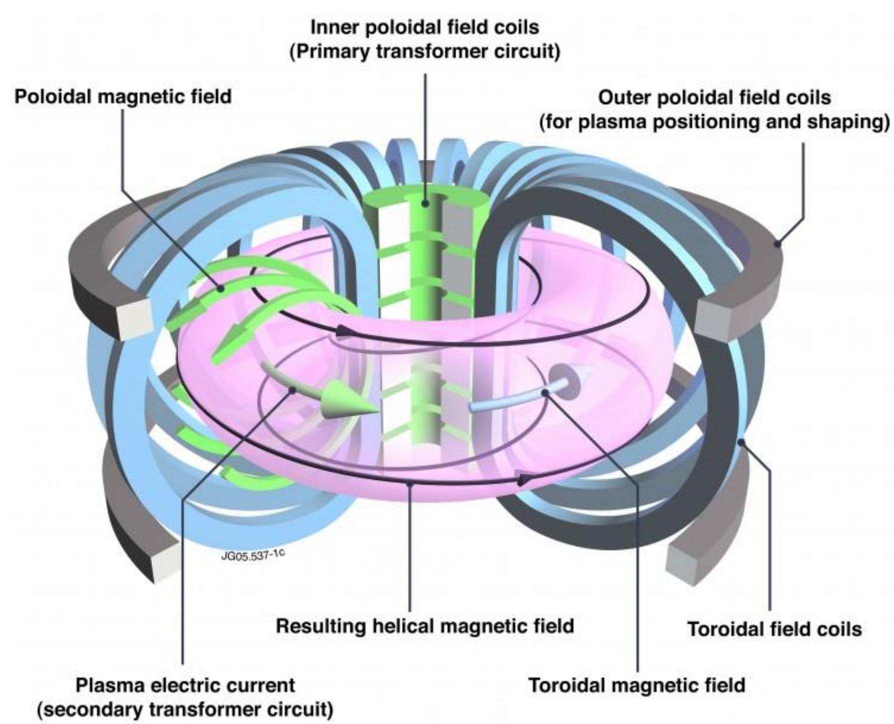




• Points with no drift

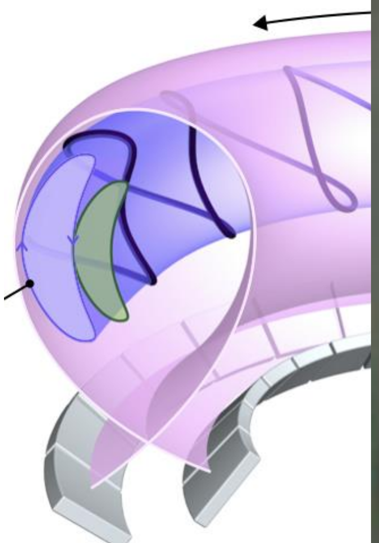
A poloidal magnetic field is required to reduce the drift across field lines

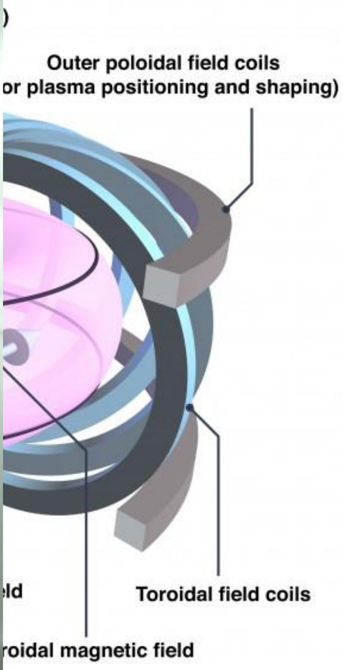




https://www.davidpace.com/keeping-fusion-plasmas-hot/
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

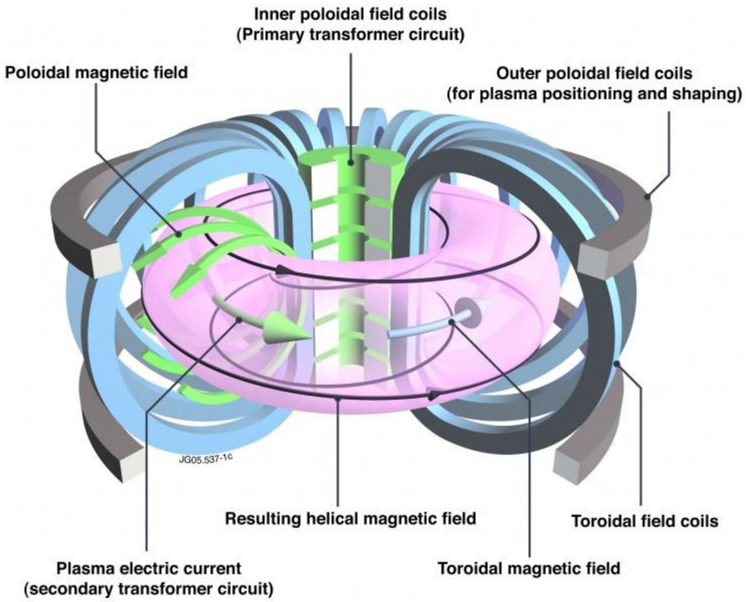
A poloidal magnetic field is required to reduce the drift across field lines



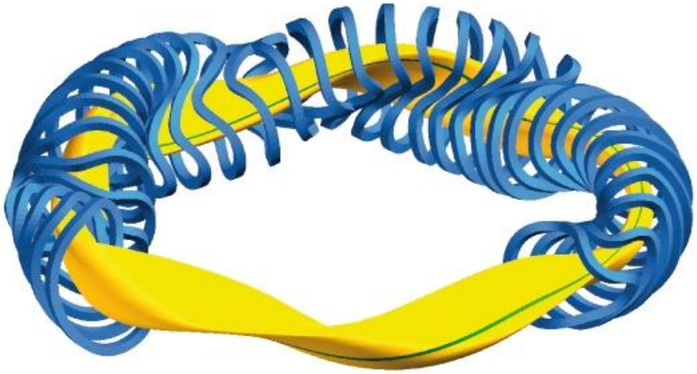


Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



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https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Magnetohydrodynamics description of plasma

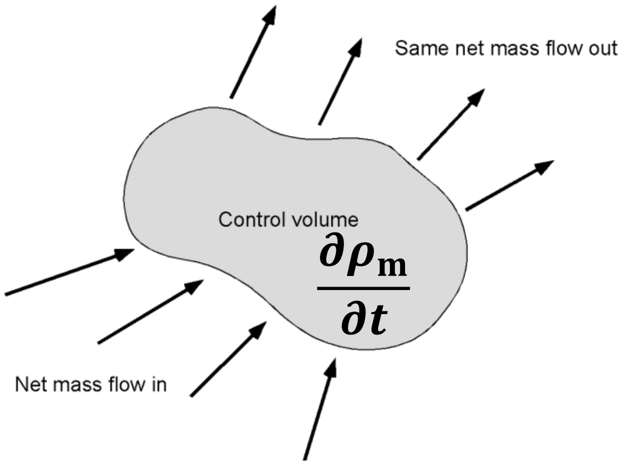
- **Continuity eq:** $\frac{\partial \rho_\mathrm{m}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
- **Momentum eq:** $\rho_\mathrm{m}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v}\right] = \rho_\mathrm{q}\,\vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$
- **Ohm's law:** $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt}\left(\frac{P}{\rho_\mathrm{m}{}^{\gamma}}\right) = 0$
- **Maxwell's eqs:**

