Introduction to Nuclear Fusion as An Energy Source

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Lecture 3

2026 spring semester

Tuesday 9:00-12:00

Materials:

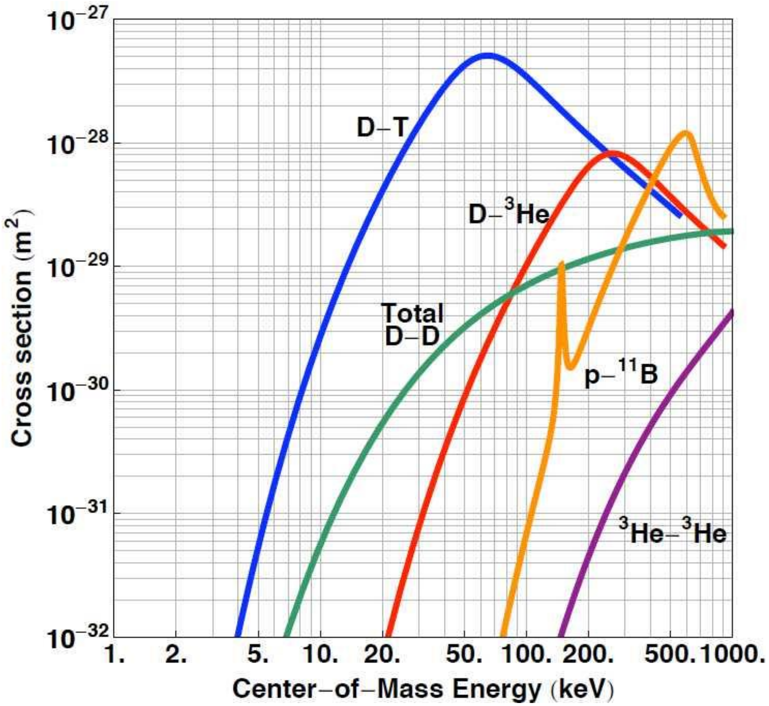
https://capst.ncku.edu.tw/PGS/index.php/teaching/

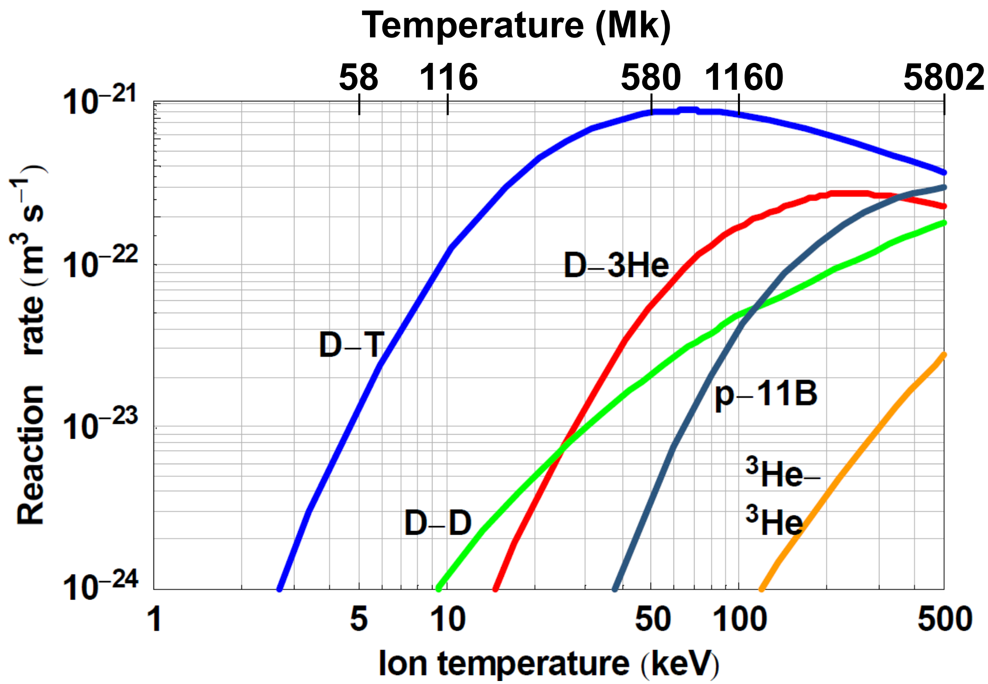
Online courses:

https://reurl.cc/MMnkOL

2026/3/10 updated

Fusion doesn’t come easy

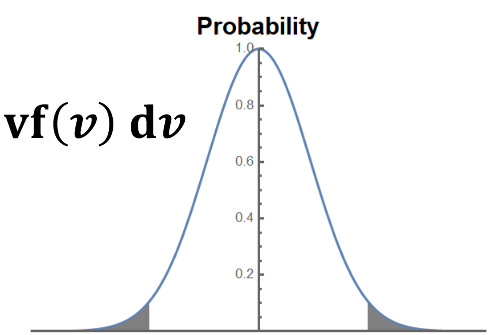




- The DT fusion reactivity is maximum at T ≈ 64 keV
- @ T = 10 keV, $<\sigma v>_{DT} \approx 100 <\sigma v>_{DD}$

- **Reaction rate:**

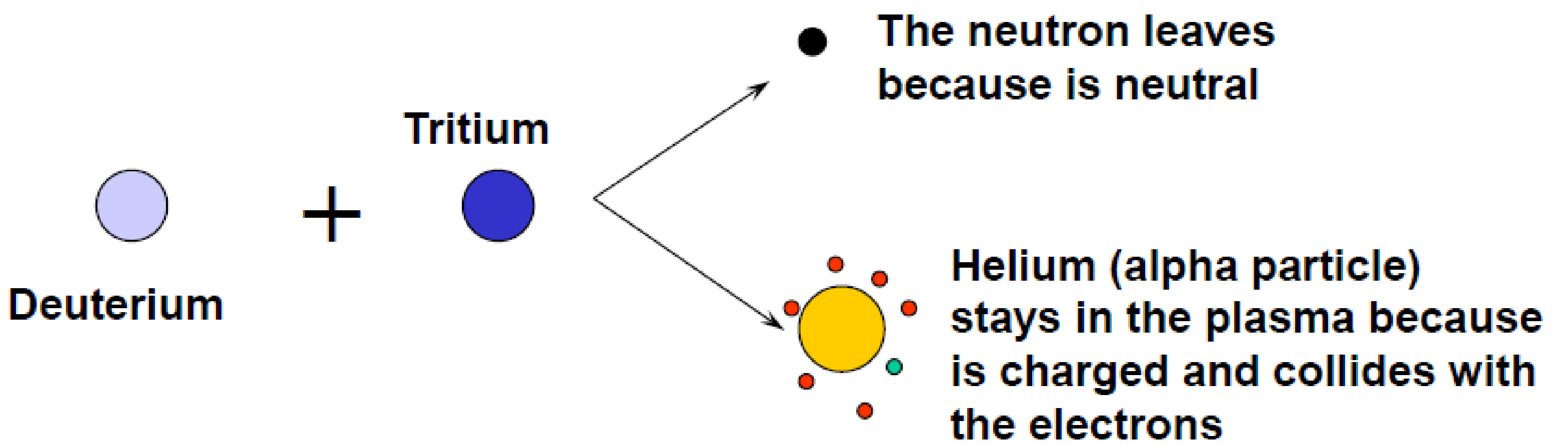
$$<\sigma v> = \int \sigma(v) v f(v)\, dv$$



https://i.stack.imgur.com/wXQD5.jpg
Santarius, J. F., “Fusion Space Propulsion – A Shorter Time Frame Than You Think”, JANNAF, Monterey, 5-8 December 2005.

It takes a lot of energy or power to keep the plasma at 100M ^{0}C

- **Let the plasma do it itself!**



- **The α-particles heat the plasma.**

*R. Betti, HEDSA HEDP Summer School, 2015

Under what conditions the plasma keeps itself hot?

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_α: α particle heating

S_h: external heating

S_B: Bremsstrahlung radiation

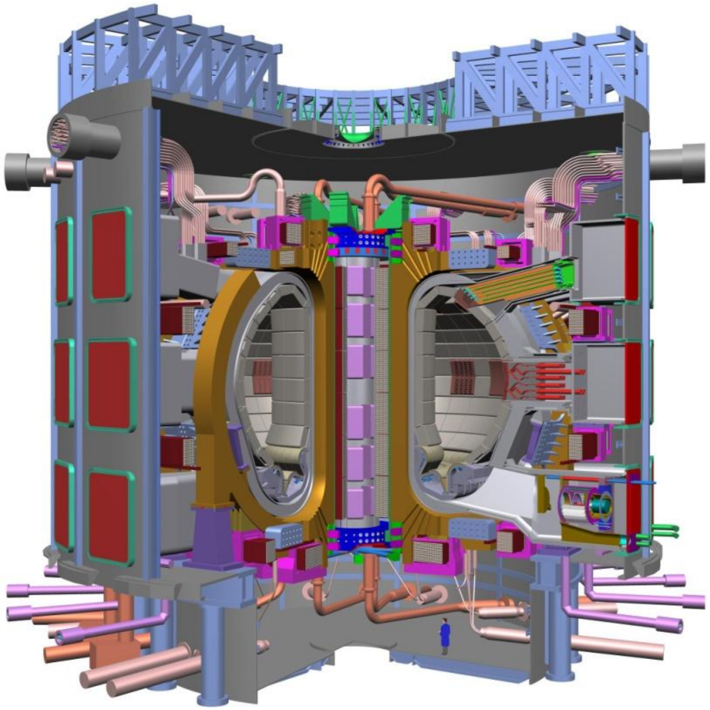
S_k: heat conduction lost

Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

- **P: pressure, or called energy density**
- **τ is confinement time**

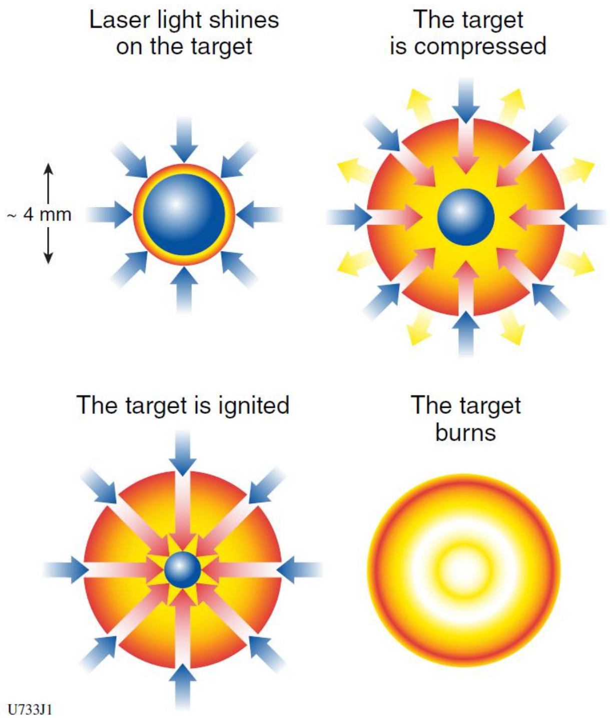
To control? Or not to control?

