Introduction to Nuclear Fusion as An Energy Source

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Lecture 4

2024 spring semester

Wednesday 9:10-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

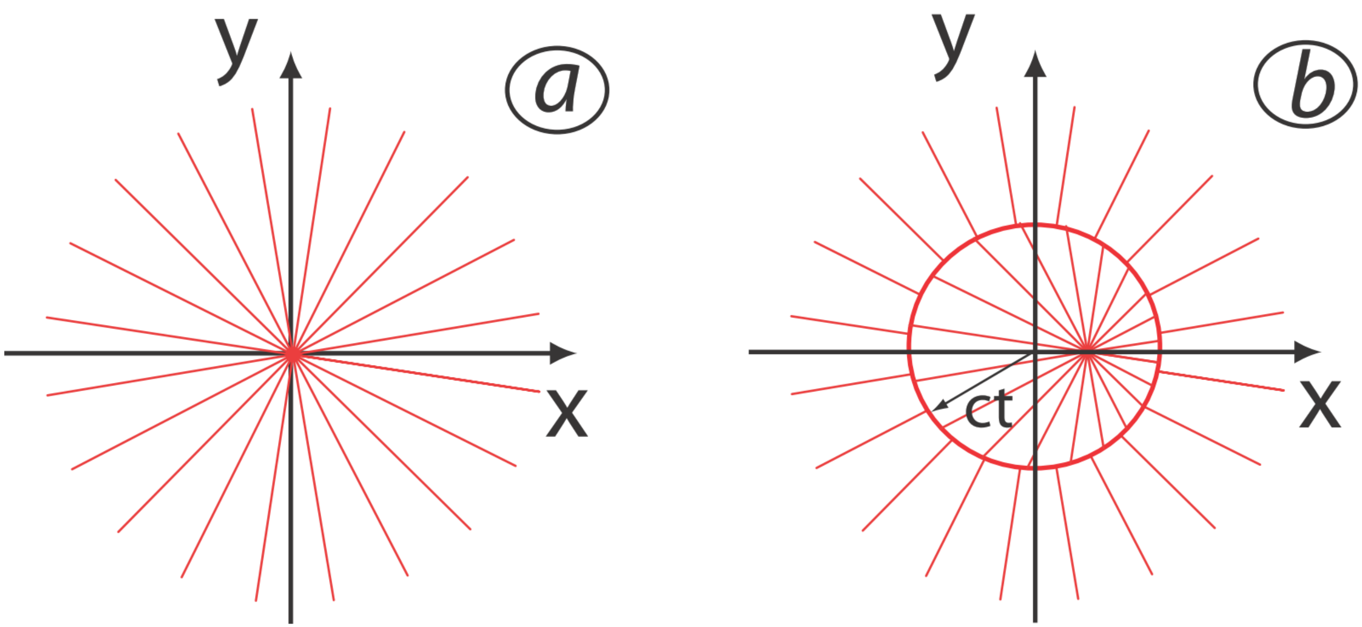
Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=ma76b50f97b1c6d72db61de9eaa9f0b27

2024/3/27 updated

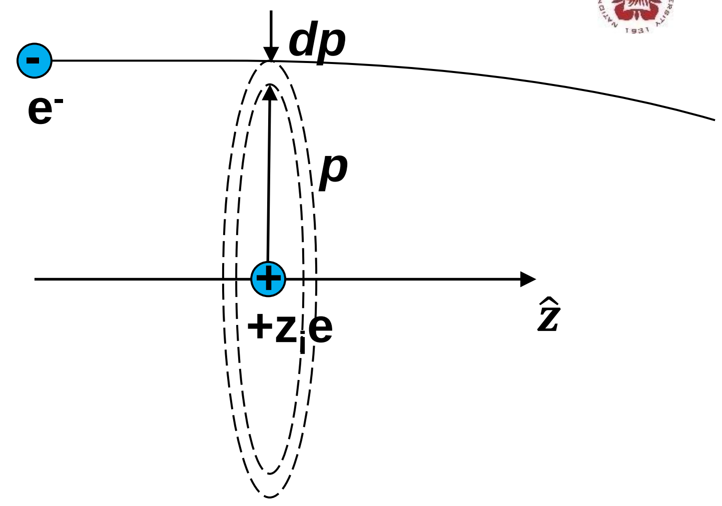
Electromagnetic wave is radiated when a charge particle is accelerated

- **The retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.**



https://jseldredphysics.files.wordpress.com/2018/04/lecture18.pdf

Bremsstrahlung emission



p: Impact parameter

- When an electron collides with a nucleus via coulomb interaction, the electron is accelerated and thus radiates, called Bremsstrahlung radiation.
- In the non-relativistic limit, the Bremsstrahlung radiation power is:

$$P_{\mathrm{B,e1,i1}} = \frac{e^2}{6\pi\epsilon_o}\frac{\dot{v}^2}{c^3}$$

- The coulomb force experienced by the electron is:

$$\dot{v} = \frac{F}{m_e} = \frac{\mathrm{z}e^2}{4\pi\epsilon_o m_e r^2} = \frac{ze^2}{4\pi\epsilon_o m_e[p^2+(vt)^2]} \approx \frac{\mathrm{z}e^2}{4\pi\epsilon_o m_e p^2}$$

=>

$$P_{\mathrm{B,e1,i1}} = \frac{z^2 e^6}{96\pi^3{\epsilon_o}^3 c^3 {m_e}^2}\frac{1}{p^4} \quad \mathrm{(W)}$$

Bremsstrahlung emission

- For multiple ion species: n_j, z_j

$$\bar{P}_B = \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) n_e \mathrm{T_e}{}^{1/2} \sum_j z_j^2 n_{i,j} \left(\frac{W}{m^3}\right)$$

$$= \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) Z_{\mathrm{eff}} n_e^2 \mathrm{T_e}{}^{1/2} \left(\frac{W}{m^3}\right)$$

where $$Z_{\mathrm{eff}} \equiv \frac{\sum_j z_j{}^2 n_j}{n_e} = \frac{\sum_j z_j{}^2 n_j}{\sum_j z_j n_j} \qquad n_e = \sum_j z_j n_j$$

$$\bar{P}_B = 5.35 \times 10^{-37} Z_{\mathrm{eff}} n^2_{\mathrm{e\,(m^{-3})}} T^{1/2}_{\mathrm{e\,(keV)}} \left(\frac{W}{m^3}\right)$$

$$\bar{P}_B \equiv C_{\mathrm{B}} Z_{\mathrm{eff}} n^2_{\mathrm{e\,(m^{-3})}} T^{1/2}_{\mathrm{e\,(keV)}} \left(\frac{W}{m^3}\right)$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_h: external heating

S_α: α particle heating

$$D + T \longrightarrow He^4\ (3.5\ \mathrm{MeV}) + n\ (14.1\ \mathrm{MeV})$$

$$S_\mathrm{f} = E_\mathrm{f} n_1 n_2 \langle \sigma \mathrm{v} \rangle \left(\mathrm{W/m^3}\right)$$

$$S_\alpha = \frac{1}{4} E_\alpha n^2 \langle \sigma \mathrm{v} \rangle = \frac{1}{16} E_\alpha \frac{p^2 \langle \sigma \mathrm{v} \rangle}{T^2}$$