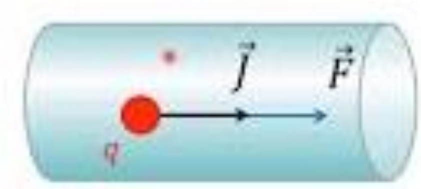
$$\nabla \cdot \vec{E} = \frac{\rho_\mathrm{q}}{\epsilon_o}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_\mathrm{o}\,\vec{j} + \epsilon_\mathrm{o}\mu_\mathrm{o}\frac{\partial \vec{E}}{\partial t}$$





https://eaglepubs.erau.edu/introductiontoaerospaceflightvehicles/chapter/conservation-of-mass-continuity-equation/
https://www.youtube.com/watch?v=lu0Ep8_Gp8U

Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation

- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ **w/ long wavelength ($\lambda >> \lambda_d$)**
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \rho_q \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$ (the term $\rho_q \vec{E}$ is struck out)
- **Ohm's law:** $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt}\left(\frac{P}{\rho_m{}^{\gamma}}\right) = 0$
- **Maxwell's eqs:**

$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o} \approx 0$ **w/ long wavelength ($\lambda >> \lambda_d$) => quasi neutral**

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$

(the term $\epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$ is struck out)

w/ low freq. ($\omega << \omega_{pe}$)

Ideal MHD

- Continuity eq: $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- Momentum eq: $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- Ohm's law: $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- Equation of state: $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$$
- Maxwell's eqs:

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

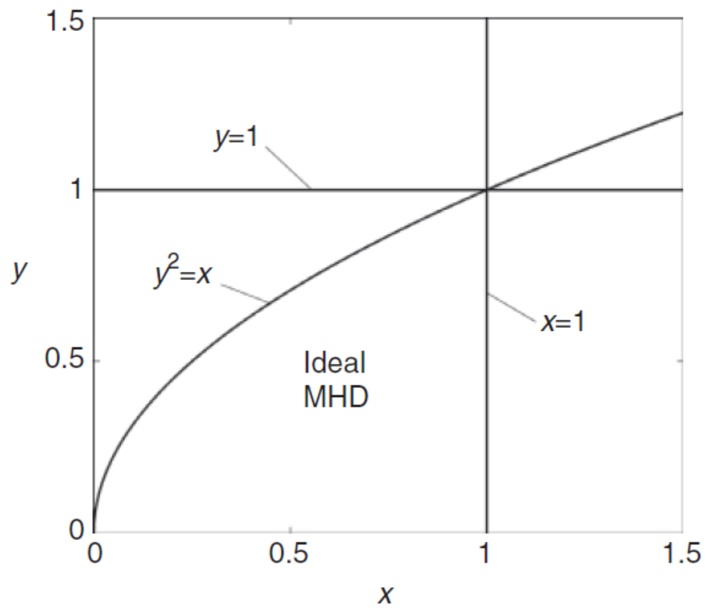
$$\nabla \cdot \vec{j} = 0$$

- Requirement:
  - High collisionality – fluid model
  - Small gyro radius – low frequency
  - Small resistivity – a perfect conductor

Conflict! (High collisionality ↔ Small resistivity)

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a} \frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

Region of validity for ideal MHD



$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad \lambda_i \sim v_{Ti}\tau_{ii}$$

$$y = \frac{r_{Li}}{a}$$

- **Requirement:**
  - **High collisionality** $x \ll 1$
  - **Small gyro radius** $y \ll 1$
  - **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement (small η)

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad \eta\,\vec{j} = \vec{E} + \vec{v} \times \vec{B} \qquad \frac{|\eta j|}{|v \times B|} \sim ?$$

$$|j \times B| \sim |\nabla p| \qquad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a}\frac{nT}{B} \sim \frac{1}{a}\frac{n m_\mathrm{i} {v_\mathrm{Ti}}^2}{B} \qquad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_\mathrm{Ti}}{a} \qquad \omega_\mathrm{ci} = \frac{v_\mathrm{Ti}}{r_\mathrm{Li}}$$

$$\eta \sim \frac{m_\mathrm{e}}{n e^2 \tau_\mathrm{ei}} \qquad \tau_\mathrm{ei} \sim \tau_\mathrm{ee} \sim \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \tau_\mathrm{ii} \qquad k \sim \nabla \sim \frac{1}{a}$$

$$\frac{|\eta j|}{|v \times B|} \sim \frac{\eta j}{v_\mathrm{Ti} B} \sim \frac{m_\mathrm{e}}{n e^2 \tau_{ei}} \frac{1}{a} \frac{n m_\mathrm{i} {v_\mathrm{Ti}}^2}{B} \frac{1}{v_\mathrm{Ti} B} = \frac{m_\mathrm{e}}{\tau_\mathrm{ei}} \frac{v_\mathrm{Ti}}{a} \frac{m_\mathrm{i}}{e^2 B^2} = \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ei}} \frac{v_\mathrm{Ti}}{a} \frac{{m_\mathrm{i}}^2}{e^2 B^2} = \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ei}} \frac{v_\mathrm{Ti}}{a {\omega_\mathrm{ci}}^2}$$

$$\sim \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ii}} \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \frac{v_\mathrm{Ti}}{a {\omega_{ci}}^2} = \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{v_\mathrm{Ti}}{\tau_\mathrm{ii} a} \frac{{r_\mathrm{Li}}^2}{{v_\mathrm{Ti}}^2} = \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{1}{\tau_\mathrm{ii} a} \frac{{r_\mathrm{Li}}^2}{v_\mathrm{Ti}} \sim \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{1}{\omega \tau_\mathrm{ii}} \left(\frac{r_\mathrm{Li}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

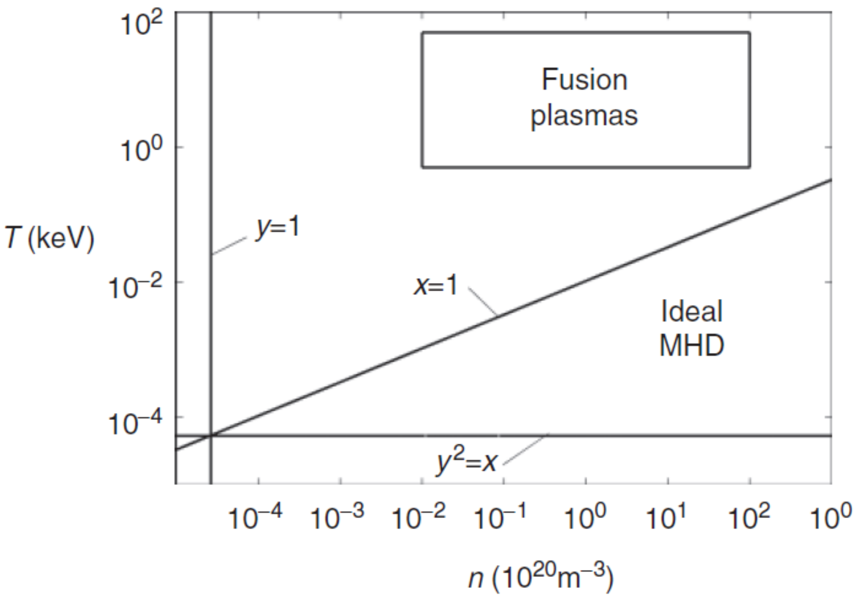
Fusion plasma is not in the ideal MHD region!