- **Magnetic confinement fusion (MCF)**



- **Plasma is confined by toroidal magnetic field.**

- **Inertial confinement fusion (ICF)**



- **A DT ice capsule filled with DT gas is imploded by laser.**

Course Outline

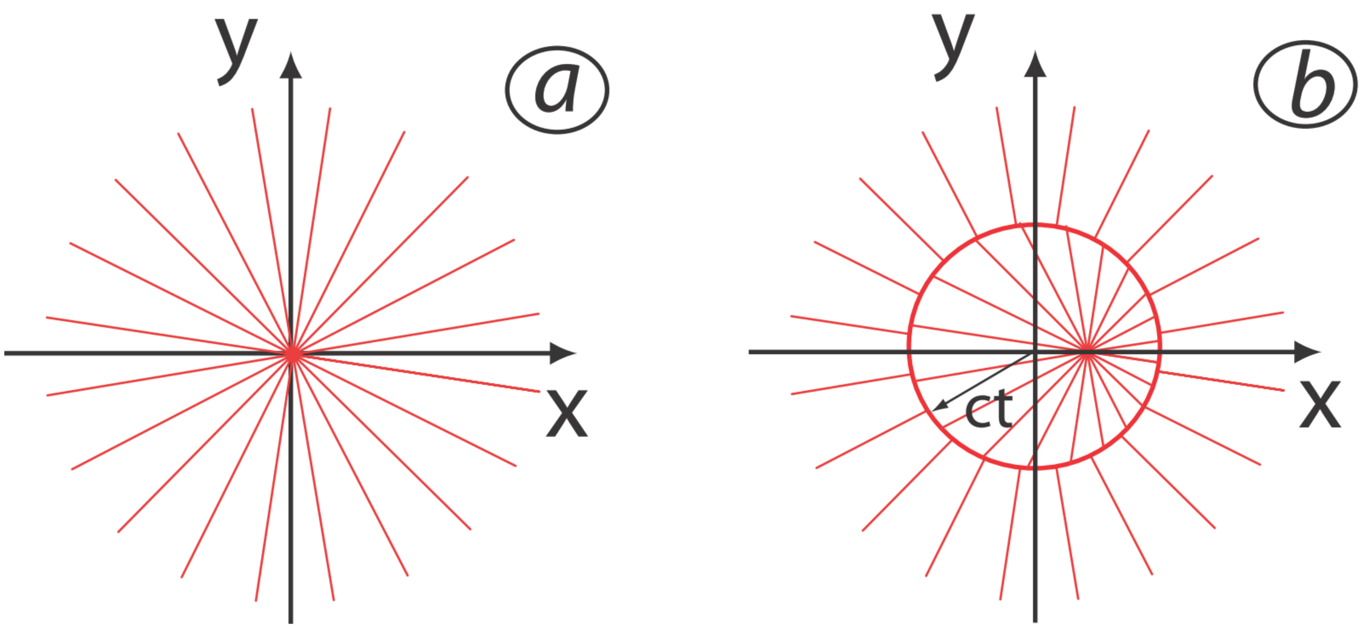
- Brief background reviews
  - Electromagnetics
  - Plasma physics
- Introduction to nuclear fusion
  - Nuclear binding energy (Fission vs Fusion)
  - Fusion reaction physics
  - Some important fusion reactions (Cross section)
    - Main controlled fusion fuels
    - Advanced fusion fuels
  - Maxwell-averaged fusion reactivities

Course Outline

- Introduction to nuclear fusion (cont.)
  - Collisions (Bremsstrahlung radiation)
  - Columb scattering. Cross section of the Columb scattering
  - Beam-target fusion vs thermonuclear fusion
  - Lawson criteria, ignition conditions
  - Magnetic confinement fusion (MCF) vs Inertial confinement fusion (ICF)

Electromagnetic wave is radiated when a charge particle is accelerated

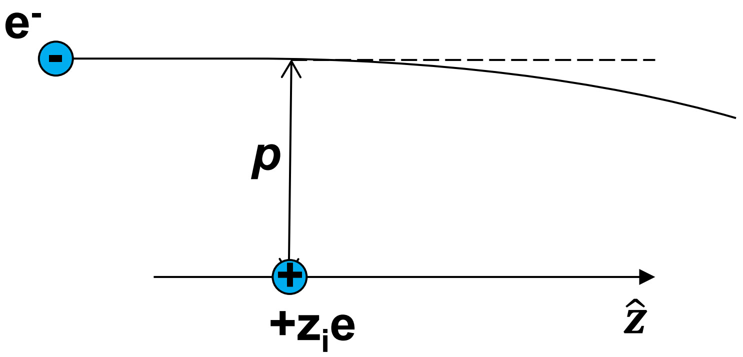
- **The retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.**



https://jseldredphysics.files.wordpress.com/2018/04/lecture18.pdf

Bremsstrahlung emission

- When an electron collides with a nucleus via coulomb interaction, the electron is accelerated and thus radiates, called Bremsstrahlung radiation.
- In the non-relativistic limit, the Bremsstrahlung radiation power is:



$$P_{\mathrm{B,e1,i1}} = \frac{e^2}{6\pi\epsilon_o}\frac{\dot{v}^2}{c^3}$$

p: Impact parameter

- The coulomb force experienced by the electron is:

$$\dot{v} = \frac{F}{m_e} = \frac{ze^2}{4\pi\epsilon_o m_e r^2} = \frac{ze^2}{4\pi\epsilon_o m_e[p^2 + (vt)^2]} \approx \frac{ze^2}{4\pi\epsilon_o m_e p^2}$$

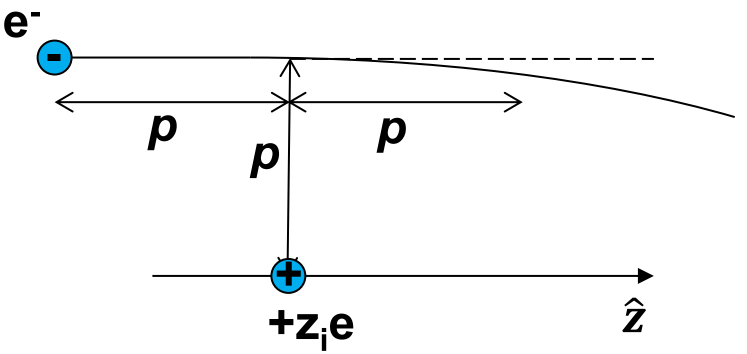
=>

$$P_{\mathrm{B,e1,i1}} = \frac{z^2 e^6}{96\pi^3{\epsilon_o}^3 c^3 {m_e}^2}\frac{1}{p^4} \quad \mathrm{(W)}$$

Bremsstrahlung emission

- The electron begins to accelerate when it is about a distance *p* from the ion. It continuous to accelerate until it travels a distance *p* away from the ion.

$$\Delta t = \frac{2p}{v}$$



- Therefore, the energy loss by one electron colliding one ion is:

$$E_{\mathrm{B,e1,i1}} \approx P_{\mathrm{B,e1,i1}} \Delta t = \frac{z^2 e^6}{48\pi^3 \epsilon_o{}^3 c^3 m_e{}^2} \frac{1}{vp^3} \quad (\mathrm{J})$$

- With careful integration:

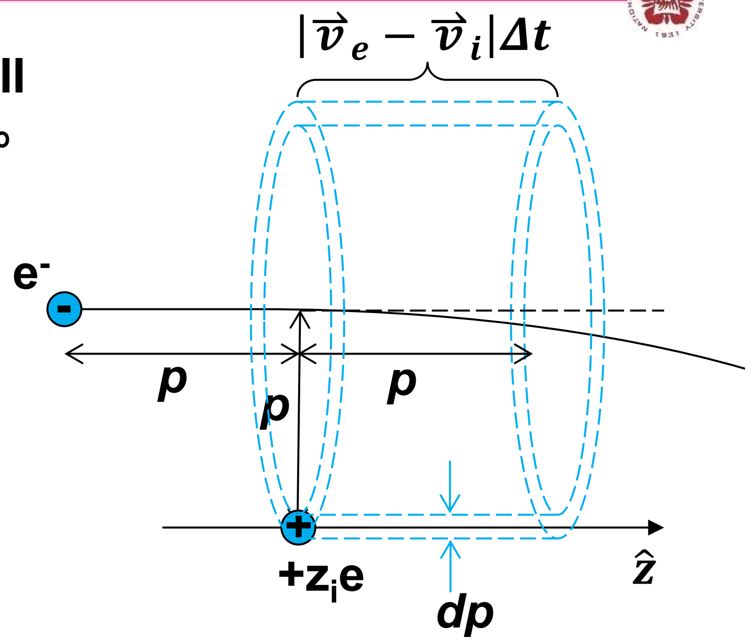
$$E_{\mathrm{B,e1,i1}} = \int_{-\infty}^{\infty} P_{\mathrm{B,e1,i1}} \, \mathrm{d}t = \frac{2z^2 e^6}{3(4\pi\epsilon_o)^3 m_e{}^2 c^3} \int_{-\infty}^{\infty} \frac{1}{[p^2 + (vt)^2]^2} \, dt$$

$$= \frac{\pi z^2 e^6}{3(4\pi\epsilon_o)^3 m_e{}^2 c^3} \frac{1}{vp^3}$$

Bremsstrahlung emission

- **To consider the electron colliding with all ions with impact parameter *p* from 0 to ∞ and include the distribution function of ions $f_i(v_i)$.**
- **Number of ions within the cylinder collided with the incident electron is:**

$$N_i = n_i 2\pi p\, dp |\vec{v}_e - \vec{v}_i| \Delta t$$
$$= \int \int_0^\infty f_i(\vec{v}_i) |\vec{v}_e - \vec{v}_i| \Delta t 2\pi p\, dp\, d\vec{v}_i$$



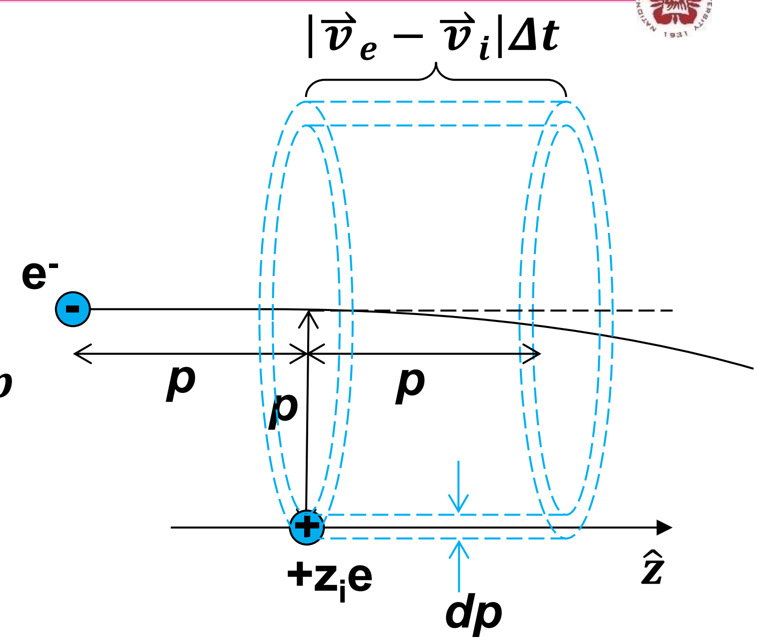
- **The averaged radiation power is:**

$$\bar{P}_{B,e1} = \frac{\bar{E}_{B,e1,i1} N_i}{\Delta t} = \int \int_0^\infty \frac{\bar{E}_{B,e1,i1} f_i(\vec{v}_i) |\vec{v}_e - \vec{v}_i| \Delta t 2\pi p\, dp\, d\vec{v}_i}{\Delta t}$$
$$= \int d\vec{v}_i \int_0^\infty \bar{E}_{B,e1,i1} |\vec{v}_e - \vec{v}_i| f_i(\vec{v}_i) 2\pi p\, dp$$

Bremsstrahlung emission

- Power of the electron colliding with all ions with impact parameter *p* from 0 to ∞ and include the distribution function of ions $f_\mathrm{i}(v_\mathrm{i})$:

$$\overline{P}_{B,e1} = \int d\vec{v}_\mathrm{i} \int_0^\infty \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}| f_\mathrm{i}(\vec{v}_\mathrm{i}) 2\pi p \, dp$$



- In addition, we need to consider the distribution function of electrons $f_\mathrm{e}(v_\mathrm{e})$. The total power loss is:

$$\overline{P}_B = \int d\vec{v}_\mathrm{i} \int d\vec{v}_\mathrm{e} \int_0^\infty \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}| f_\mathrm{i}(\vec{v}_\mathrm{i}) f_\mathrm{e}(\vec{v}_\mathrm{e}) 2\pi p \, dp$$

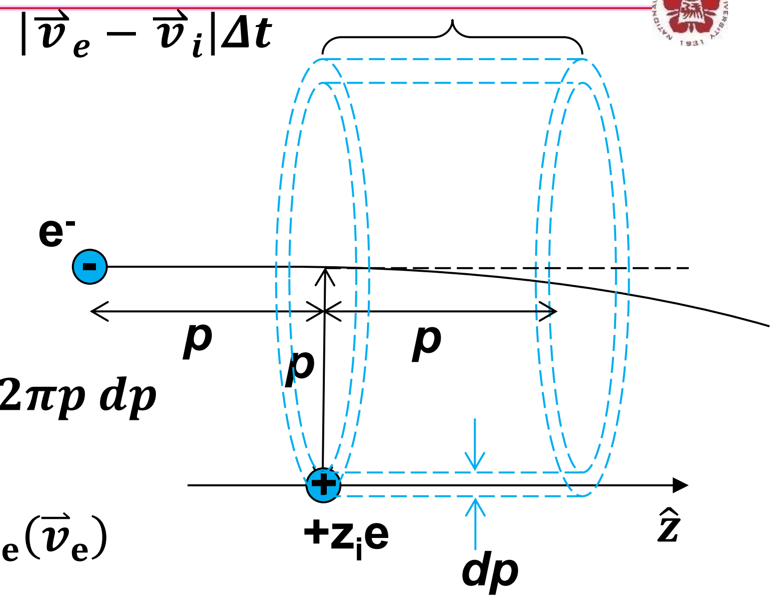
- Since $|v_\mathrm{e}| >> |v_\mathrm{i}|$, $\quad |\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}| \approx v_\mathrm{e}$.
- In addition:

$$\int f_i(\vec{v}_i) \, \mathrm{d}\vec{v}_i \equiv n_i$$

$$d\vec{v}_e = dv_x dv_y dv_z = v_\mathrm{e}^2 \sin\theta \, dv d\theta d\phi \rightarrow 4\pi v_\mathrm{e}^2 dv_\mathrm{e}$$

$$f_\mathrm{e} = n_\mathrm{e} \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \exp\left(-\frac{m_\mathrm{e} v_\mathrm{e}^2}{2T_\mathrm{e}}\right)$$