E_α = 3.5 MeV

$$p = p_\mathrm{e} + p_\mathrm{i} = 2p_\mathrm{e} = 2n_e T \equiv 2nT$$

S_B: Bremsstrahlung radiation

$$S_B = C_\mathrm{B} Z_\mathrm{eff} n_{\mathrm{e}\ (\mathrm{m}^{-3})}^2 T_{\mathrm{e}\ (\mathrm{keV})}^{1/2} \left(\frac{W}{m^3}\right)$$

$$= \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}}$$

S_k: heat conduction lost

$$S_\kappa = \frac{3}{2} \frac{p}{\tau}$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma \mathrm{v} \rangle}{T^2} \geqslant \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$$S_\alpha + S_h = S_B + S_k$$

$$\frac{1}{16} E_\alpha \frac{\cancel{p^2}\langle\sigma v\rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{\cancel{p^2}}{T^{3/2}} + \frac{3}{2}\frac{\cancel{p}}{\tau}$$

$$p\tau \geqslant \frac{6}{\frac{1}{4} E_\alpha \frac{\langle\sigma v\rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

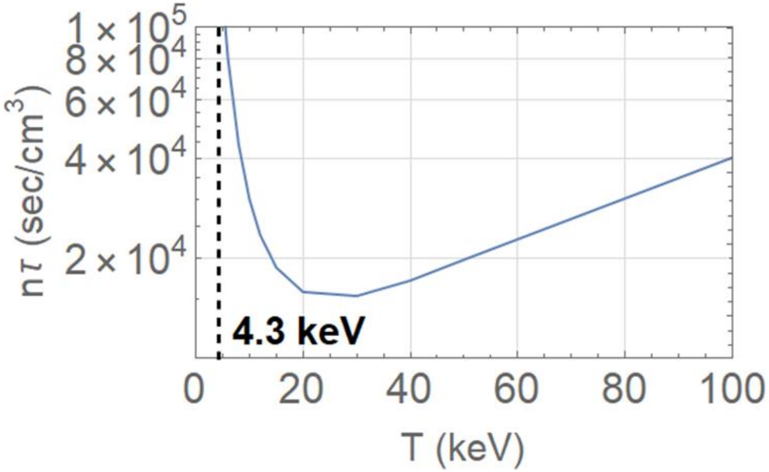
$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

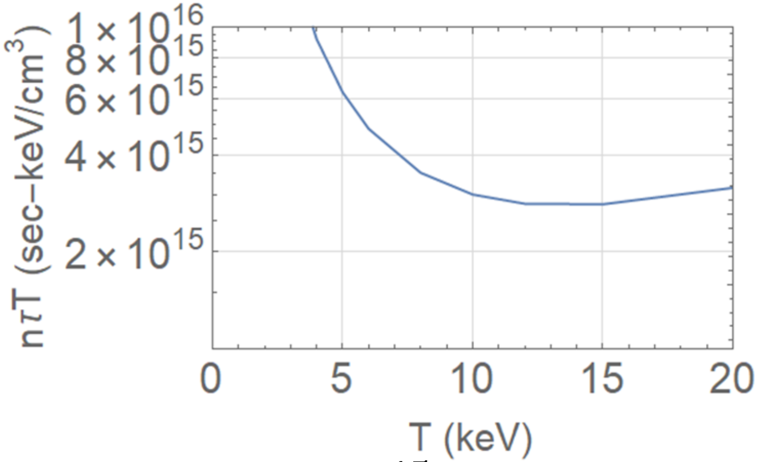
$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$

$$nT\tau > \frac{3T^2}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$

Temperature needs to be greater than ~5 keV to ignite

$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$





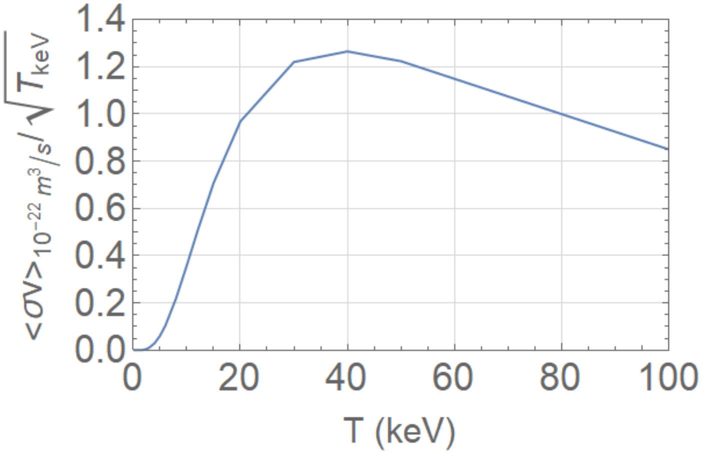
$$n\tau > 2\times10^4\ \mathrm{sec/cm^3}$$

$$nT\tau > 3.5\times10^{15}\ \mathrm{keV-sec/cm^3}$$

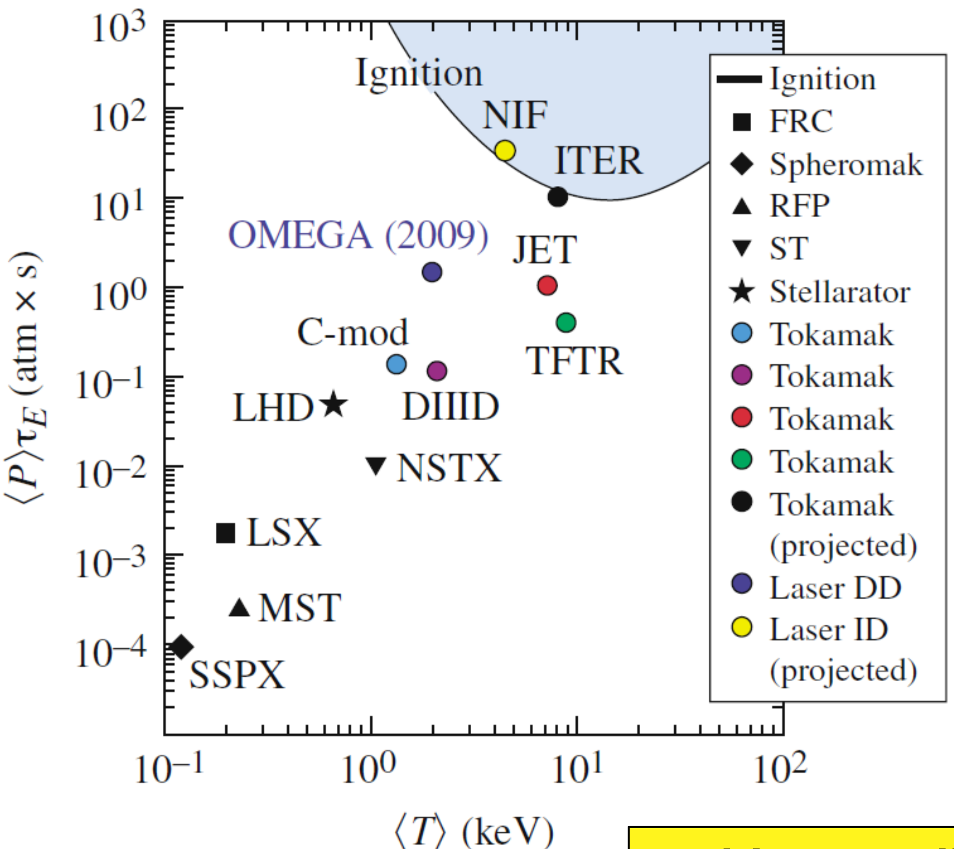
$S_\alpha > S_B$ $$\frac{1}{4}E_\alpha n^2\langle\sigma v\rangle > C_B n^2 T^{1/2}$$