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad y = \frac{r_{Li}}{a}$$

$$10^{18}\ \mathrm{m^{-3}} < n < 10^{22}\ \mathrm{m^{-3}}$$

$$0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$$

$$\beta \equiv \frac{2\mu_o nT}{B^2}$$



- **Requirement:**
  - **High collisionality** $x = 3 \times 10^3 \frac{T^2}{an} << 1$
  - **Small gyro radius** $y = 2.3 \times 10^{-2} \left(\frac{\beta}{na^2}\right)^{1/2} << 1$
  - **Small resistivity** $\frac{y^2}{x} = 1.8 \times 10^{-7} \frac{\beta}{aT^2} << 1$

- **With strong B, the gyromotion mimic the collisional characteristics.**

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Additional simplification of the momentum equation

- **Momentum eq:** $\boldsymbol{\rho}_{\mathrm{m}}\left[\frac{\partial \vec{\boldsymbol{v}}}{\partial t}+(\vec{\boldsymbol{v}}\cdot\boldsymbol{\nabla})\,\vec{\boldsymbol{v}}\right]=\vec{\boldsymbol{j}}\times\vec{\boldsymbol{B}}-\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{P}}$

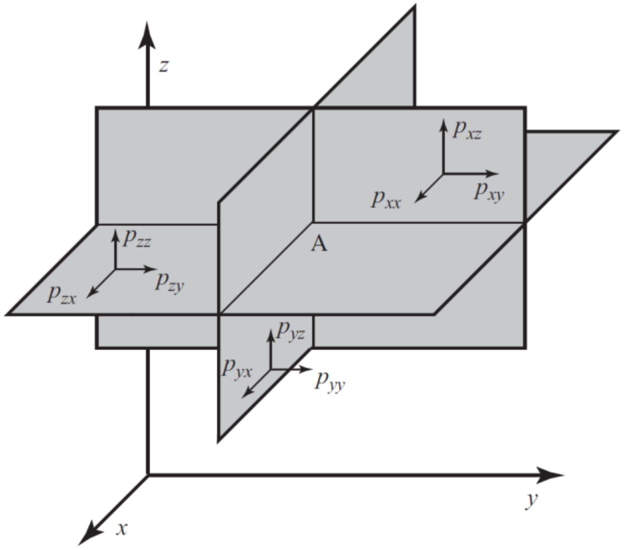
$$\nabla\cdot\overleftrightarrow{P}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{\mathrm{xx}} & p_{\mathrm{xy}} & p_{\mathrm{xz}}\\ p_{\mathrm{yx}} & p_{\mathrm{yy}} & p_{\mathrm{yz}}\\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & p_{\mathrm{zz}}\end{pmatrix}=\begin{pmatrix}\frac{\partial p_{\mathrm{xx}}}{\partial x}+\frac{\partial p_{\mathrm{yx}}}{\partial y}+\frac{\partial p_{\mathrm{zx}}}{\partial z}\\ \frac{\partial p_{\mathrm{xy}}}{\partial x}+\frac{\partial p_{\mathrm{yy}}}{\partial y}+\frac{\partial p_{\mathrm{zy}}}{\partial z}\\ \frac{\partial p_{\mathrm{xz}}}{\partial x}+\frac{\partial p_{\mathrm{yz}}}{\partial y}+\frac{\partial p_{\mathrm{zz}}}{\partial z}\end{pmatrix}$$

$$\nabla\cdot\overleftrightarrow{P}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}p_{\mathrm{xx}} & 0 & 0\\ 0 & p_{\mathrm{yy}} & 0\\ 0 & 0 & p_{\mathrm{zz}}\end{pmatrix}+\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}0 & p_{\mathrm{xy}} & p_{\mathrm{xz}}\\ p_{\mathrm{yx}} & 0 & p_{\mathrm{yz}}\\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & 0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\partial p_{\mathrm{xx}}}{\partial x}\\ \frac{\partial p_{\mathrm{yy}}}{\partial y}\\ \frac{\partial p_{\mathrm{zz}}}{\partial z}\end{pmatrix}+\begin{pmatrix}\frac{\partial p_{\mathrm{yx}}}{\partial y}+\frac{\partial p_{\mathrm{zx}}}{\partial z}\\ \frac{\partial p_{\mathrm{xy}}}{\partial x}+\frac{\partial p_{\mathrm{zy}}}{\partial z}\\ \frac{\partial p_{\mathrm{xz}}}{\partial x}+\frac{\partial p_{\mathrm{yz}}}{\partial y}\end{pmatrix}$$

Additional simplification of the momentum equation

$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$



$$= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{pmatrix}}_{\text{Viscosity } \nabla \cdot \overleftrightarrow{\Pi}}$$

- **Isotropic plasma:** $p_{xx} = p_{yy} = p_{zz} \equiv p$

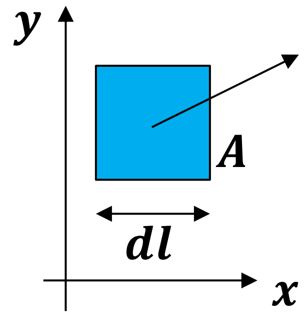
$$\begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} \equiv p\overleftrightarrow{1}$$

$$\nabla \cdot \overleftrightarrow{P} = \nabla p + \nabla \cdot \overleftrightarrow{\Pi}$$

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

Viscosity is negligible in a collision-dominated plasma

- Y component of momentum transfer through the surface A.