Bremsstrahlung emission



$$\overline{P}_B = \int d\vec{v}_\mathrm{i} \int d\vec{v}_\mathrm{e} \int_0^\infty \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}| f_\mathrm{i}(\vec{v}_\mathrm{i}) f_\mathrm{e}(\vec{v}_\mathrm{e}) 2\pi p\, dp$$

$$= 2\pi \int f_\mathrm{i}(\vec{v}_\mathrm{i})\, d\vec{v}_i \int 4\pi {v_\mathrm{e}}^2 dv_\mathrm{e} \int_0^\infty p\, dp E_{\mathrm{B,e1,i1}} v_\mathrm{e} f_\mathrm{e}(\vec{v}_\mathrm{e})$$

$$= 8\pi^2 n_\mathrm{i} \int {v_\mathrm{e}}^3 dv_\mathrm{e} \int_0^\infty p\, dp E_{\mathrm{B,e1,i1}} n_\mathrm{e} \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right)$$

$$= 8\pi^2 n_\mathrm{i} n_\mathrm{e} \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \int_0^\infty {v_\mathrm{e}}^3 dv_\mathrm{e} \int_0^\infty p\, dp \left(\frac{z^2 e^6}{48\pi^3 {\epsilon_\mathrm{o}}^3 c^3 {m_\mathrm{e}}^2} \frac{1}{v_\mathrm{e} p^3}\right) \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right)$$

$$= 8\pi^2 n_\mathrm{i} n_\mathrm{e} \left(\frac{z^2 e^6}{48\pi^3 {\epsilon_\mathrm{o}}^3 c^3 {m_\mathrm{e}}^2}\right) \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \int_0^\infty {v_\mathrm{e}}^2 \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right) dv_\mathrm{e} \int_0^\infty \frac{dp}{p^2}$$

Bremsstrahlung emission

- Notice that we are using classical physics. We are not taking account of quantum effects which happen on a length scale of deBroglie wavelength $\Delta x = \hbar/(m_e v)$. Therefore, we have $p_{\min} = \hbar/(m_e v)$.

$$\int_0^\infty \frac{dp}{p^2} \rightarrow \int_{p_{\min}}^\infty \frac{dp}{p^2} = \frac{1}{p_{min}} = \frac{m_e v_e}{\hbar} = \frac{2\pi m_e v_e}{h}$$

$$\overline{P}_B = 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_o{}^3 c^3 m_e{}^2}\right)\left(\frac{m_e}{2\pi T_e}\right)^{3/2} \int_0^\infty v_e{}^2 \exp\left(-\frac{m_e v_e{}^2}{2T_e}\right) dv_e \int_0^\infty \frac{dp}{p^2}$$

$$= 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_o{}^3 c^3 m_e{}^2}\right)\left(\frac{m_e}{2\pi T_e}\right)^{3/2} \frac{2\pi m_e}{h} \int_0^\infty v_e{}^3 \exp\left(-\frac{m_e v_e{}^2}{2T_e}\right) dv_e$$

- With $\int_0^\infty x^3 e^{-x^2}\, dx = \frac{1}{2}$, a better value: $\left(\frac{2^{1/2}}{3\pi^{5/2}}\right)$

$$\overline{P}_B = \left(\frac{2^{1/2}}{6\pi^{3/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) z^2 n_i n_e T_e{}^{1/2} \quad \left(\frac{W}{m^3}\right)$$

Bremsstrahlung emission

- For multiple ion species: n_j, z_j

$$\bar{P}_B = \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) n_e T_e{}^{1/2} \sum_j z_j^2 n_{i,j} \left(\frac{W}{m^3}\right)$$

$$= \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) Z_{eff} n_e^2 T_e{}^{1/2} \left(\frac{W}{m^3}\right)$$

where $$Z_{eff} \equiv \frac{\sum_j z_j{}^2 n_j}{n_e} = \frac{\sum_j z_j{}^2 n_j}{\sum_j z_j n_j} \qquad n_e = \sum_j z_j n_j$$

$$\bar{P}_B = 5.35 \times 10^{-37} Z_{eff} n^2_{e\,(m^{-3})} T^{1/2}_{e\,(keV)} \left(\frac{W}{m^3}\right)$$

$$\bar{P}_B \equiv C_B Z_{eff} n^2_{e\,(m^{-3})} T^{1/2}_{e\,(keV)} \left(\frac{W}{m^3}\right)$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_h: external heating

S_α: α particle heating

$$D + T \longrightarrow He^4\ (3.5\ \mathrm{MeV}) + n\ (14.1\ \mathrm{MeV})$$

$$S_\mathrm{f} = E_\mathrm{f} n_1 n_2 \langle \sigma \mathrm{v} \rangle \left(\mathrm{W/m^3}\right) \quad n_\mathrm{D} = n_\mathrm{T} = \frac{1}{2} n$$

$$S_\alpha = \frac{1}{4} E_\alpha n^2 \langle \sigma \mathrm{v} \rangle = \frac{1}{16} E_\alpha \frac{p^2 \langle \sigma \mathrm{v} \rangle}{T^2}$$

E_α = 3.5 MeV

$$p = p_\mathrm{e} + p_\mathrm{i} = 2p_\mathrm{e} = 2n_e T \equiv 2nT$$

S_B: Bremsstrahlung radiation

$$S_B = C_\mathrm{B} Z_\mathrm{eff} n^2_{\mathrm{e}\ (\mathrm{m}^{-3})} T^{1/2}_{\mathrm{e}\ (\mathrm{keV})} \left(\frac{W}{m^3}\right)$$

$$= \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}}$$

S_k: heat conduction lost

$$S_\kappa = \frac{3}{2} \frac{p}{\tau}$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma \mathrm{v} \rangle}{T^2} \geqslant \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$$S_\alpha + S_h = S_B + S_k$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

$$p\tau \geqslant \frac{6}{\frac{1}{4} E_\alpha \frac{\langle \sigma v \rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

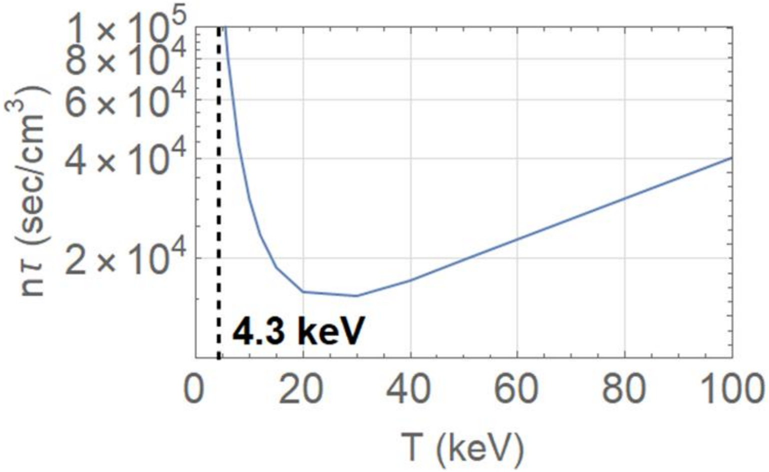
$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

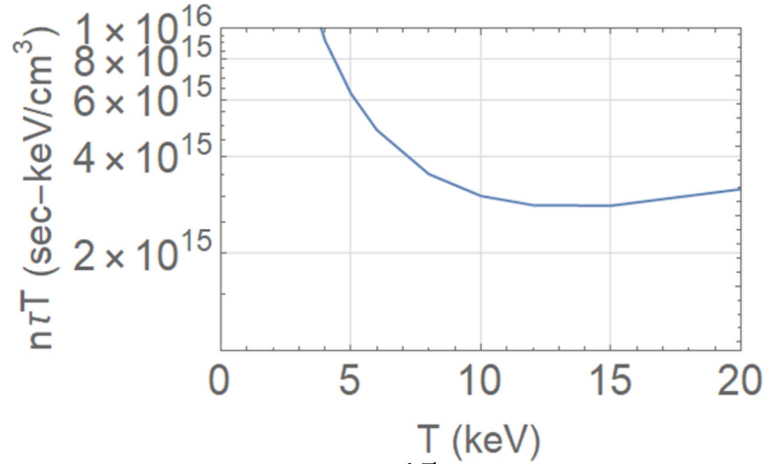
$$nT\tau > \frac{3T^2}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

$$n\tau > \frac{3T}{\frac{1}{4} \langle \sigma v \rangle E_\alpha - C_B \sqrt{T}}$$

Temperature needs to be greater than ~5 keV to ignite

$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$





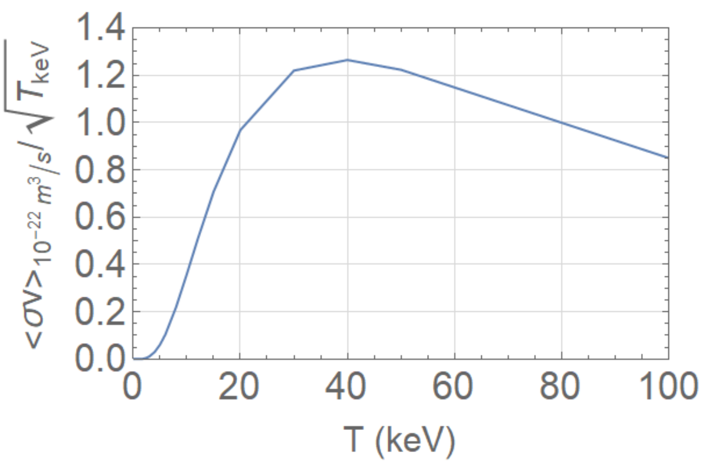
$$n\tau > 2 \times 10^4\ \mathrm{sec/cm^3}$$

$$nT\tau > 3.5 \times 10^{15}\ \mathrm{keV-sec/cm^3}$$

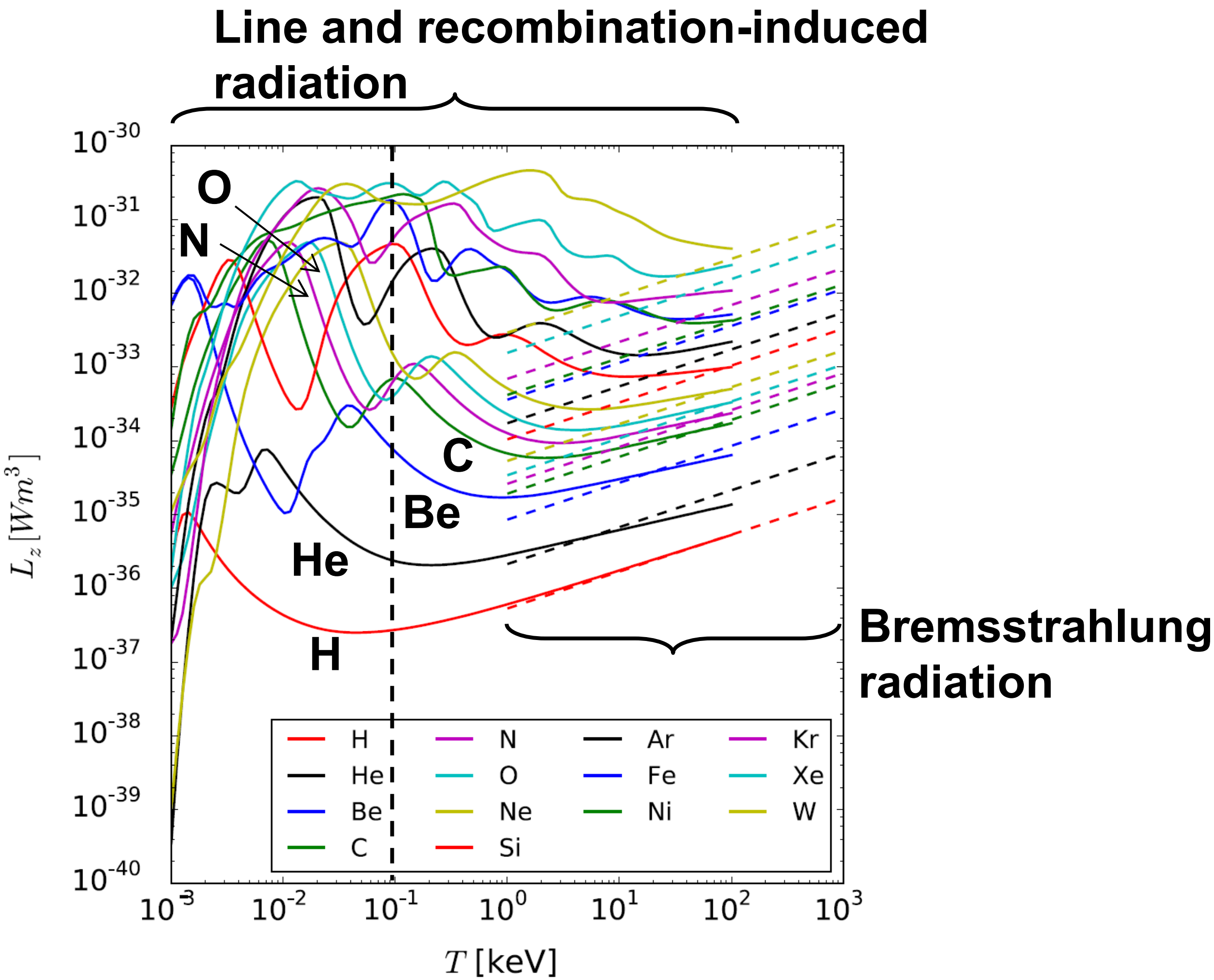
S_α>S_B $$\frac{1}{4}E_\alpha n^2\langle\sigma v\rangle > C_B n^2 T^{1/2}$$

$$\frac{\langle\sigma v\rangle}{T^{1/2}} > \frac{4C_B}{E_\alpha}$$

$$T > 4.3\ \mathrm{keV}$$



Temperature of 100 eV is the threshold of radiation barrier by impurities



H. Lux, etc., Fusion Eng. Des. **42**, 101 (2015)

Under what conditions the plasma keeps itself hot?