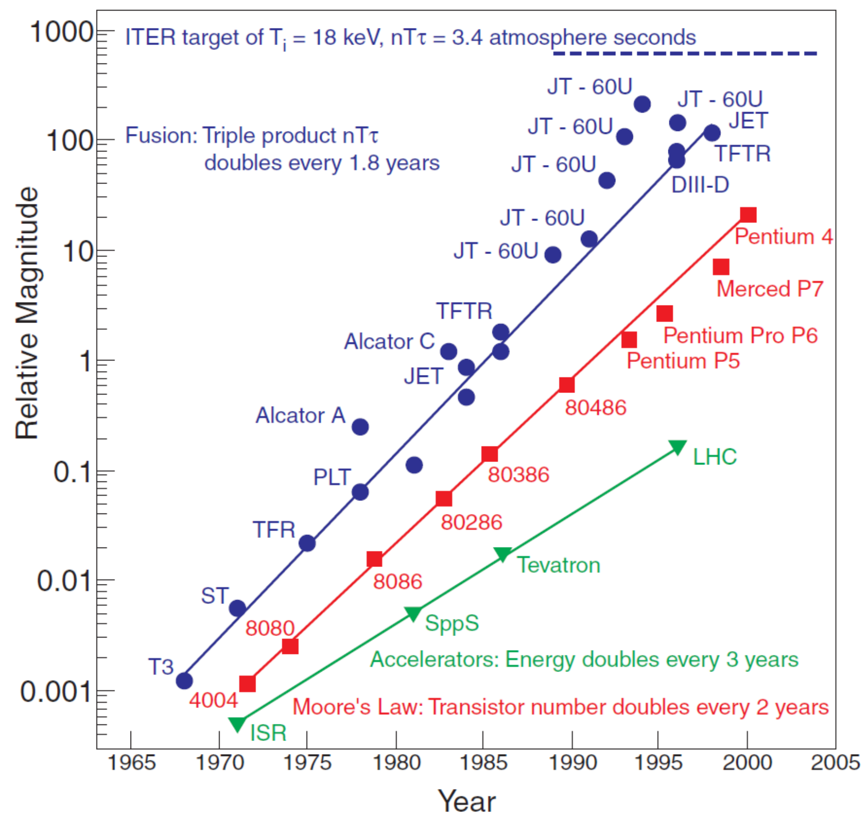
$$\frac{\langle\sigma v\rangle}{T^{1/2}} > \frac{4C_B}{E_\alpha}$$

$$T > 4.3\ \mathrm{keV}$$



We are closed to ignition!





Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

A. J. Webster, Phys. Educ. **38**, 135 (2003)
R. Betti, etc., Phys. Plasmas, **17**, 058102 (2010)

Under what conditions the plasma keeps itself hot?

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_α: α particle heating

S_h: external heating

S_B: Bremsstrahlung radiation

S_k: heat conduction lost

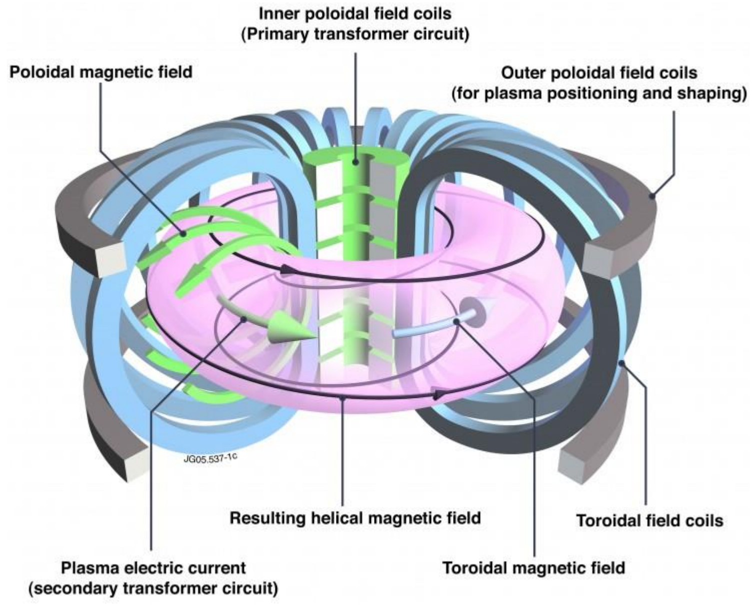
Ignition condition: $P\tau$ > 10 atm-s = 10 Gbar - ns

- **P: pressure, or called energy density**
- **τ is confinement time**

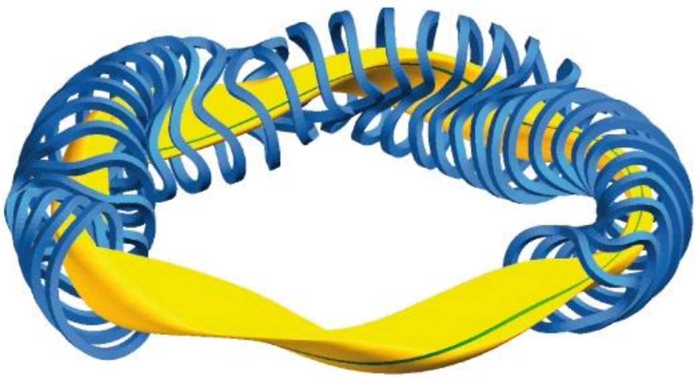
The plasma is too hot to be contained

- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. P~atm, τ~sec, T~10 keV (10^8 ºC)**