$$\pi_{\mathrm{xy}}^{\mathrm{ii}} \sim (m v_{\mathrm{y}} n) \frac{d v_{\mathrm{x}}}{dx} dl \sim \mu \frac{d v_x}{dx}$$

$$\mu \sim mnvdl \sim mnv(v\tau_{\mathrm{ii}}) \sim mn \left(\sqrt{\frac{T_{\mathrm{i}}}{m}} \right)^2 \tau_{\mathrm{ii}} \sim n T_{\mathrm{i}} \tau_{\mathrm{ii}}$$

$$\overleftrightarrow{\Pi} \sim \mu \left(2\nabla_{\parallel} \cdot \vec{v}_{\parallel} - \frac{2}{3} \nabla \cdot \vec{v} \right) \sim \mu \frac{v_{\mathrm{Ti}}}{a}$$

$$\left| \frac{\nabla \cdot \overleftrightarrow{\Pi}}{\nabla p} \right| \sim \frac{\overleftrightarrow{\Pi} a}{ap} \sim \frac{n T_{\mathrm{i}} \tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{ap} \sim \frac{\tau_{\mathrm{ii}} v_{\mathrm{Ti}}}{a} \sim \frac{\lambda_i}{a} \ll 1 \qquad \lambda_i \sim v_{\mathrm{Ti}} \tau_{\mathrm{ii}}$$

$$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

⇩

$$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"

- **Equilibrium state:**

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p \equiv 0$$

$$\vec{j} \times \vec{B} = \nabla p$$

$$\nabla \times \vec{B} = \mu_o\, \vec{j}$$

$$\vec{j} \times \vec{B} = \frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{\mu_o}\left[(\vec{B} \cdot \vec{\nabla})\, \vec{B} - \frac{1}{2}\, \vec{\nabla}\, B^2 \right] = \nabla p$$

$$\nabla \left(P + \frac{B^2}{2\mu_o} \right) = \frac{1}{\mu_o} (\vec{B} \cdot \vec{\nabla})\, \vec{B}$$

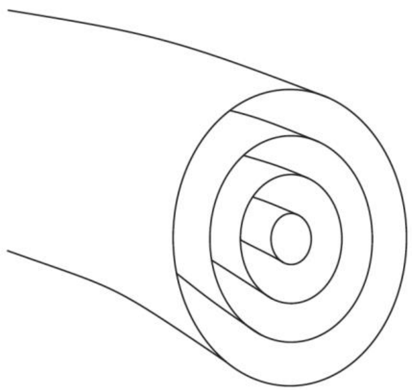
Magnetic pressure **Magnetic tension** ← **Forces caused by curvature of the field lines**

$$\vec{j} \perp \nabla p \qquad \vec{B} \perp \nabla p \quad \Rightarrow \quad \vec{j} \cdot \nabla p = 0 \qquad \vec{B} \cdot \nabla p = 0$$

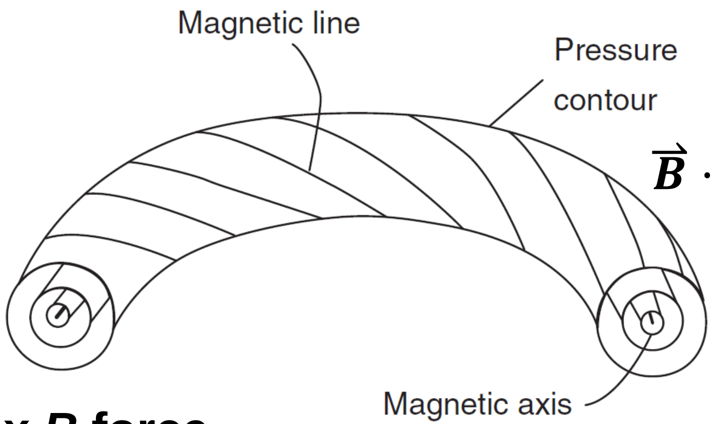
- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

Magnetic lines lying on pressure contour

- **Contours of constant pressure**



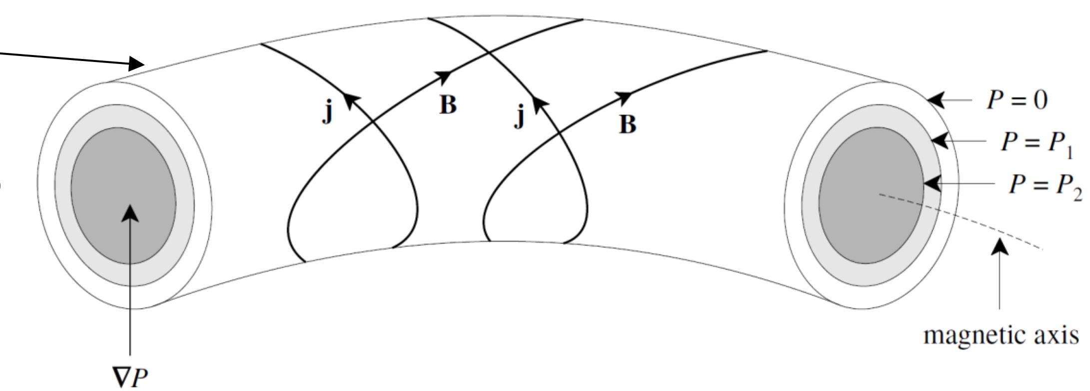
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

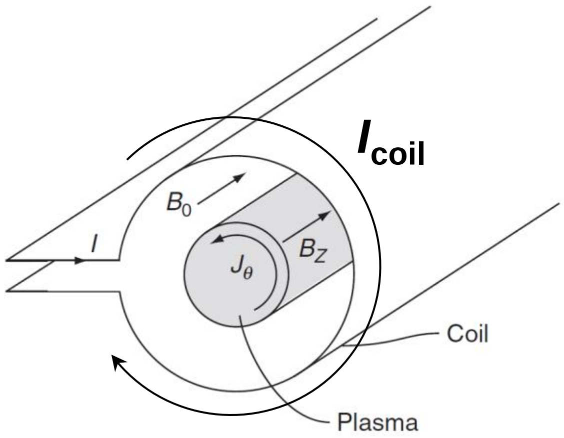
$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z\hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

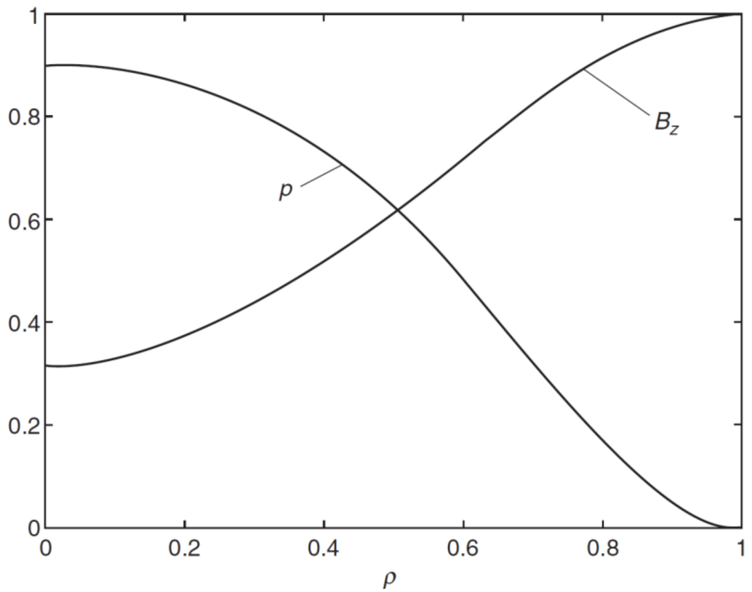
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma

- **Example:**

$$\frac{2\mu_o p(r)}{B_o^{\,2}} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$



$$j_\theta B_z = \frac{dp}{dr} \quad \Rightarrow \quad \frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{B_o^{\,2}} \qquad \rho = \frac{r}{a}$$

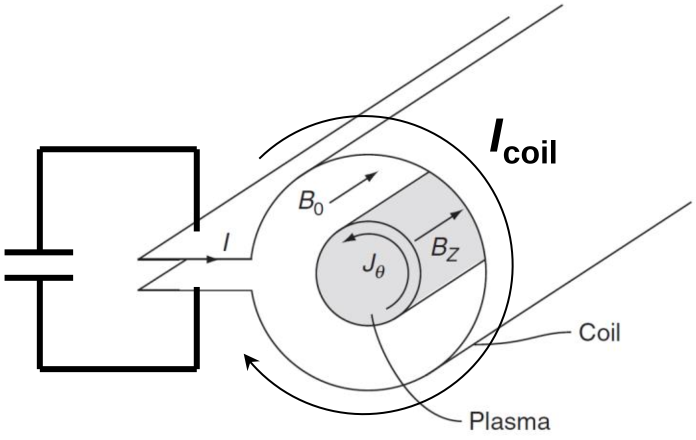
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{B_o^{\,2}} = \frac{4\mu_o}{a^2 B_o^{\,2}}\int_0^a p r\,dr = 2\int_0^1\left(1-\frac{B_z^{\,2}}{B_o^{\,2}}\right)\rho\,d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad \Rightarrow \quad \hat{\beta} \approx \frac{\beta_o}{2}\ ,\beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad \Rightarrow \quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$$0 < \beta < 1$$

Theta pinches provide good radial confinement but NOT axially

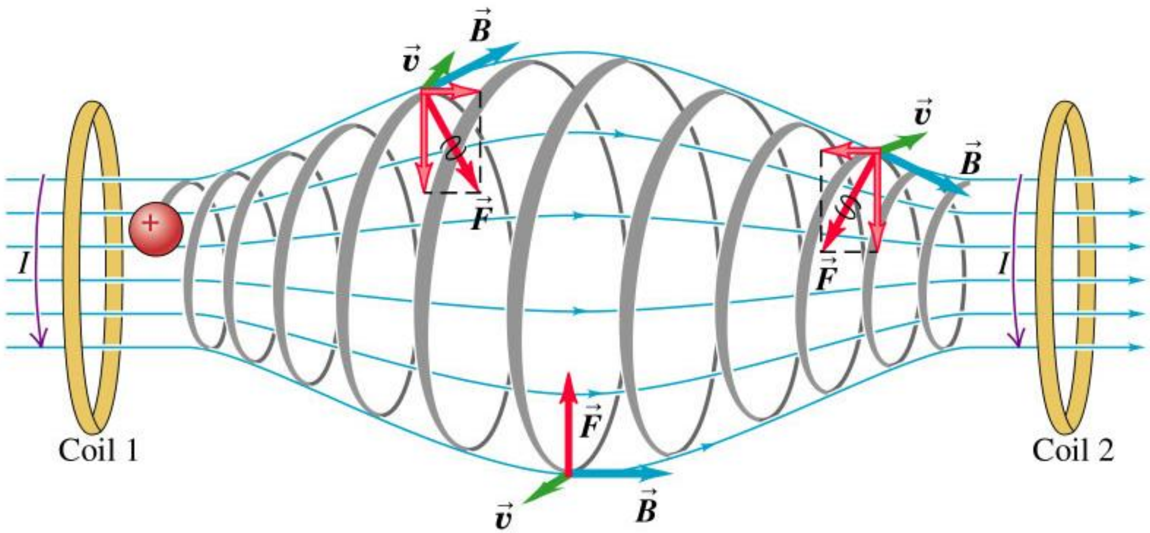


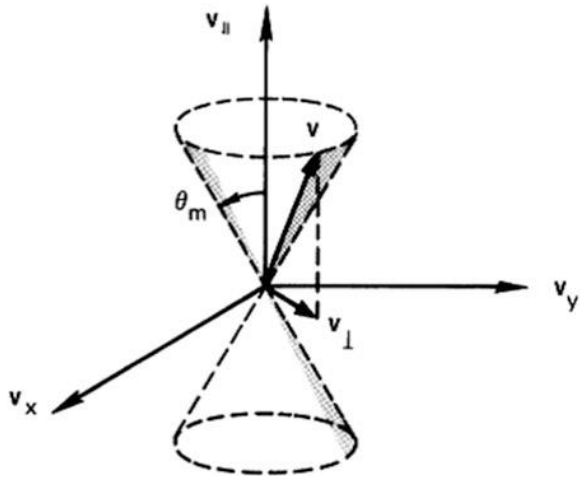
- **The gas is initially preionized.**
- **The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.**
- **The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.**
- **This energy is ultimately converted to heat after repeated reflections off the converging piston.**
- **T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.**
- **The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{Ti} \sim 10\mu s$ for L = 5 m.**

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- **Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.**





$$v'_{\perp}{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_{\perp}{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv \sin^2\theta$$

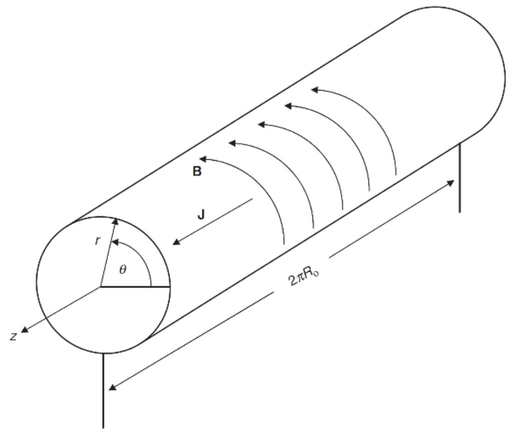
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_{\perp}{}^2 \qquad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_{\perp}{}^2}{B}$$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = \sin^2\theta_m$$

- **Large $v_{||}$ may occur from collisions between particles.**
- **Those confined charged particle are eventually lost due to collisions.**

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

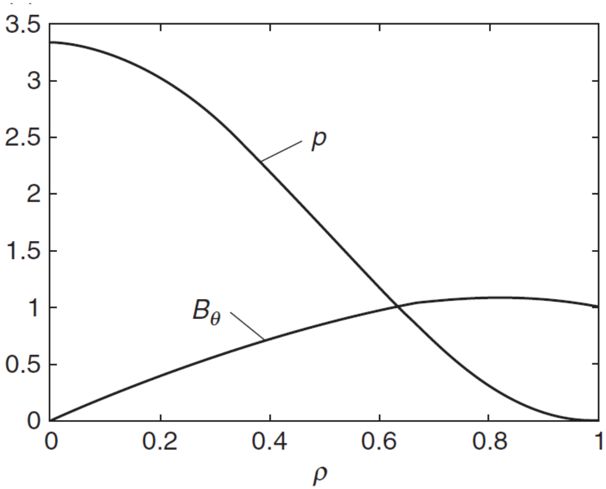
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