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  S_α: α particle heating

  S_h: external heating

  S_B: Bremsstrahlung radiation

  S_k: heat conduction lost

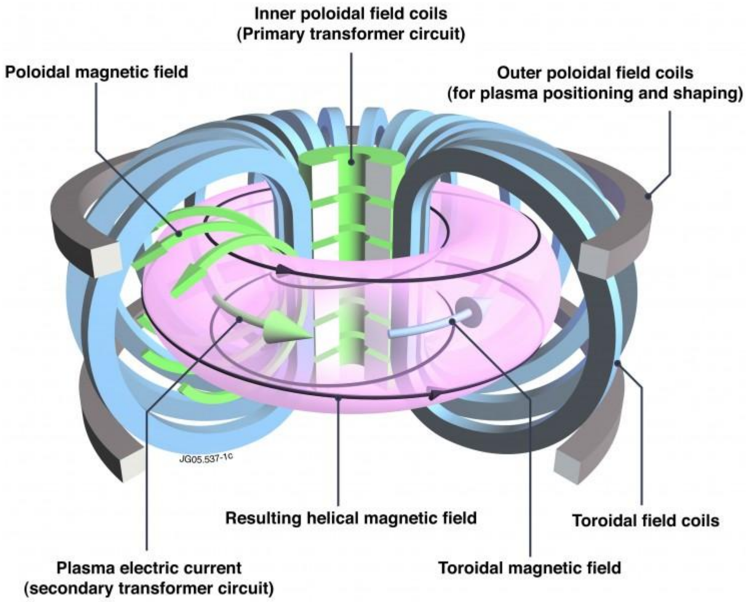
Ignition condition: $P\tau$ > 10 atm-s = 10 Gbar - ns

- **P: pressure, or called energy density**
- **τ is confinement time**

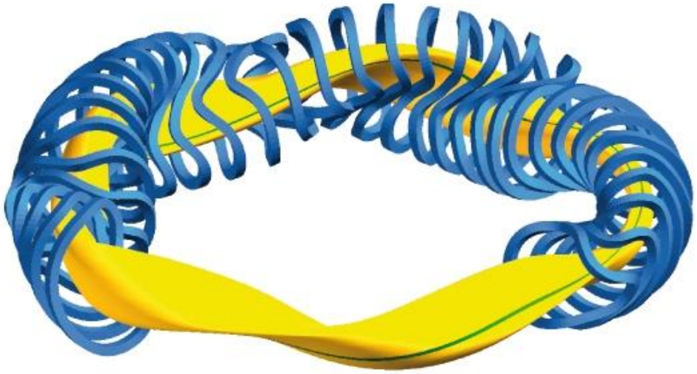
The plasma is too hot to be contained

- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. P~atm, τ~sec, T~10 keV (10^8 ºC)**

Tokamak



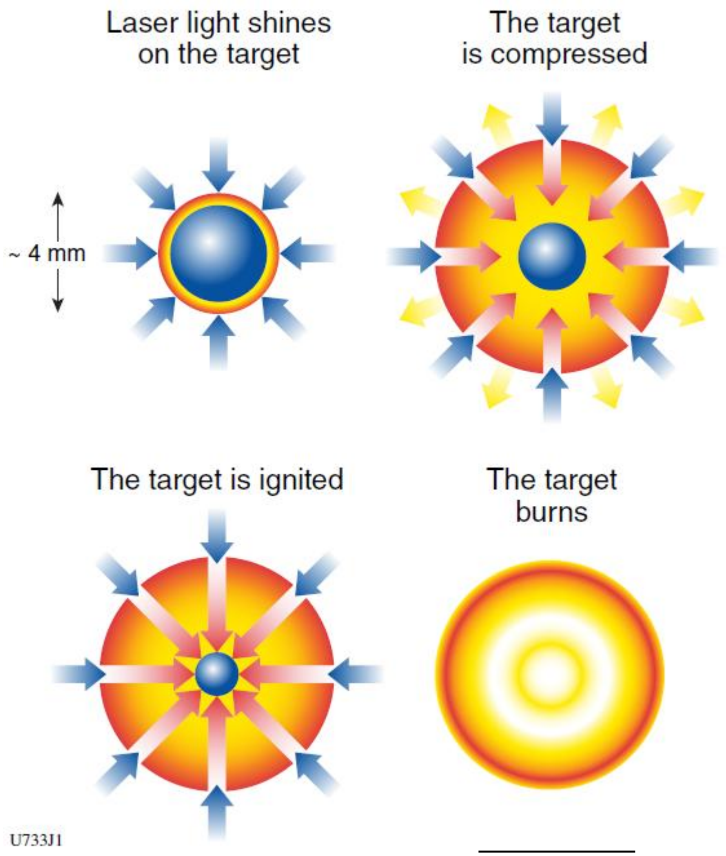
Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Don’t confine it!

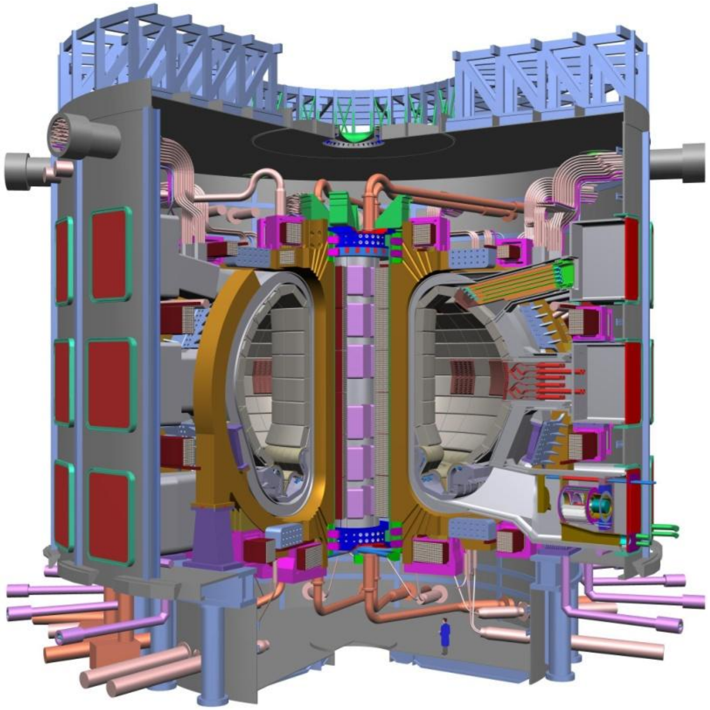
- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ~nsec, T~10 keV (10^8 ^{0}C)**



Inertial confinement fusion: an introduction, Laboratory for Laser Energetics, University of Rochester

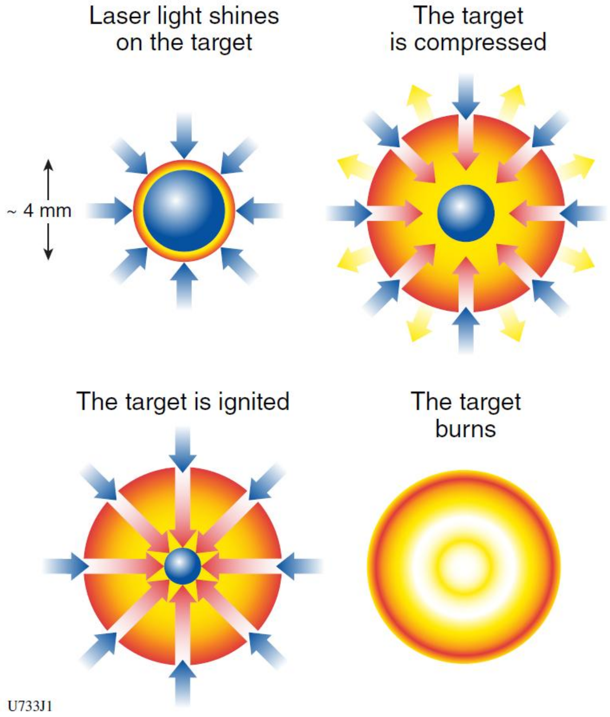
To control? Or not to control?

- **Magnetic confinement fusion (MCF)**



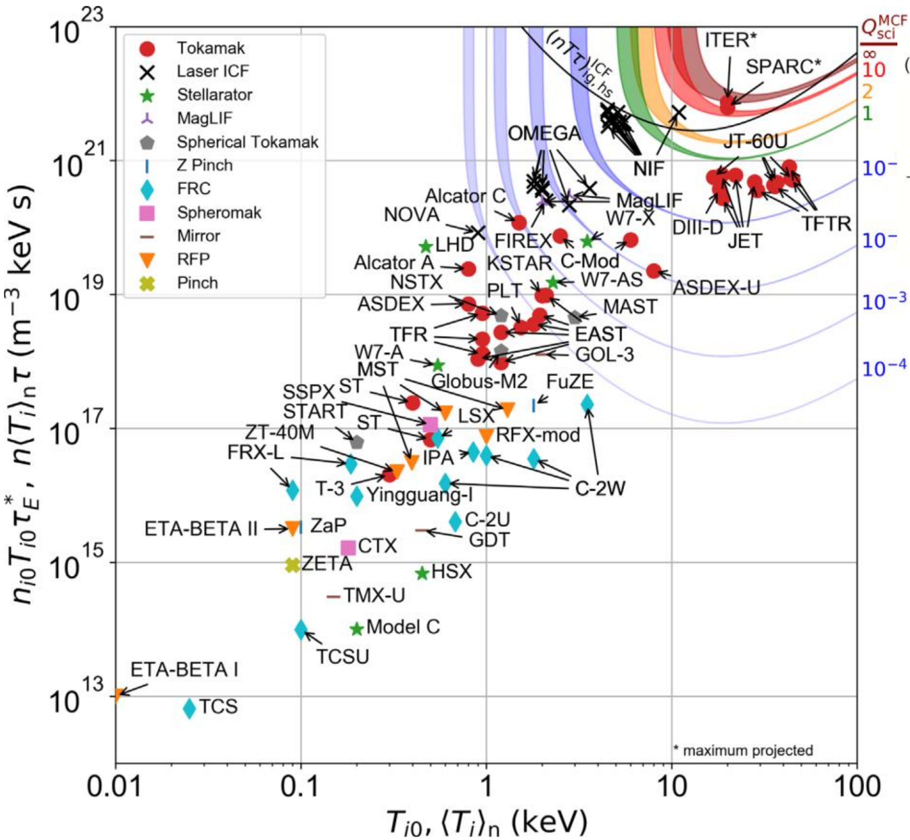
  - **Plasma is confined by toroidal magnetic field.**

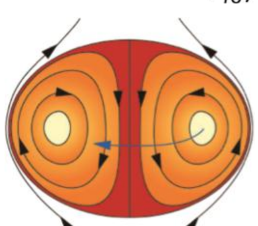
- **Inertial confinement fusion (ICF)**

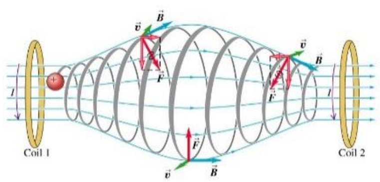


  - **A DT ice capsule filled with DT gas is imploded by laser.**

We are closed to ignition!