Tokamak



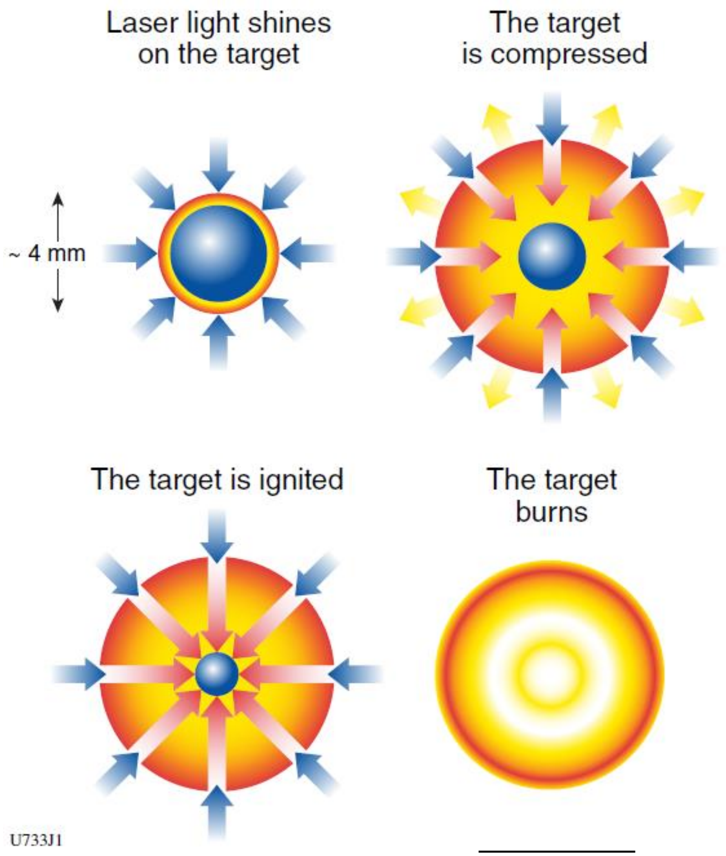
Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Don't confine it!

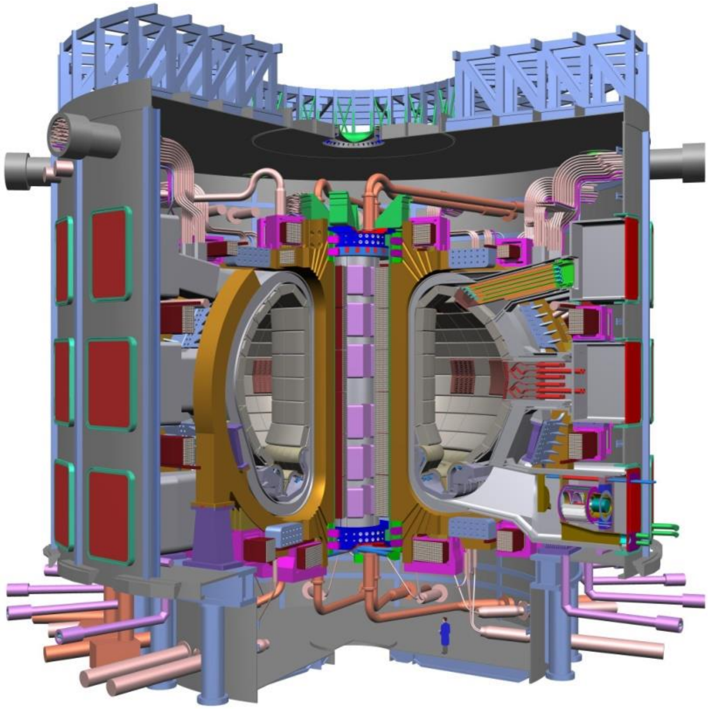
- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ~nsec, T~10 keV (10^8 ^{0}C)**



Inertial confinement fusion: an introduction, Laboratory for Laser Energetics, University of Rochester

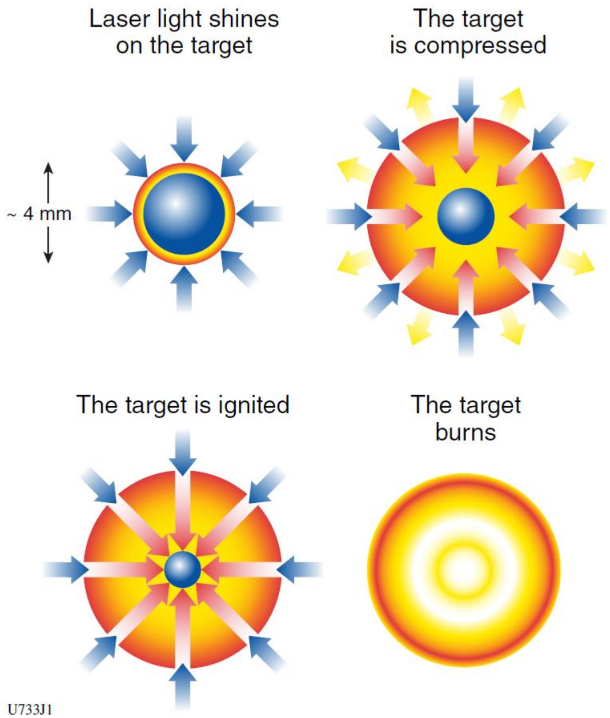
To control? Or not to control?

- **Magnetic confinement fusion (MCF)**



  - **Plasma is confined by toroidal magnetic field.**

- **Inertial confinement fusion (ICF)**

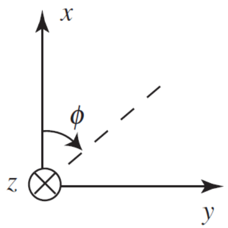


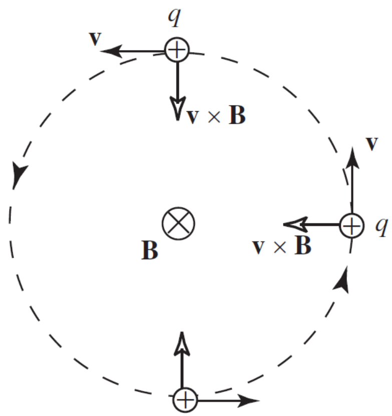
  - **A DT ice capsule filled with DT gas is imploded by laser.**

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Charged particles gyro around the magnetic field line





$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

- **Assuming** $\vec{B} = B\hat{z}$ **and the electron oscillates in x-y plane**

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \qquad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