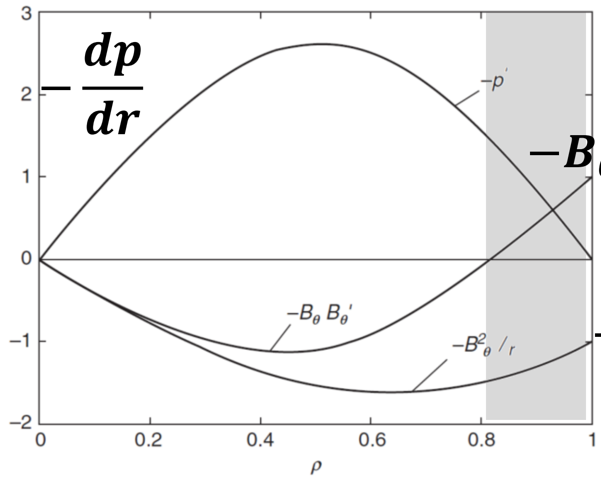
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β





$$-\frac{dp}{dr}$$

$$-B_\theta\frac{dB_\theta}{dr}$$

$$-\frac{B_\theta^2}{\mu_o r}$$

$$\frac{d}{dr}\left(p+\frac{B_\theta^2}{2\mu_o}\right)+\frac{B_\theta^2}{\mu_o r}=0$$

- **Example:**

$$\frac{2\mu_o p(r)}{{B_{\theta a}}^2}=\frac{2}{3}(5-2\rho^2)(1-\rho^2)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}}=2\rho\left(1-\frac{\rho^2}{2}\right)$$

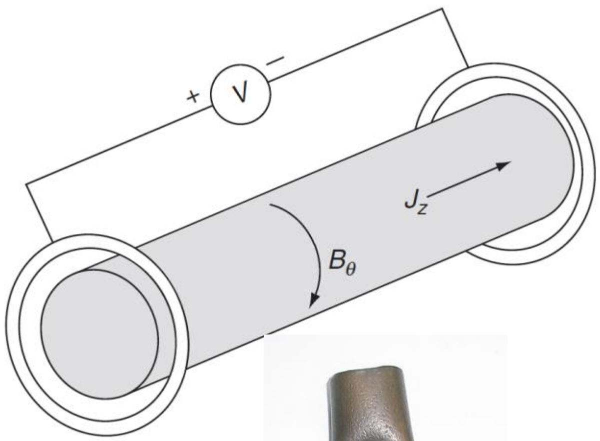
$$j_z=\frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) \Rightarrow \frac{a\mu_o j_z(r)}{B_{\theta a}}=4(1-\rho^2)$$

$$B_{\theta a}\equiv B_\theta(a)=\frac{\mu_o I}{2\pi a}$$

$$\beta\equiv\beta_p=\frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2}=\frac{4\mu_o}{a^2{B_{\theta a}}^2}\int_0^a prdr=1$$

Bennett pinch relation: $\beta=1$

Huge instabilities occur in a z pinch

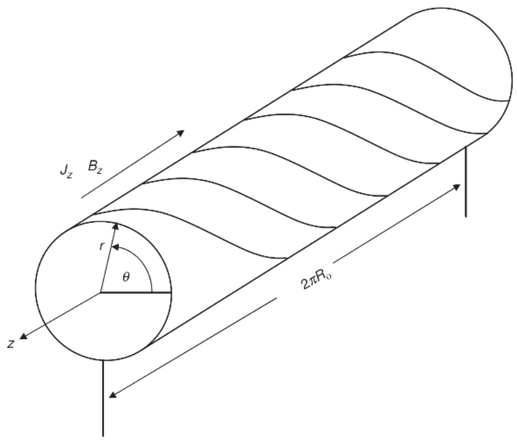


- **A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.**
- **The gas is ionized by the high voltage and produces a z current flowing along the plasma.**
- **Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.**

Main issue: unstable.

https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$(\nabla \times \vec{B})_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$(\nabla \times \vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$(\nabla \times \vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

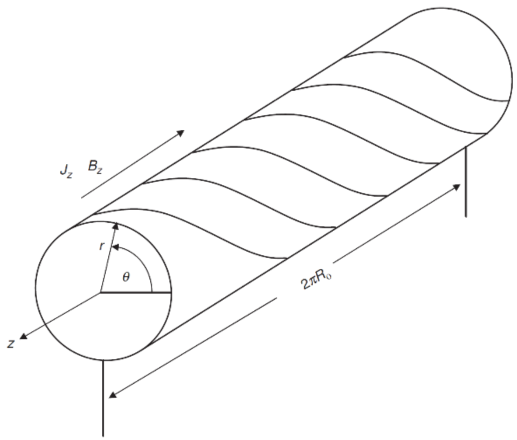
$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta \hat{\theta} + B_z \hat{z}$

$$\vec{j} = j_\theta \hat{\theta} + j_z \hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_z}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

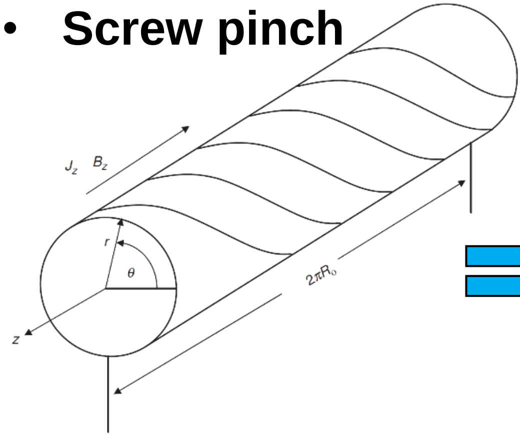
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch is flexible with varies range of β

- **Screw pinch**



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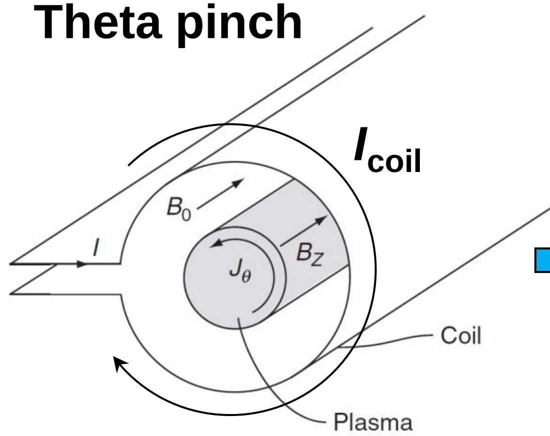
- **Theta pinch**