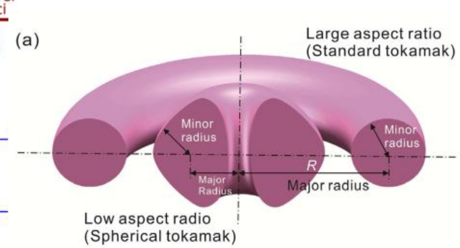




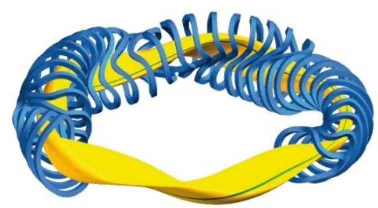


Mirror

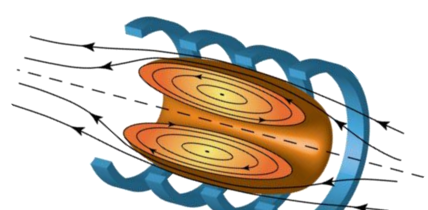
MCF



(Spherical) tokamak

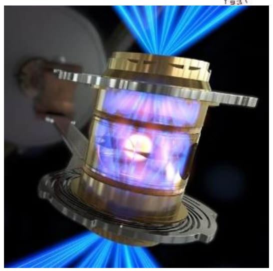


Stellerator

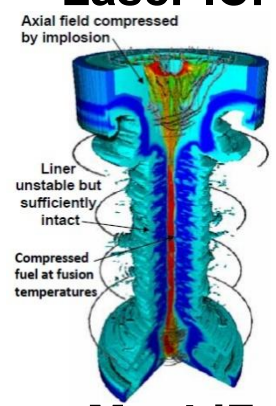


FRC

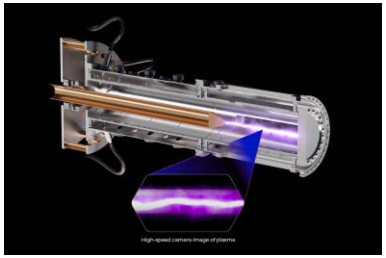
ICF



Laser ICF



MagLIF

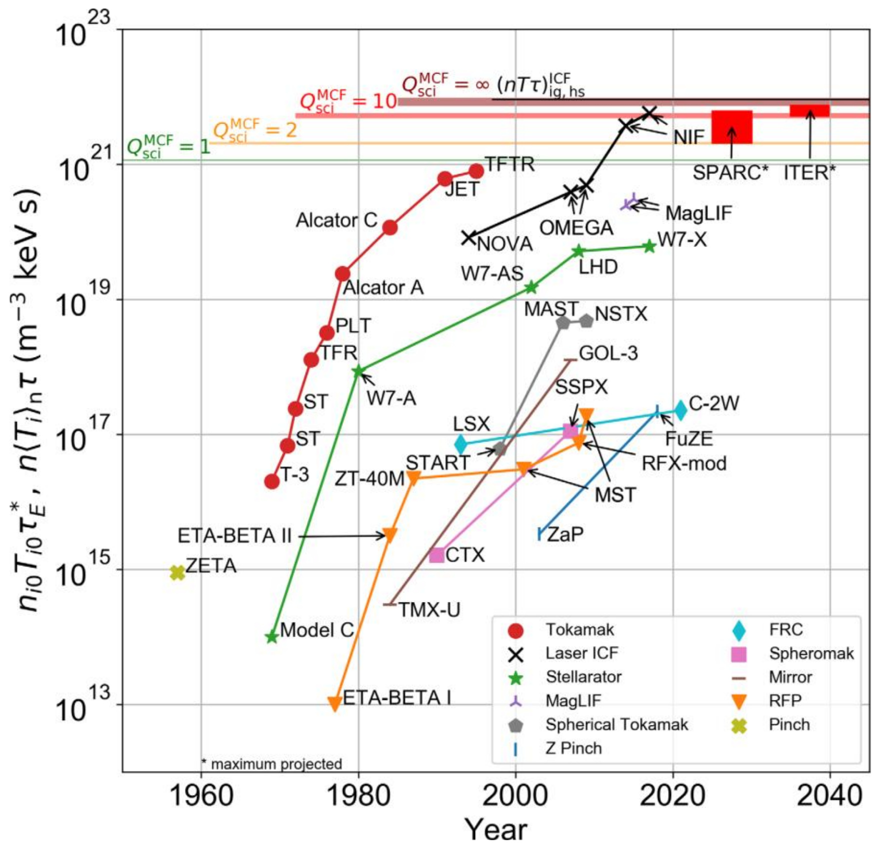


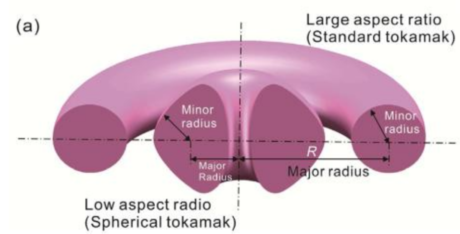
Sheared-flow Z pinch

Samuel E. Wurzel and Scott C. Hsu, Phys. Plasmas, **29**, 062103 (2022)

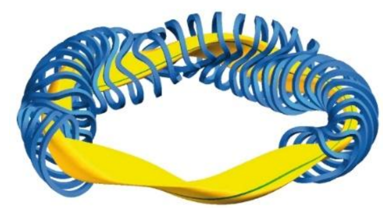
We are closed to ignition!

MCF | ICF

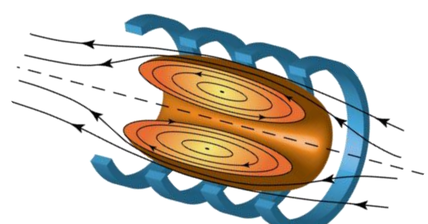




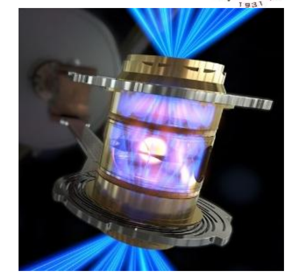
(Spherical) tokamak



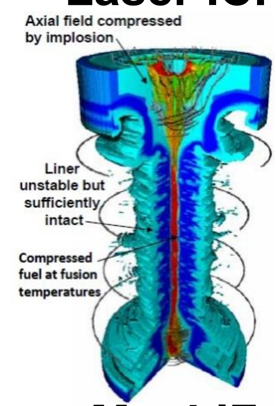
Stellerator



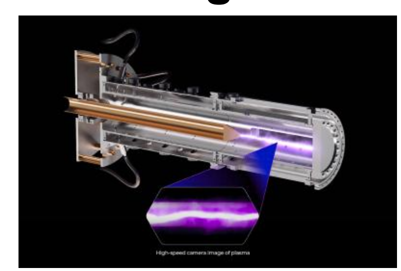
FRC



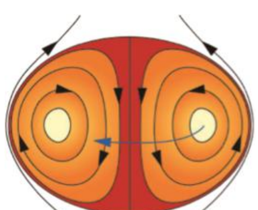
Laser ICF



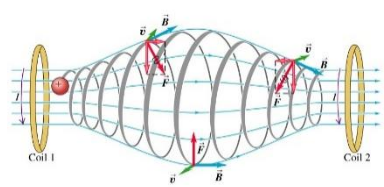
MagLIF



Sheared-flow Z pinch



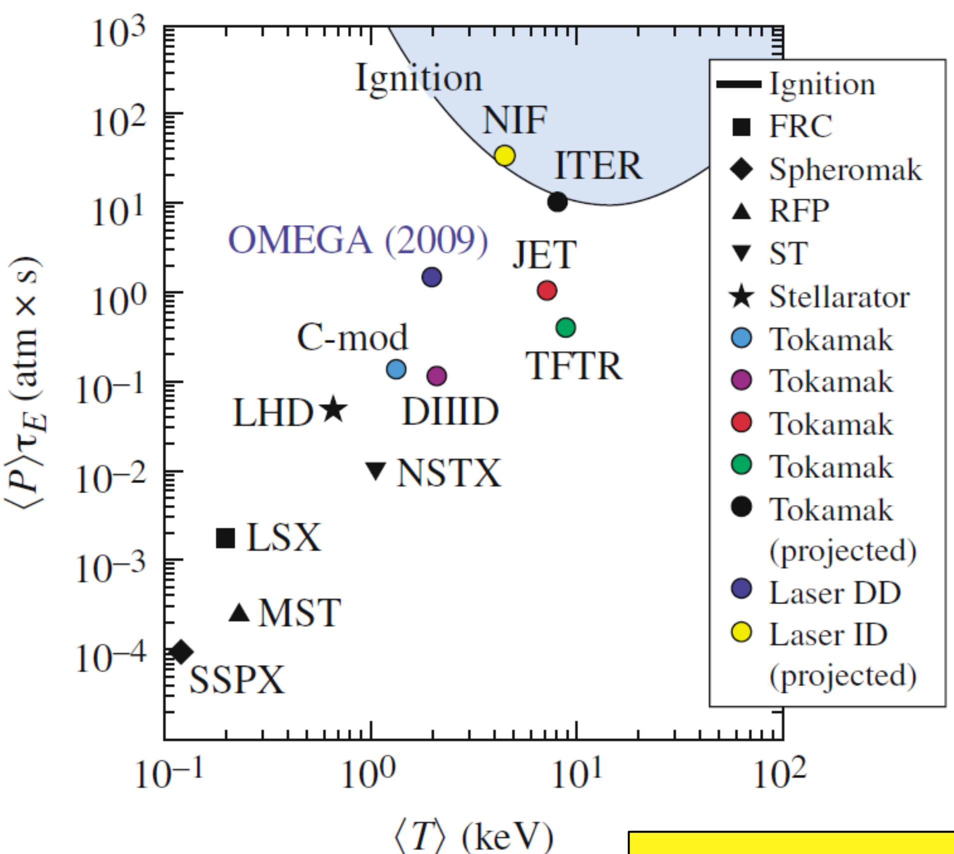
Spheromak

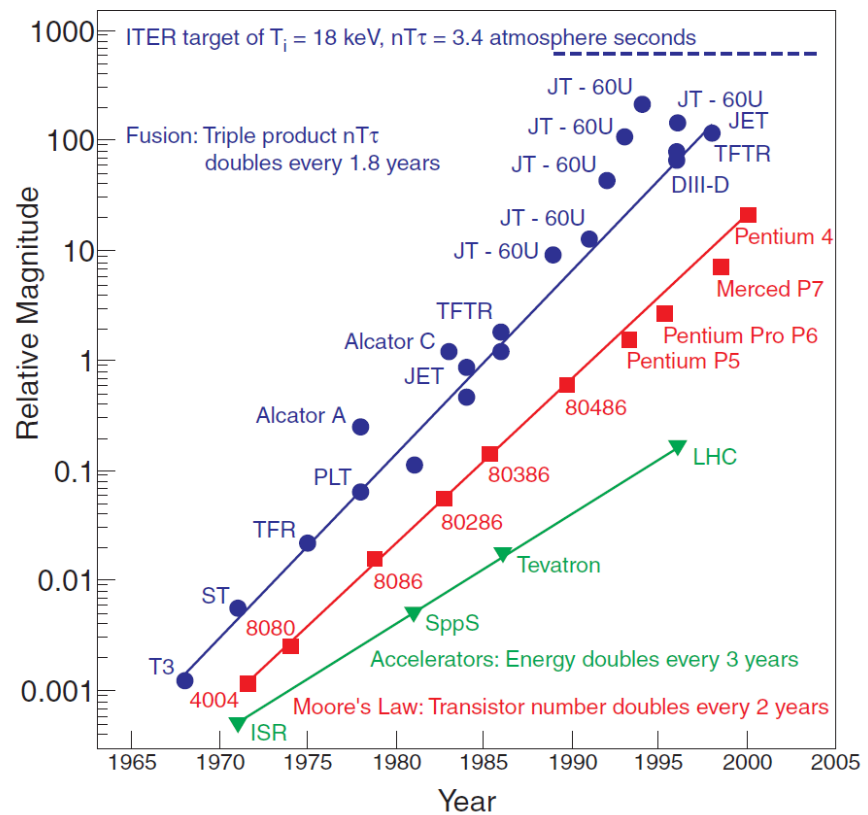


Mirror

Samuel E. Wurzel and Scott C. Hsu, Phys. Plasmas, **29**, 062103 (2022)

We are closed to ignition!