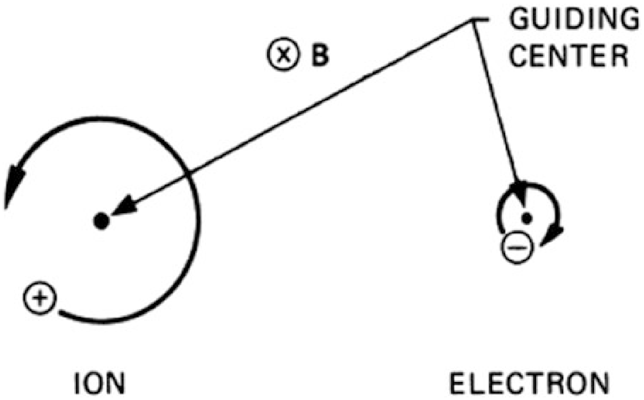
$$\omega_c \equiv \frac{|q|B}{m}$$ **Cyclotron frequency or gyrofrequency**

$$\ddot{v}_x + \omega_c^{\,2} v_x = 0$$

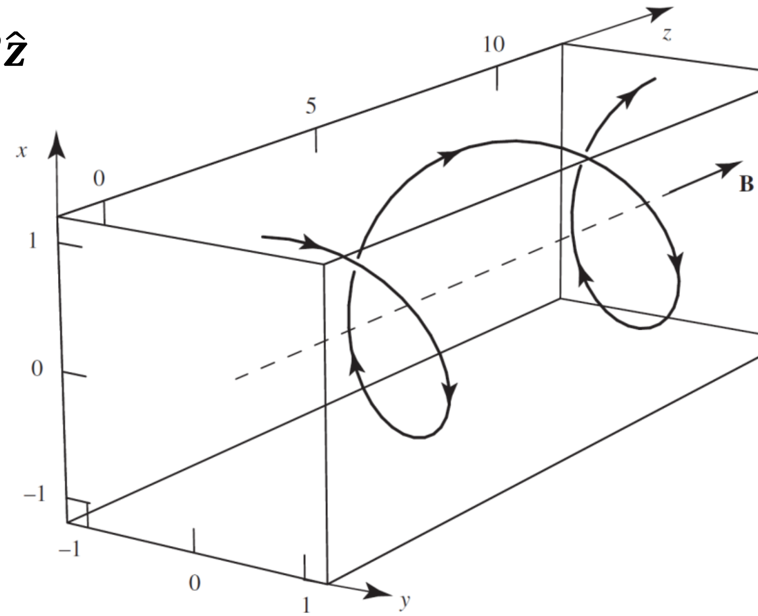
$$\ddot{v}_y + \omega_c^{\,2} v_y = 0$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$\vec{B} = B\hat{z}$$



$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = v_\perp \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

$$x = \mp r_c sin(\pm\omega_c t + \psi) + (x_o - r_c sin\psi)$$

$$y = \pm r_c cos(\pm\omega_c t + \psi) + (y_o + r_c cos\psi)$$

$$z = z_o + v_{||} t$$

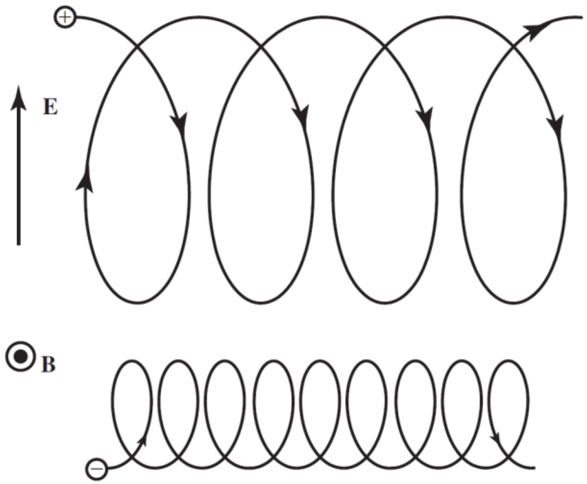
$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left|\frac{m v_\perp{}^2}{r}\right| = \left|q\,\vec{v} \times \vec{B}\right| = |q v_\perp B|$$

$$r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$$ **Larmor radius or gyroradius**

ExB drift

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{||} = \hat{x}E_{\perp} + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$v_{||}(t) = \frac{qE_{||}}{m}t + v_{||,o}$$

$$\vec{v}_{\perp}(t) = \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$m\frac{d}{dt}\left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) = q\left[\hat{x}E_{\perp} + \left(\vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)\right) \times \hat{z}B\right]$$

$$\underline{m\frac{d\,\vec{v}_{\mathrm{ac}}(t)}{dt}} = q[\underline{\hat{x}E_{\perp} + \vec{v}_{\mathrm{E}} \times \hat{z}B} + \underline{\vec{v}_{\mathrm{ac}}(t) \times \hat{z}B}]$$

No E field case: $m\frac{d\,\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$

$$\hat{x}E_{\perp} + \vec{v}_{\mathrm{E}} \times \hat{z}B = 0$$

$$\boxed{\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}}$$ ExB drift velocity

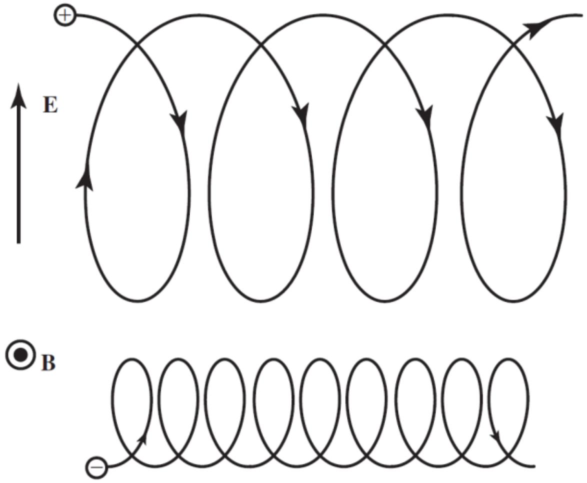
$$m\frac{d\,\vec{v}_{\mathrm{ac}}(t)}{dt} = q\,\vec{v}_{\mathrm{ac}}(t) \times \hat{z}B$$ Gyro motion

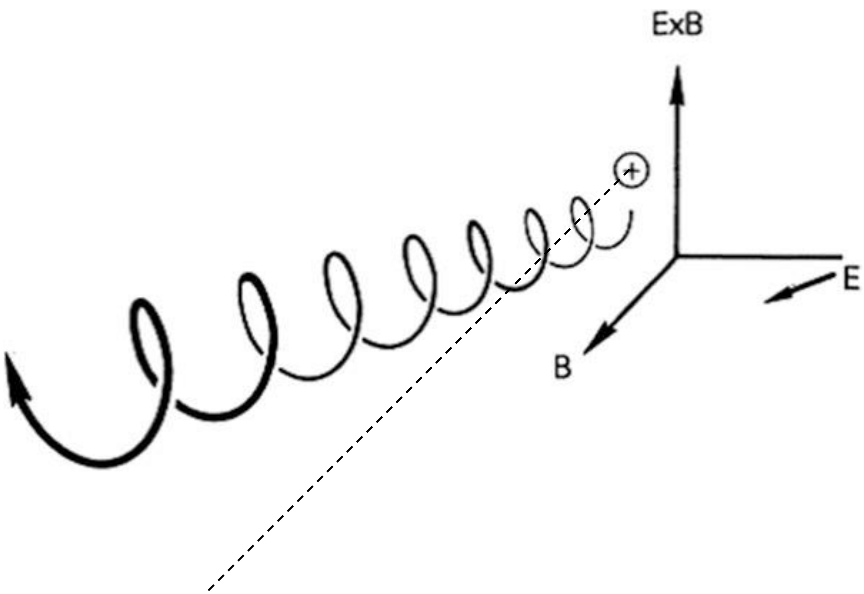
$$\vec{v}(t) = \hat{z}v_{||}(t) + \vec{v}_{\mathrm{E}} + \vec{v}_{\mathrm{ac}}(t)$$

$$\langle\vec{v}(t)\rangle = \frac{1}{T_c}\int_0^{T_c}\vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

- Electrons and ions drift in the same direction.