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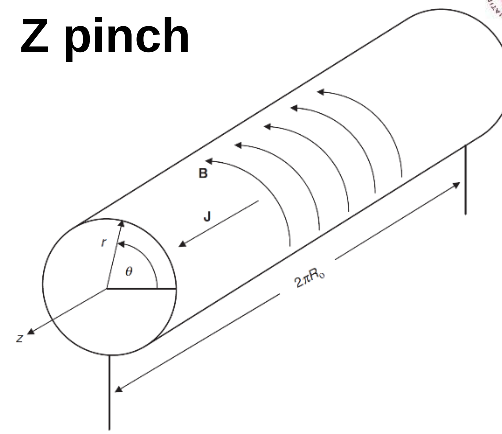
- **Z pinch**



$$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$$
$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

$$\vec{B} = B_z\hat{z}$$
$$\vec{j} = j_\theta\hat{\theta}$$

$$\vec{B} = B_\theta\hat{\theta}$$
$$\vec{j} = j_z\hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

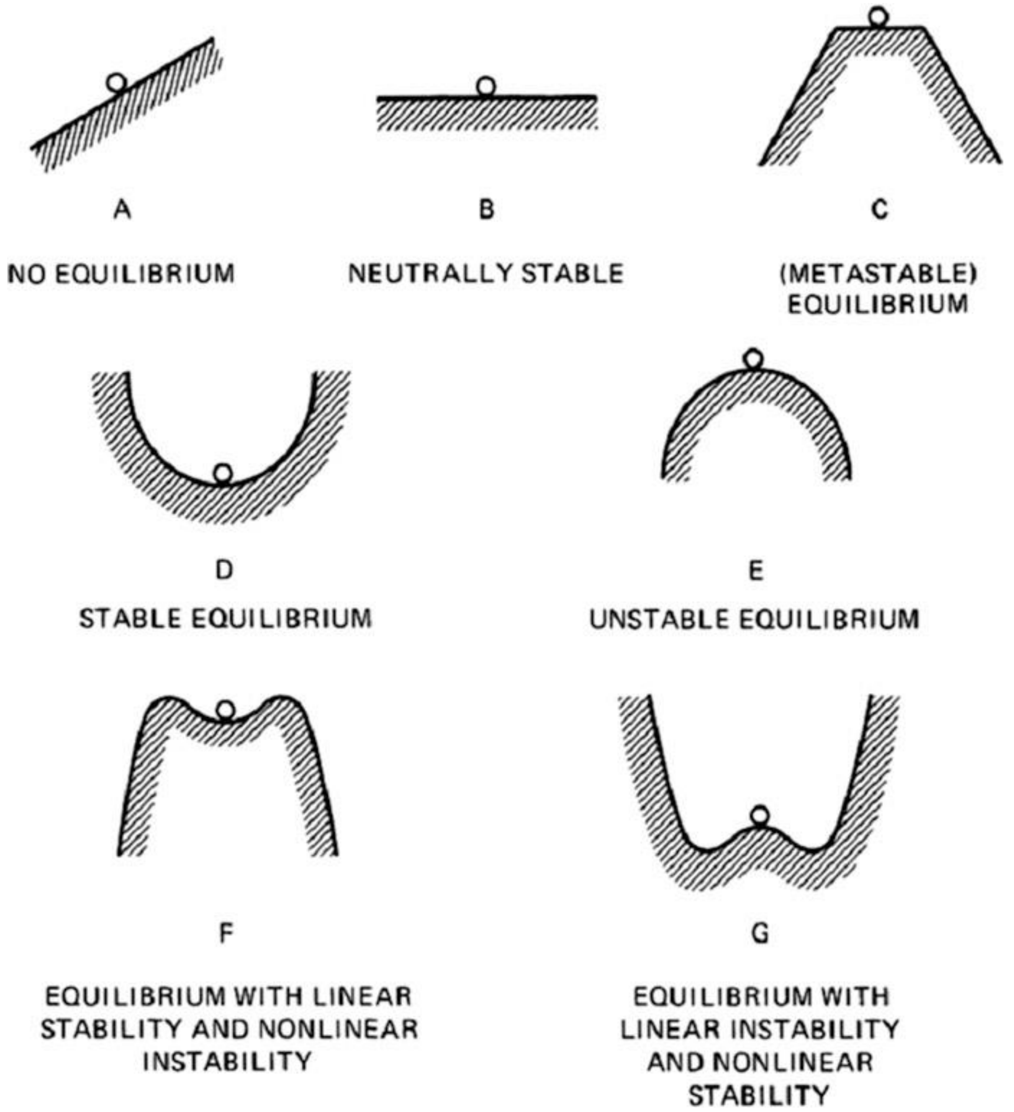
$$\int_0^a \pi r^2 dr\left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0$$

$$\langle p\rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2\rangle\right)$$

$$\beta_t = \frac{2\mu_o\langle p\rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t\beta_p}{\beta_t + \beta_p} = \frac{2\mu_o\langle p\rangle}{B_o^2 + {B_{\theta a}}^2}$$

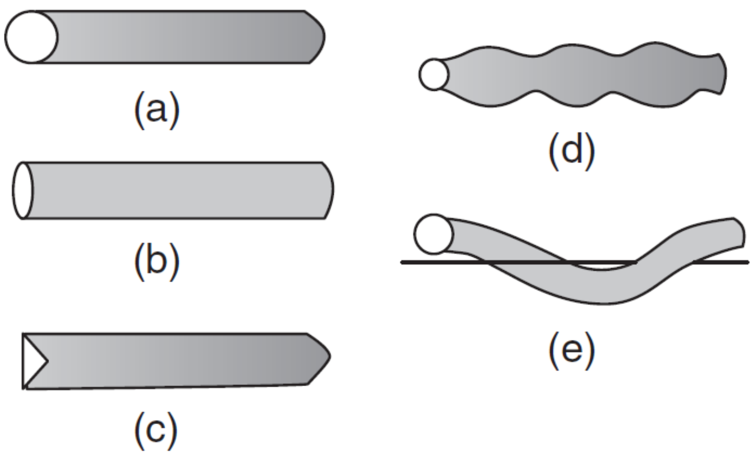
$$0 \le \langle\beta\rangle \le 1$$

An equilibrium state may not be stable



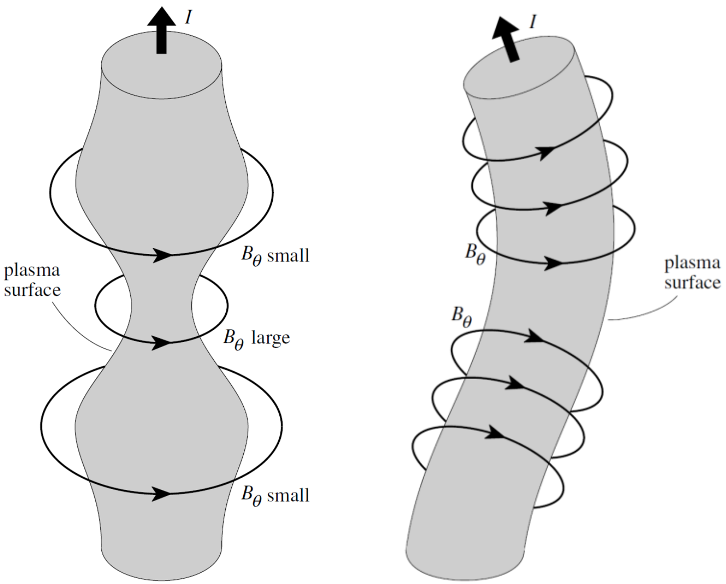
A cylindrical plasma column may not be stable