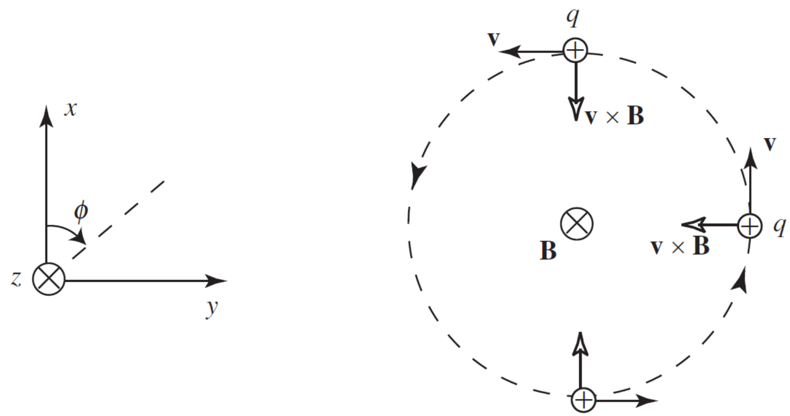
Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

A. J. Webster, Phys. Educ. **38**, 135 (2003)
R. Betti, etc., Phys. Plasmas, **17**, 058102 (2010)

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Charged particles gyro around the magnetic field line



$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

- **Assuming** $\vec{B} = B\hat{z}$ **and the electron oscillates in x-y plane**

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \qquad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

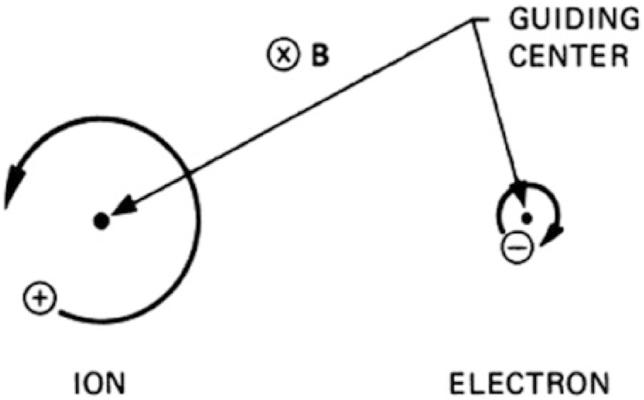
$$\omega_c \equiv \frac{|q|B}{m}$$ **Cyclotron frequency or gyrofrequency**

$$\ddot{v}_x + {\omega_c}^2 v_x = 0$$

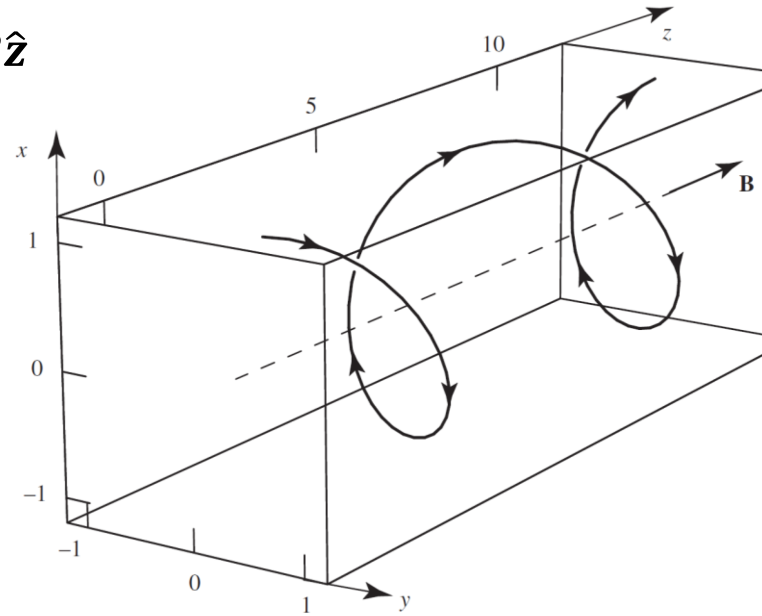
$$\ddot{v}_y + {\omega_c}^2 v_y = 0$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$\vec{B} = B\hat{z}$$



$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = v_\perp \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

$$x = \mp r_c sin(\pm\omega_c t + \psi) + (x_o - r_c sin\psi)$$

$$y = \pm r_c cos(\pm\omega_c t + \psi) + (y_o + r_c cos\psi)$$

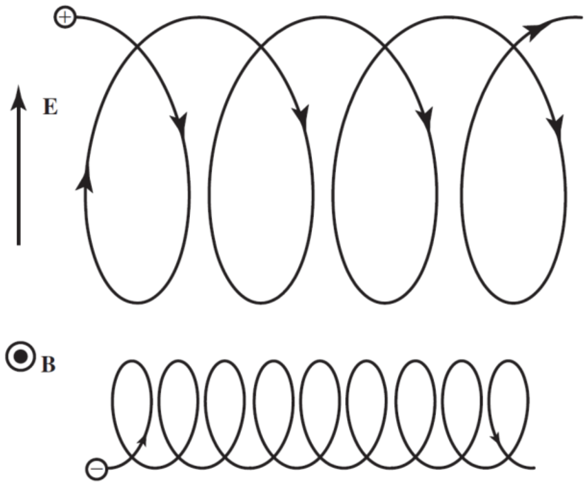
$$z = z_o + v_{||} t$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left|\frac{m v_\perp{}^2}{r}\right| = \left|q\,\vec{v} \times \vec{B}\right| = |q v_\perp B|$$

$$r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$$ **Larmor radius or gyroradius**

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$v_{||}(t) = \frac{qE_{||}}{m}t + v_{||,o}$$

$$\vec{v}_\perp(t) = \vec{v}_\mathrm{E} + \vec{v}_\mathrm{ac}(t)$$

$$m\frac{d}{dt}\left(\vec{v}_\mathrm{E} + \vec{v}_\mathrm{ac}(t)\right) = q\left[\hat{x}E_\perp + \left(\vec{v}_\mathrm{E} + \vec{v}_\mathrm{ac}(t)\right) \times \hat{z}B\right]$$

$$m\frac{d\vec{v}_\mathrm{ac}(t)}{dt} = q[\hat{x}E_\perp + \vec{v}_\mathrm{E} \times \hat{z}B + \vec{v}_\mathrm{ac}(t) \times \hat{z}B]$$

No E field case: $m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$

$$\hat{x}E_\perp + \vec{v}_\mathrm{E} \times \hat{z}B = 0$$

$\times \hat{z}B$

$$(\vec{C} \times \vec{B}) \times \vec{A} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

$$\vec{v}_\mathrm{E} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$m\frac{d\vec{v}_\mathrm{ac}(t)}{dt} = q\,\vec{v}_\mathrm{ac}(t) \times \hat{z}B$$

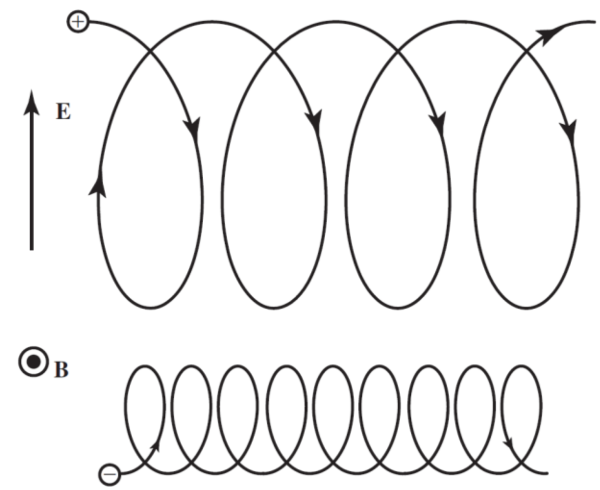
Gyro motion

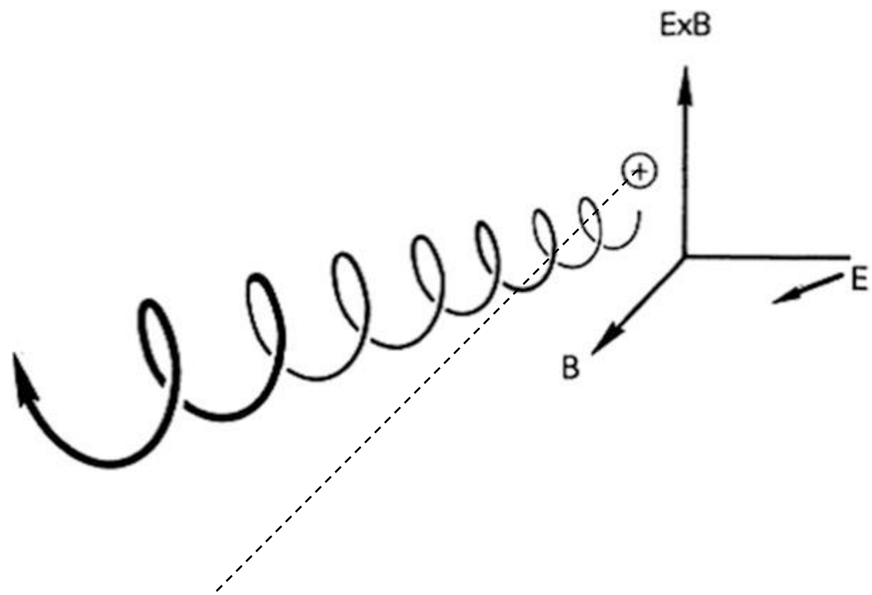
$$\vec{v}(t) = \hat{z}v_{||}(t) + \vec{v}_\mathrm{E} + \vec{v}_\mathrm{ac}(t)$$

$$\langle\vec{v}(t)\rangle = \frac{1}{T_c}\int_0^{T_c}\vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

- Electrons and ions drift in the same direction.

No current is generated in ExB drift





$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t)\, dt = \hat{z} v_{||}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x} E_\perp \times \hat{z} B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$ **ExB drift velocity**

- **Electrons and ions drift in the same direction.**

Gravitational drift

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs

$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E}\times\vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_\perp + \hat{z}F_{||} = \hat{x}F_\perp + \hat{z}F_{||}$$

$$m\frac{dv_{||}}{dt} = F_{||}$$

$$m\frac{d\,\vec{v}_\perp}{dt} = q\left(\hat{x}\frac{F_\perp}{q} + \vec{v}_\perp \times \hat{z}B\right)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_F$$

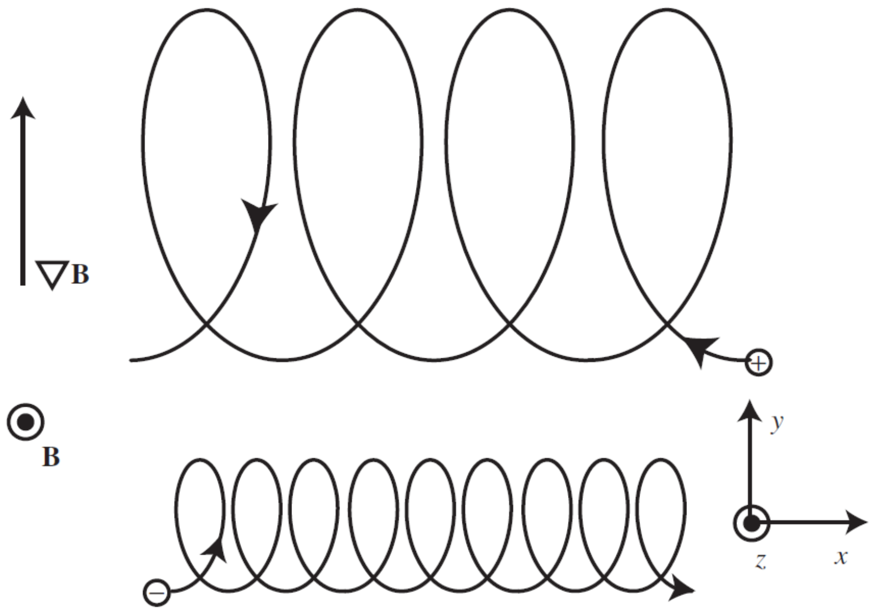
$$\vec{v}_{\mathrm{F}} = \frac{\hat{x}(F_\perp/q) \times \hat{z}B}{B^2} = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2}$$