No current is generated in ExB drift





$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$ **ExB drift velocity**

- **Electrons and ions drift in the same direction.**

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs

$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{||} = \hat{x}E_{\perp} + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_{\perp} + \hat{z}F_{||} = \hat{x}F_{\perp} + \hat{z}F_{||}$$

$$m\frac{dv_{||}}{dt} = F_{||}$$

$$m\frac{d\,\vec{v}_{\perp}}{dt} = q\left(\hat{x}\frac{F_{\perp}}{q} + \vec{v}_{\perp} \times \hat{z}B\right)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_F$$

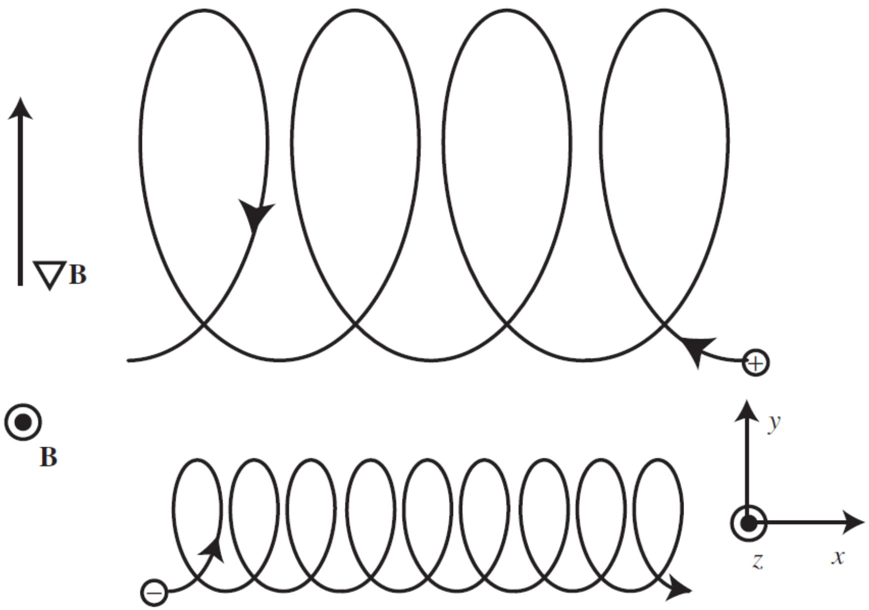
$$\vec{v}_{\mathrm{F}} = \frac{\hat{x}(F_{\perp}/q) \times \hat{z}B}{B^2} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

Gravitational drift velocity

- **Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.**

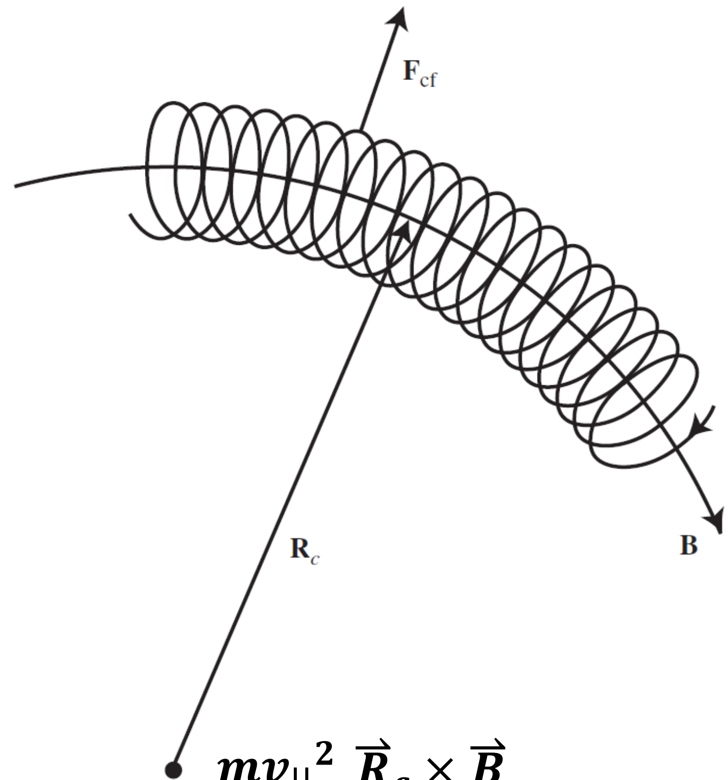
Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**



$$\vec{v}_{\nabla} = \frac{m{v_{\perp}}^{2}}{2q}\frac{\vec{B}\times\nabla B}{B^{3}}$$

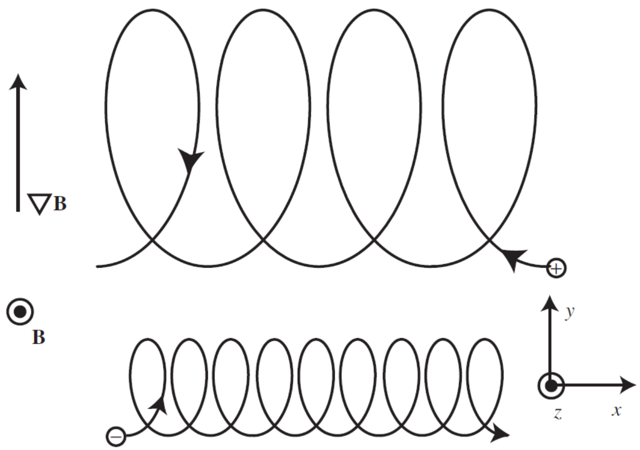
- **Curvature drift**



$$\vec{v}_{R} = \frac{m{v_{||}}^{2}}{2q}\frac{\vec{R}_{c}\times\vec{B}}{R_{c}B^{2}}$$

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_{c}B^{2}}\left({v_{||}}^{2}+\frac{1}{2}{v_{\perp}}^{2}\right) = \frac{m}{q}\frac{\vec{R}_{c}\times\vec{B}}{{R_{c}}^{2}B^{2}}\left({v_{||}}^{2}+\frac{1}{2}{v_{\perp}}^{2}\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x} q v_y B_z - \hat{y} q v_x B_z$$

$$\simeq \hat{x} q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right) - \hat{y} q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$B_z(y) = B_o + y\frac{\partial B_z}{\partial y} + y^2 \frac{1}{2}\frac{\partial^2 B_z}{\partial y^2} + \ldots$$

$$F_x = q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm\omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm\omega_c t + \psi)$$