- **Instabilities of theta pinch**



(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0

- **Instabilities of z pinch**



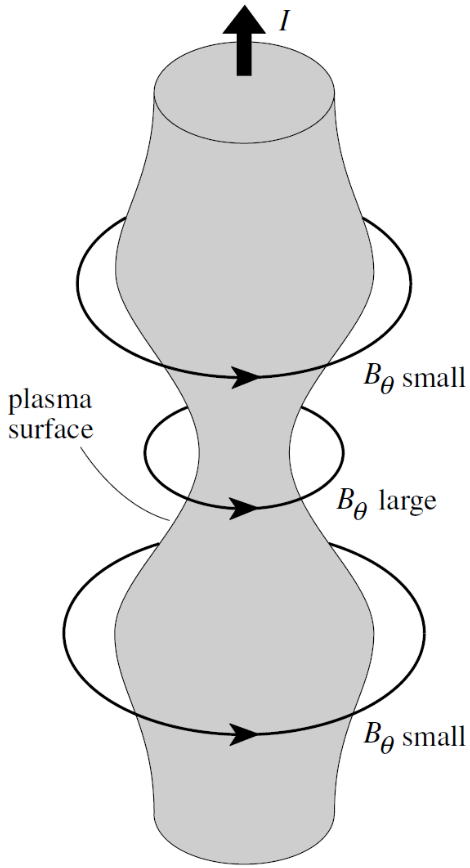
Sausage instability (m=0)

Kink instability (m=1)

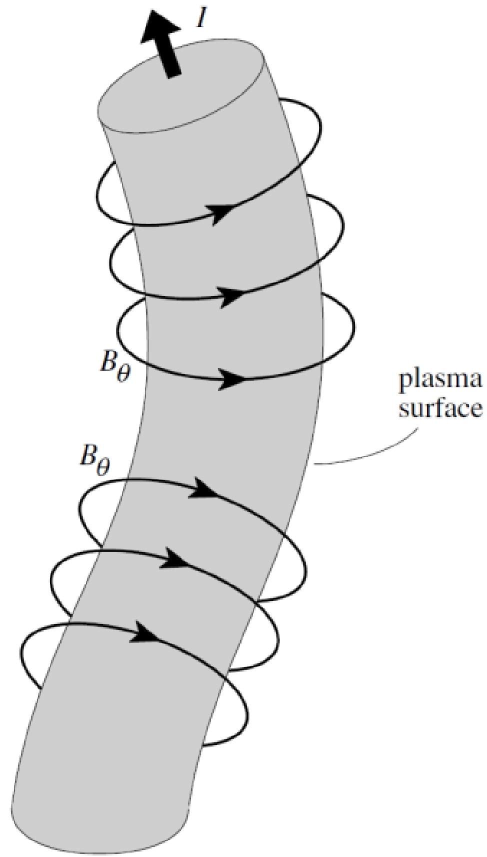
$$\zeta(\vec{r}) = \zeta(r)exp(im\theta + ikz)$$

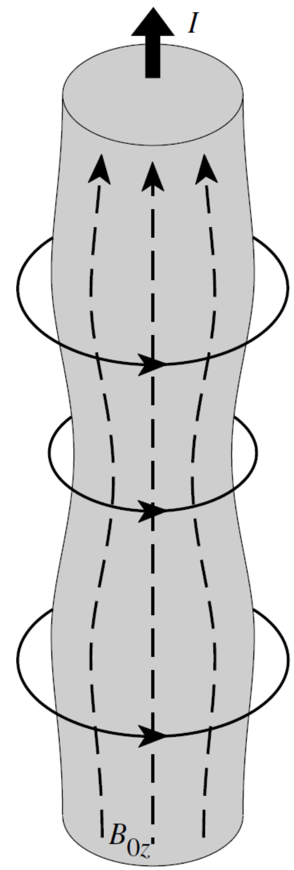
A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**



- **Kink instability**

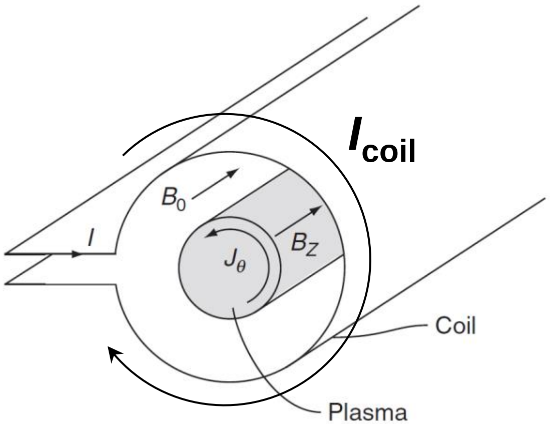


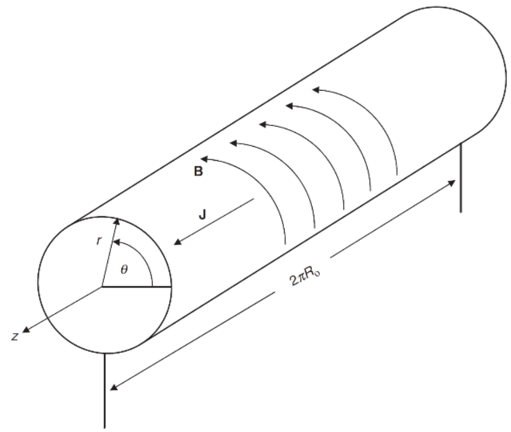


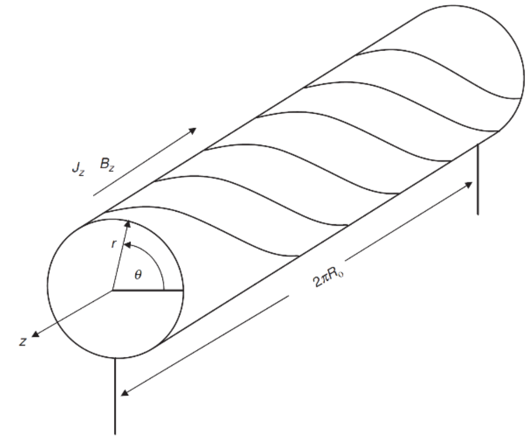
- **MHD Safety factor:** $$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$ **Kruskal–Shafranov limit**

Theta pinch is stable while z pinch is unstable

- **Theta pinch**
- **Z pinch**
- **Screw pinch**







$\vec{B} = B_z\hat{z}$

$\vec{B} = B_\theta\hat{\theta}$

$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$

$q_\theta = \infty$

$q_z = 0$

q can be controlled.

Stable

Unstable

Stable/Unstable

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$