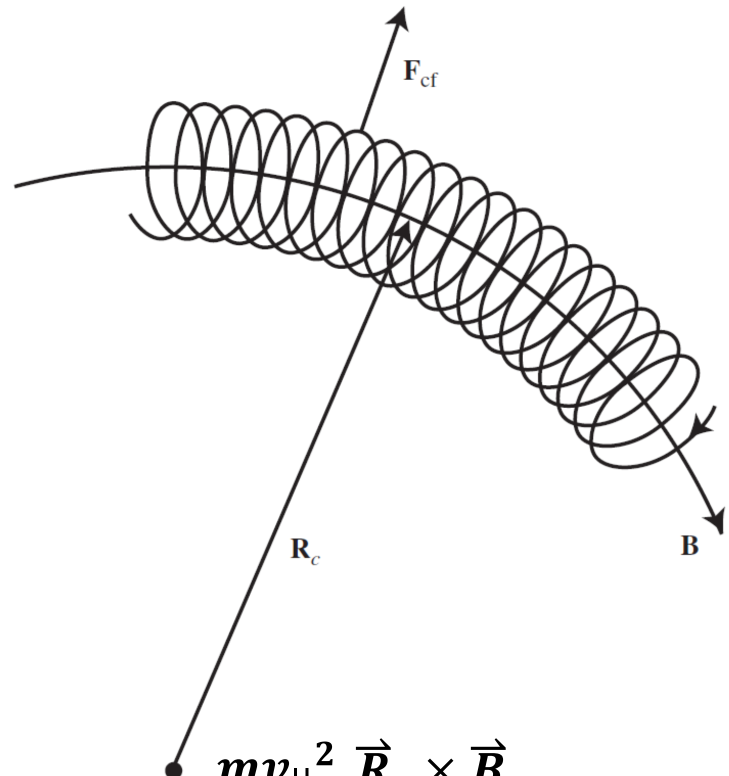
Gravitational drift velocity

- **Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.**

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**
- **Curvature drift**



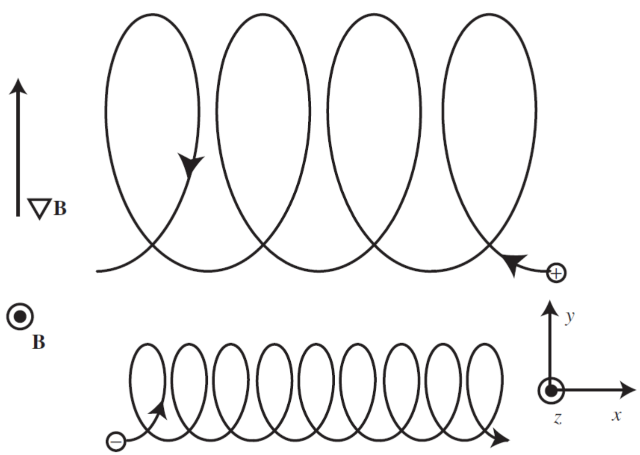


$$\vec{v}_{\nabla} = \frac{m{v_{\perp}}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$$

$$\vec{v}_R = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_{\text{R}} + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm\omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm\omega_c t + \psi)$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$

$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x} q v_y B_z - \hat{y} q v_x B_z$$

$$\simeq \hat{x} q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right) - \hat{y} q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$B_z(y) = B_o + y\frac{\partial B_z}{\partial y} + y^2 \frac{1}{2}\frac{\partial^2 B_z}{\partial y^2} + \ldots$$

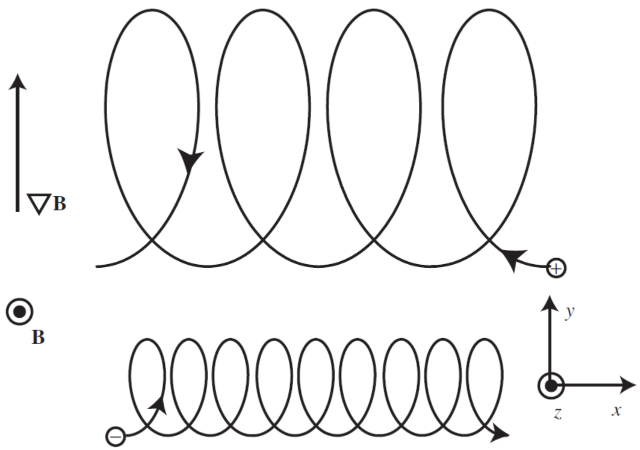
$$F_x = q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$F_x \simeq -q v_\perp \sin(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_\perp \cos(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_\perp \sin(\pm\omega_c t + \psi)\left[B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right]$$

$$F_y \simeq -qv_\perp \cos(\pm\omega_c t + \psi)\left[B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right]$$

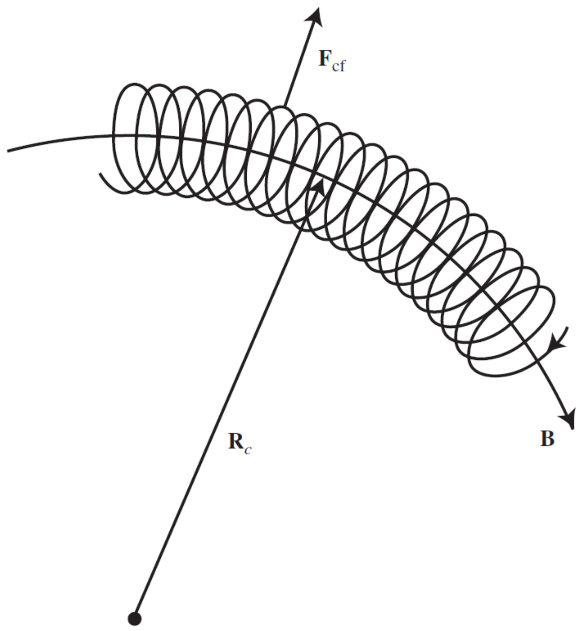
$$\langle F_x \rangle = 0$$

$$\langle F_y \rangle = \mp \frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} = -\frac{m{v_\perp}^2}{2B}\frac{\partial B_z}{\partial y} \qquad r_c = \frac{v_\perp}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \qquad \vec{v}_\nabla = \frac{1}{q}\frac{\langle F_y\rangle\hat{y}\times\hat{z}B_z}{{B_z}^2} = -\frac{m{v_\perp}^2}{2q{B_z}^2}\frac{\partial B_z}{\partial y}\hat{x}$$

- **More general:** $\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$

Charge particles drift across magnetic field lines when the magnetic field line is curved

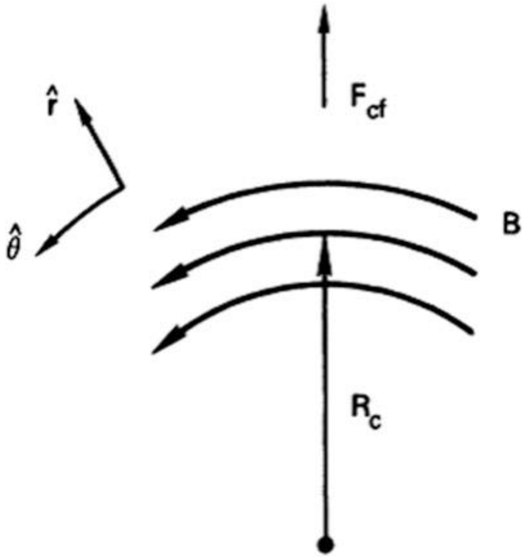


$$\vec{F}_{cf} = m v_{||}^2 \frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m v_{||}^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B\hat{r}$$

Cylindrical coordinate

$$\vec{v}_\nabla = \frac{m v_\perp^2}{2q}\frac{\vec{B}\times\nabla B}{B^3} \qquad \vec{v}_R = \frac{m v_{||}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\nabla\times\vec{B} = 0$$

$$\left(\nabla\times\vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z} = 0 \quad \left(\nabla\times\vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial\theta}$$

$$\nabla\times\vec{B} = \left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \qquad B_\theta \propto \frac{1}{r}$$

$$\frac{\nabla|B|}{|B|} = -\frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_{total} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left(v_{||}^2 + \frac{1}{2}v_\perp^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{R_c^2 B^2}\left(v_{||}^2 + \frac{1}{2}v_\perp^2\right)$$

- **Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.**

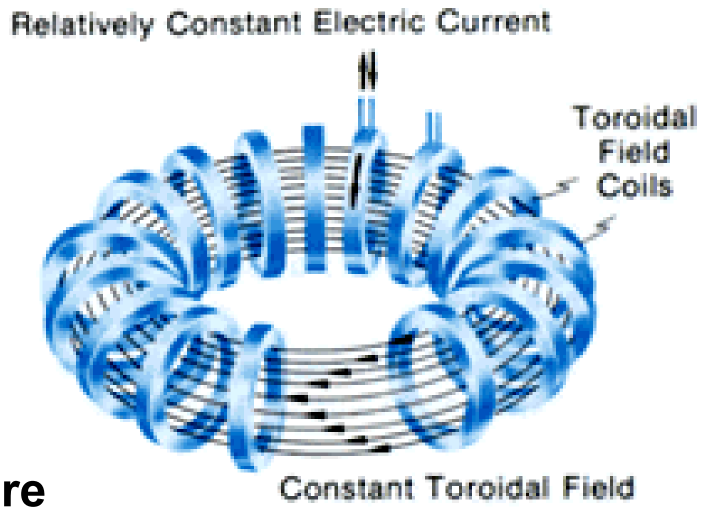
Quick summary of different drifts

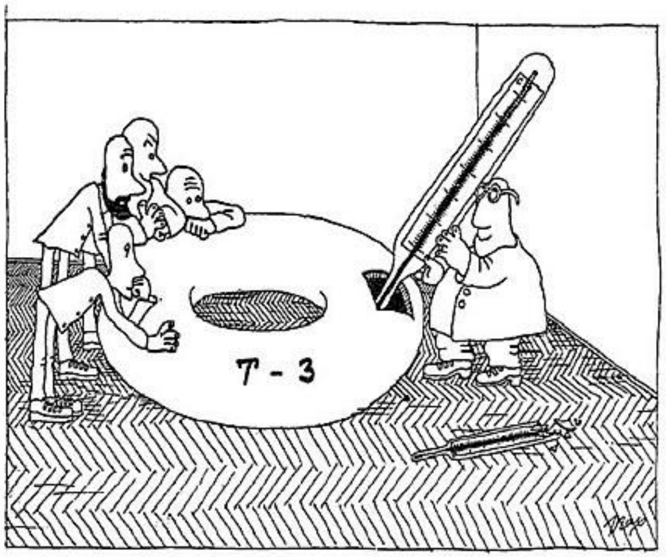
- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m{v_\perp}^2}{2q}\frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{R_{\mathrm{c}}B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}}B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right)$$

Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field

- **Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)**





Nature

Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

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Electron temperatures of 100 eV up to 1 keV and densities in the range 1–3 × 10[13] cm[−3] have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

T_e = 100 ~ 1 keV
n_e = 1-3 x 10^{13} cm^{-3}

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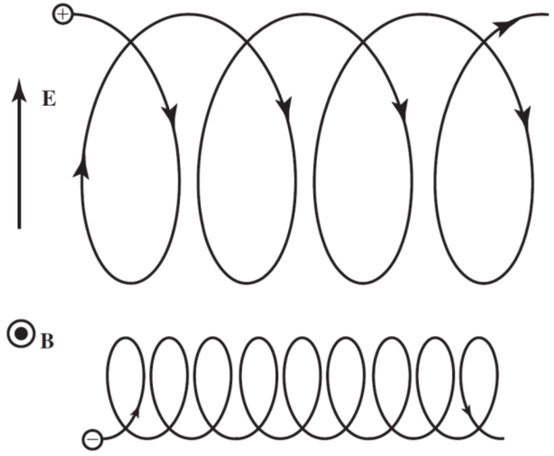
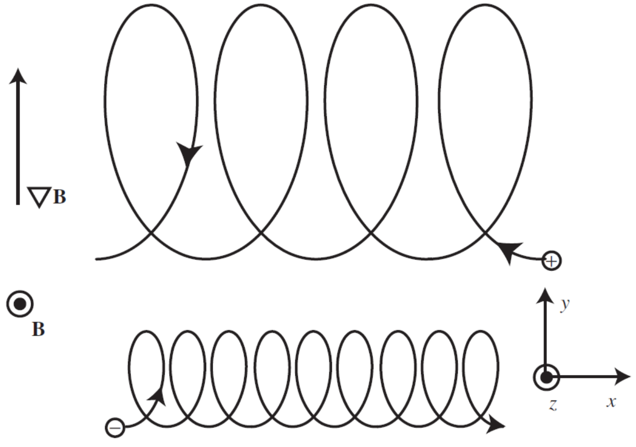
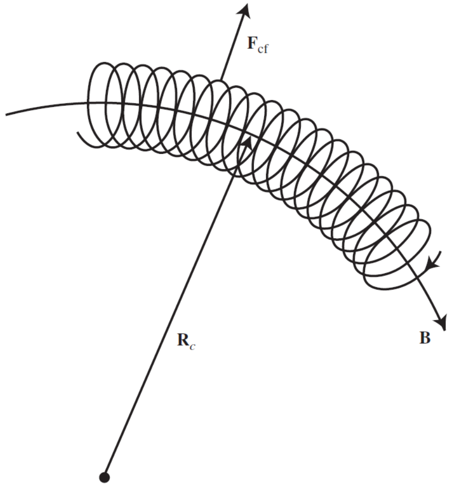
https://www.iter.org/mach/tokamak
https://en.wikipedia.org/wiki/Tokamak#cite_ref-4
Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.
N. J. Peacock,et al., Nature **224**, 488 (1969)

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \dfrac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \dfrac{m v_{\perp}{}^2}{2q} \dfrac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \dfrac{m v_{||}{}^2}{q} \dfrac{\vec{R}_{\mathrm{c}} \times \vec{B}}{R_c{}^2 B^2}$ **Depended on charge**