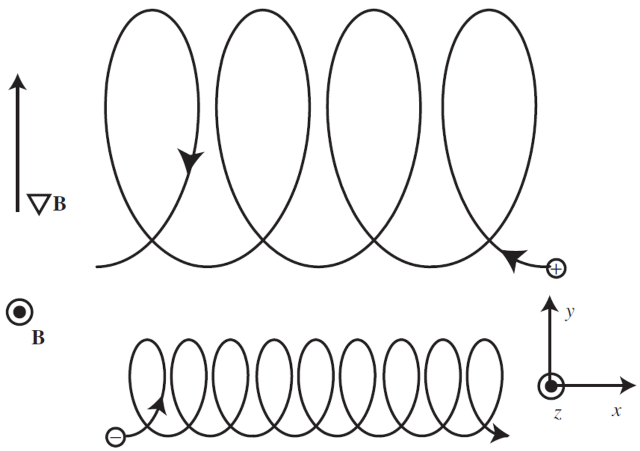
$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$

$$F_x \simeq -q v_\perp \sin(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_\perp \cos(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_\perp \sin(\pm\omega_c t + \psi)\left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$F_y \simeq -qv_\perp \cos(\pm\omega_c t + \psi)\left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

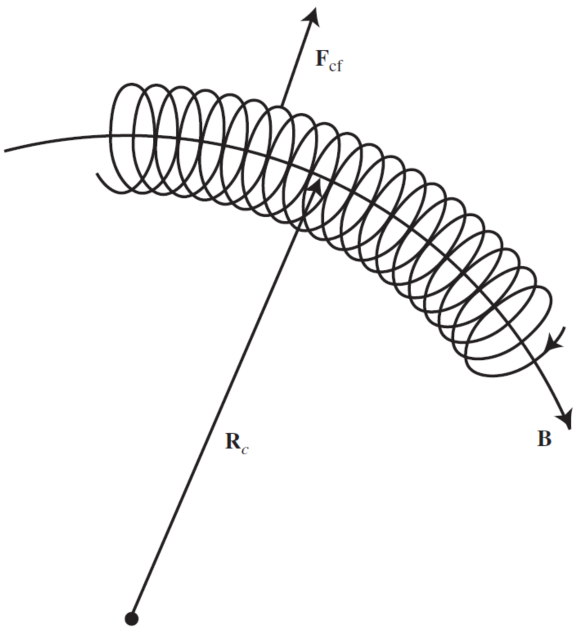
$$\langle F_x \rangle = 0$$

$$\langle F_y \rangle = \mp\frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} = -\frac{m{v_\perp}^2}{2B}\frac{\partial B_z}{\partial y} \qquad r_c = \frac{v_\perp}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \qquad \vec{v}_\nabla = \frac{1}{q}\frac{\langle F_y\rangle\hat{y}\times\hat{z}B_z}{{B_z}^2} = -\frac{m{v_\perp}^2}{2q{B_z}^2}\frac{\partial B_z}{\partial y}\hat{x}$$

- More general: $\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$

Charge particles drift across magnetic field lines when the magnetic field line is curved

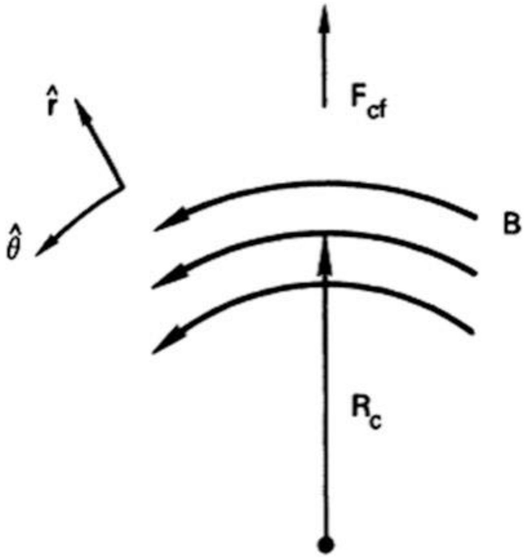


$$\vec{F}_{cf} = m{v_{||}}^2 \frac{\vec{R}_c}{{R_c}^2}$$

$$\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q}\frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m{v_{||}}^2}{q}\frac{\vec{R}_c \times \vec{B}}{{R_c}^2 B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{v}_\nabla = \frac{m v_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3} \qquad \vec{v}_R = \frac{m v_{||}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$$

$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B \hat{r}$$

$$\nabla \times \vec{B} = 0$$

$$\left(\nabla \times \vec{B}\right)_r = \left(\nabla \times \vec{B}\right)_\theta = 0$$

$$\nabla \times \vec{B} = \left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = 0 \qquad B_\theta \propto \frac{1}{r}$$

$$\frac{\nabla |B|}{|B|} = -\frac{\vec{R}_c}{R_c^2}$$

Cylindrical coordinate

$$\vec{v}_{\text{total}} = \vec{v}_{\text{R}} + \vec{v}_\nabla = \frac{\vec{B} \times \nabla B}{\omega_c B^2}\left(v_{||}^2 + \frac{1}{2} v_\perp^2\right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}\left(v_{||}^2 + \frac{1}{2} v_\perp^2\right)$$

- **Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.**

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m{v_\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \frac{m{v_{||}}^2}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_c}^2 B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}} B^2} \left({v_{||}}^2 + \frac{1}{2} {v_\perp}^2 \right) = \frac{m}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2 B^2} \left({v_{||}}^2 + \frac{1}{2} {v_\perp}^2 \right)$$

Magnetohydrodynamics description of plasma

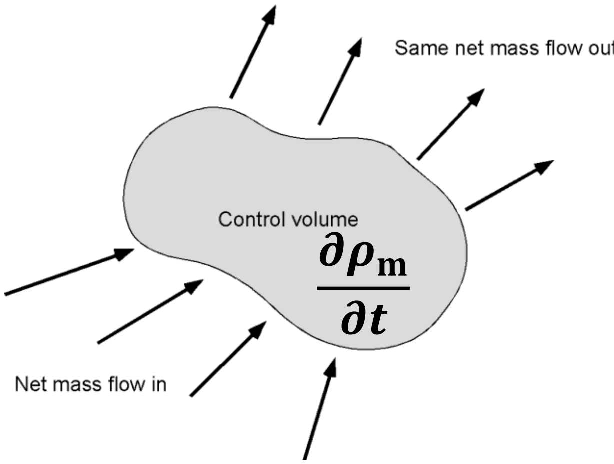
- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \rho_q \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_m}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

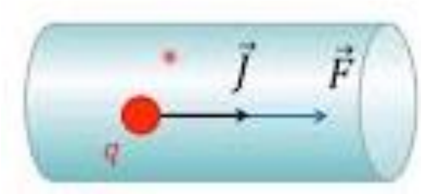
$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$





https://eaglepubs.erau.edu/introductiontoaerospaceflightvehicles/chapter/conservation-of-mass-continuity-equation/
https://www.youtube.com/watch?v=lu0Ep8_Gp8U

Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$ **w/ long wavelength ($\lambda >> \lambda_d$)**
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v}\right] = \cancel{\rho_q \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{\rho_m{}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o} \approx 0$$ **w/ long wavelength ($\lambda >> \lambda_d$) => quasi neutral**

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \cancel{\epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega << \omega_{pe}$)

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

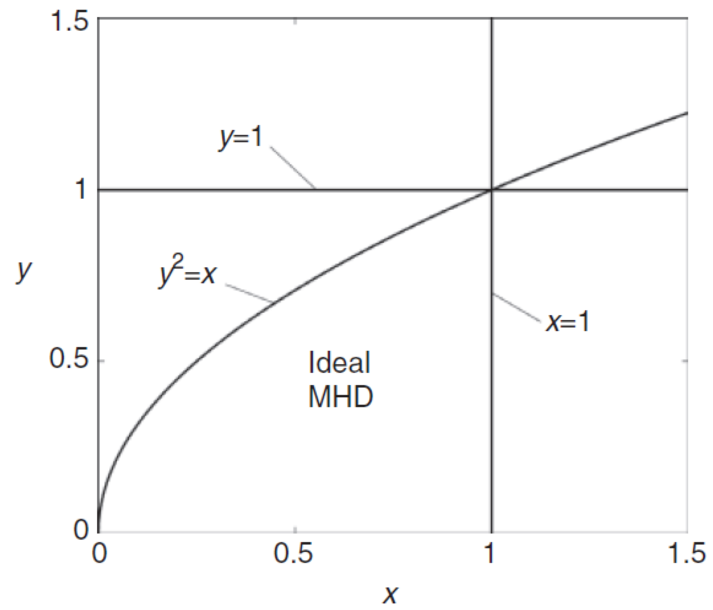
$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Conflict!

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a}\frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

Region of validity for ideal MHD



$$x = \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad \lambda_i \sim v_{\mathrm{Ti}}\tau_{\mathrm{ii}}$$

$$y = \frac{r_{\mathrm{Li}}}{a}$$

- **Requirement:**
  - **High collisionality** $x \ll 1$
  - **Small gyro radius** $y \ll 1$
  - **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad \eta \vec{j} = \vec{E} + \vec{v} \times \vec{B} \qquad \frac{|\eta j|}{|v \times B|} \sim ?$$

$$|j \times B| \sim |\nabla p| \qquad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a}\frac{nT}{B} \sim \frac{1}{a}\frac{n m_\mathrm{i} {v_\mathrm{Ti}}^2}{B} \qquad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}}$$

$$\eta \sim \frac{m_\mathrm{e}}{n e^2 \tau_\mathrm{ei}} \qquad \tau_\mathrm{ei} \sim \tau_\mathrm{ee} \sim \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \tau_\mathrm{ii} \qquad k \sim \nabla \sim \frac{1}{a}$$

$$\frac{|\eta j|}{|v \times B|} \sim \frac{\eta j}{v_{Ti} B} \sim \frac{m_\mathrm{e}}{n e^2 \tau_\mathrm{ei}} \frac{1}{a} \frac{n m_\mathrm{i} {v_\mathrm{Ti}}^2}{B} \frac{1}{v_{Ti} B} = \frac{m_\mathrm{e}}{\tau_\mathrm{ei}} \frac{v_{Ti}}{a} \frac{m_\mathrm{i}}{e^2 B^2} = \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ei}} \frac{v_{Ti}}{a} \frac{{m_\mathrm{i}}^2}{e^2 B^2} = \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ei}} \frac{v_{Ti}}{a {\omega_{ci}}^2}$$

$$\sim \frac{m_\mathrm{e}}{m_\mathrm{i} \tau_\mathrm{ii}} \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \frac{v_{Ti}}{a {\omega_{ci}}^2} = \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{v_{Ti}}{\tau_\mathrm{ii} a} \frac{{r_{Li}}^2}{{v_{Ti}}^2} = \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{1}{\tau_\mathrm{ii} a} \frac{{r_{Li}}^2}{v_{Ti}} \sim \left(\frac{m_\mathrm{e}}{m_\mathrm{i}}\right)^{1/2} \frac{1}{\omega \tau_\mathrm{ii}} \left(\frac{r_{Li}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

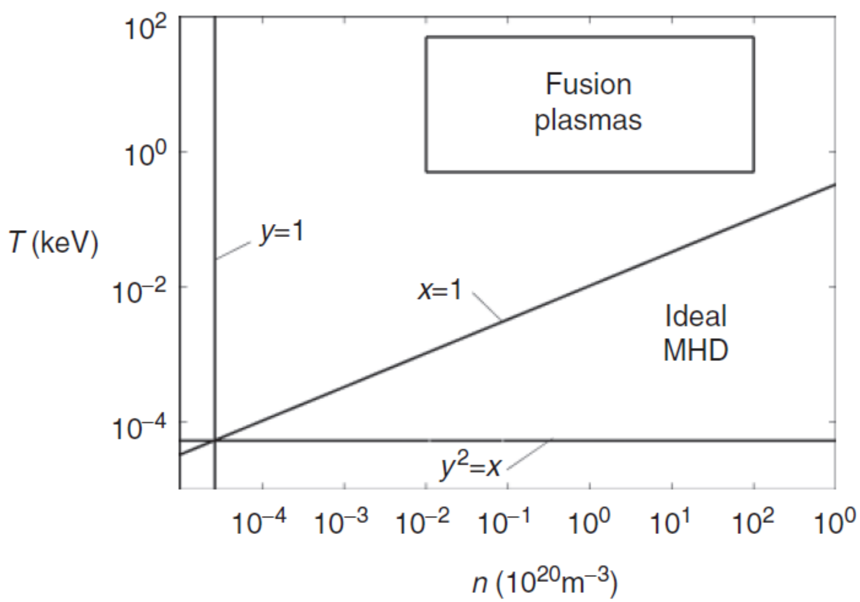
Fusion plasma is not in the ideal MHD region!

$$x = \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad y = \frac{r_\mathrm{Li}}{a}$$

$$10^{18}\ \mathrm{m}^{-3} < n < 10^{22}\ \mathrm{m}^{-3}$$

$$0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$$

$$\beta \equiv \frac{2\mu_o n T}{B^2}$$



- **Requirement:**
  - **High collisionality** $x = 3 \times 10^3 \frac{T^2}{an} << 1$
  - **Small gyro radius** $y = 2.3 \times 10^{-2} \left(\frac{\beta}{na^2}\right)^{1/2} << 1$
  - **Small resistivity** $\frac{y^2}{x} = 1.8 \times 10^{-7} \frac{\beta}{aT^2} << 1$

- **With strong B, the gyromotion mimic the collisional characteristics.**

Ideal MHD

- Continuity eq: $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- Momentum eq: $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- Ohm's law: $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- Equation of state: $$\frac{d}{dt} \left(\frac{P}{\rho_{\mathrm{m}}{}^{\gamma}} \right) = 0$$
- Maxwell's eqs:

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- Requirement:
  - High collisionality – fluid model
  - Small gyro radius – low frequency
  - Small resistivity – a perfect conductor