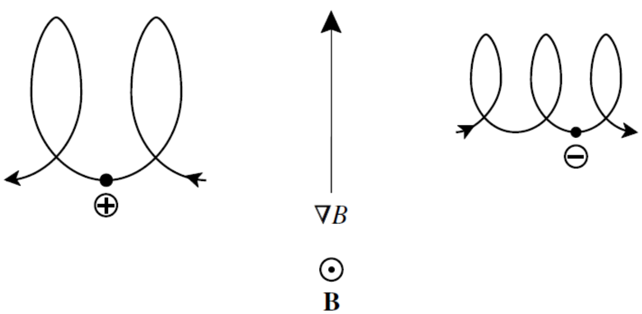




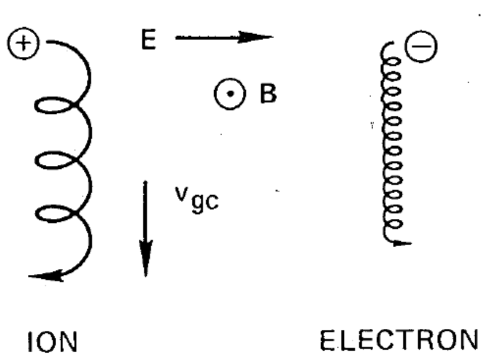


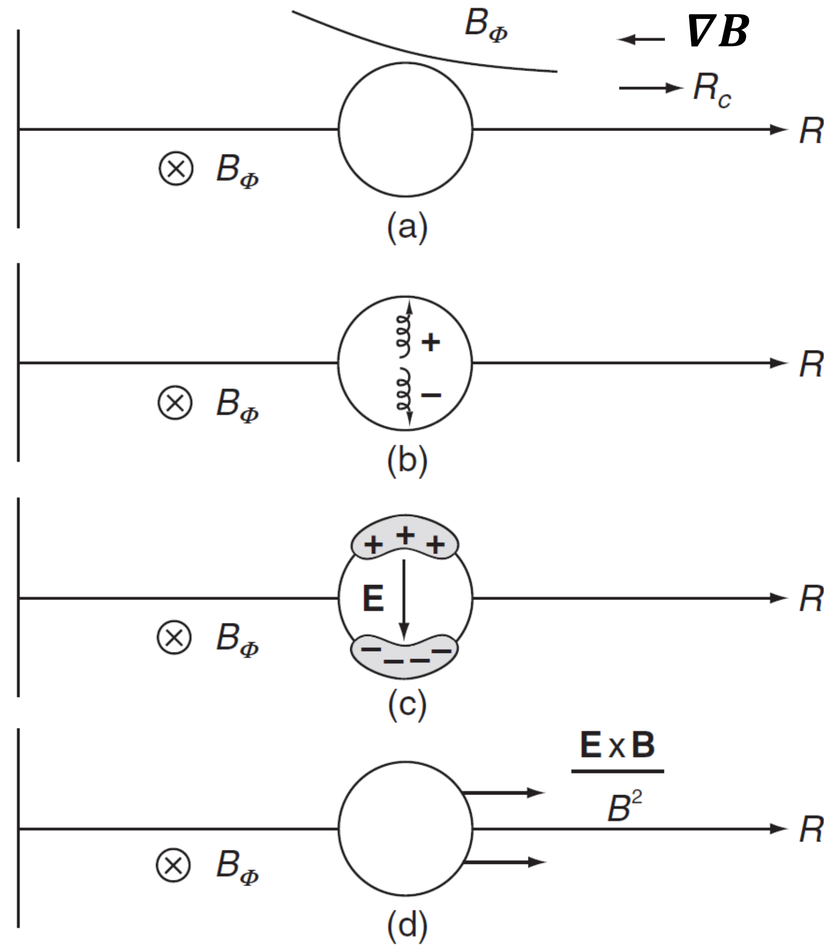
Charged particles drift across field lines

- **Grad-B drift**



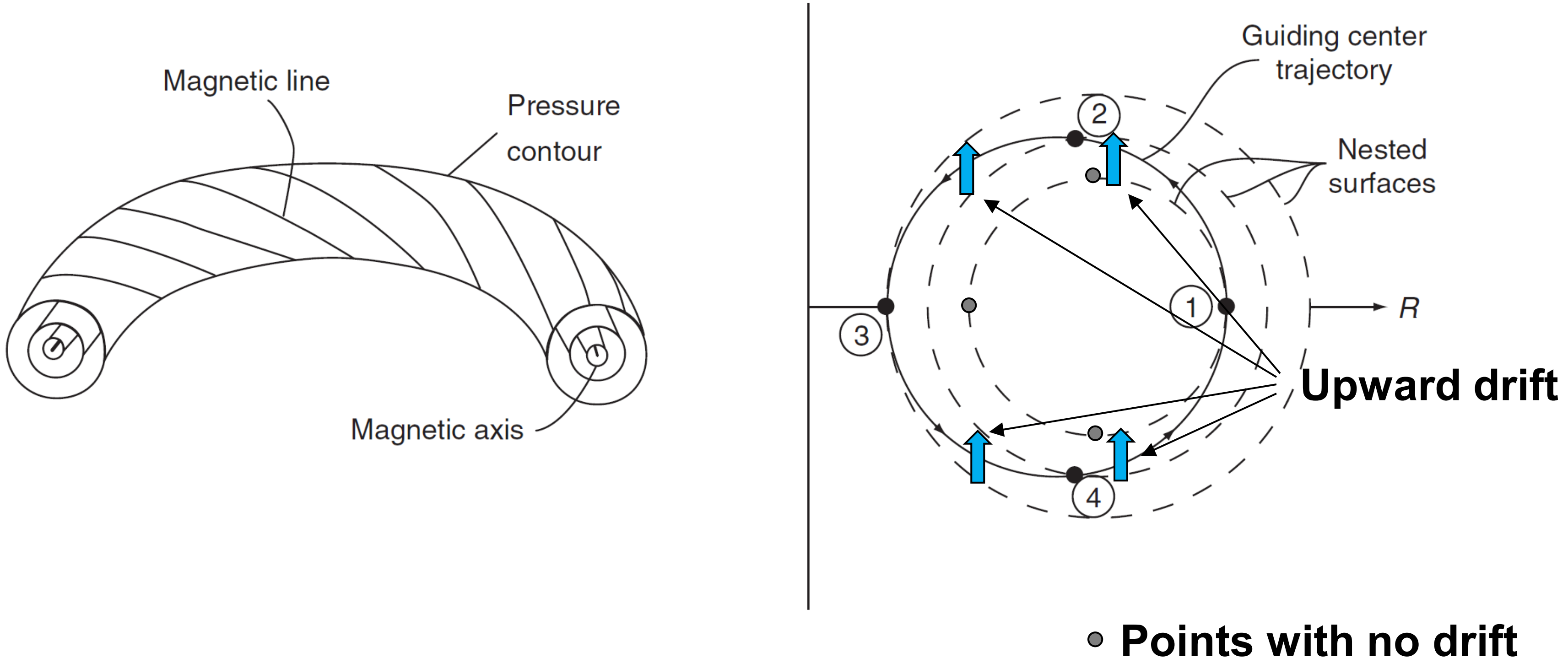
- **ExB drift**



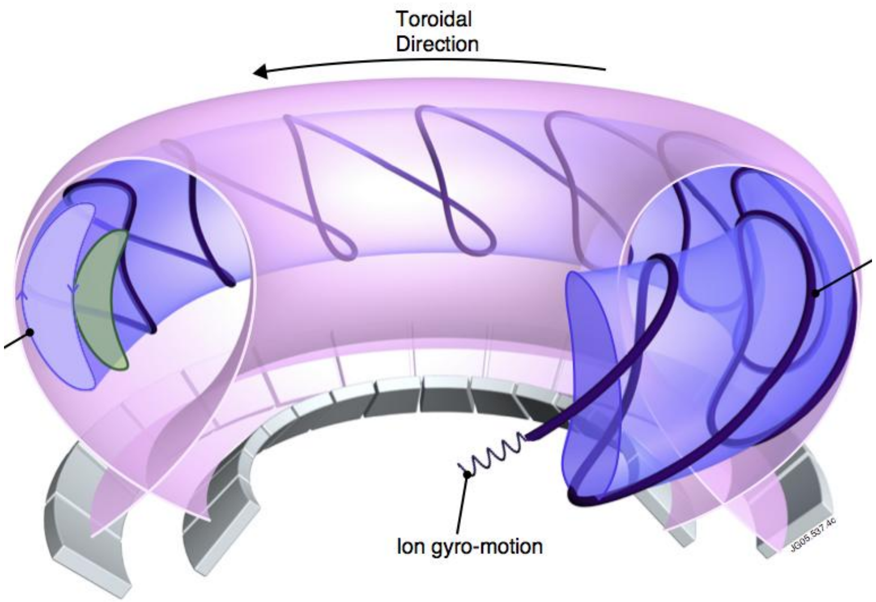


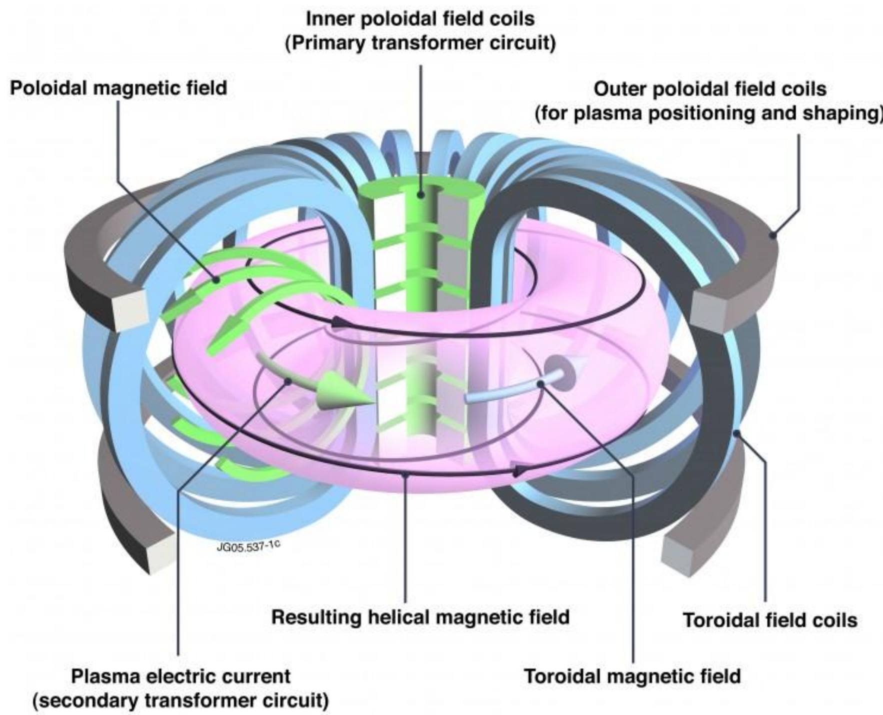
J. P. Freidberg, Ideal Magnetohydrodynamics

The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field



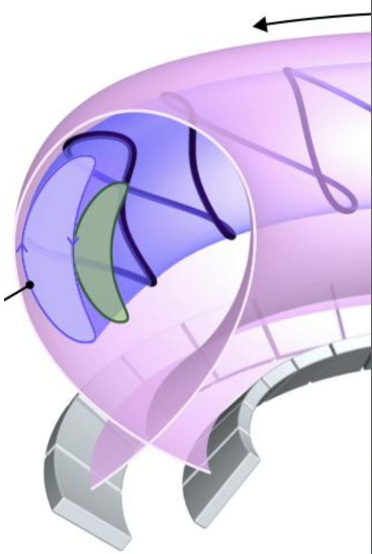
A poloidal magnetic field is required to reduce the drift across field lines

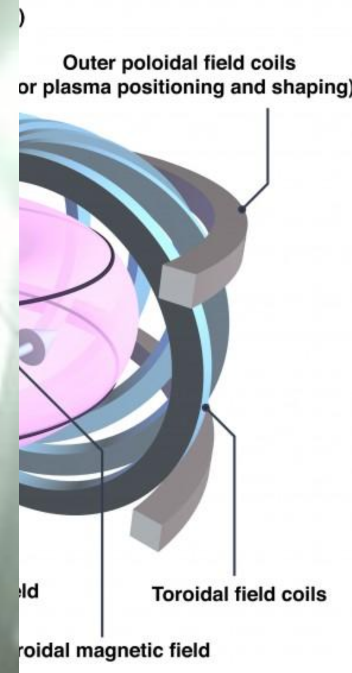




https://www.davidpace.com/keeping-fusion-plasmas-hot/
https://www.euro-fusion.org/2011/09/tokamak-principle-2/

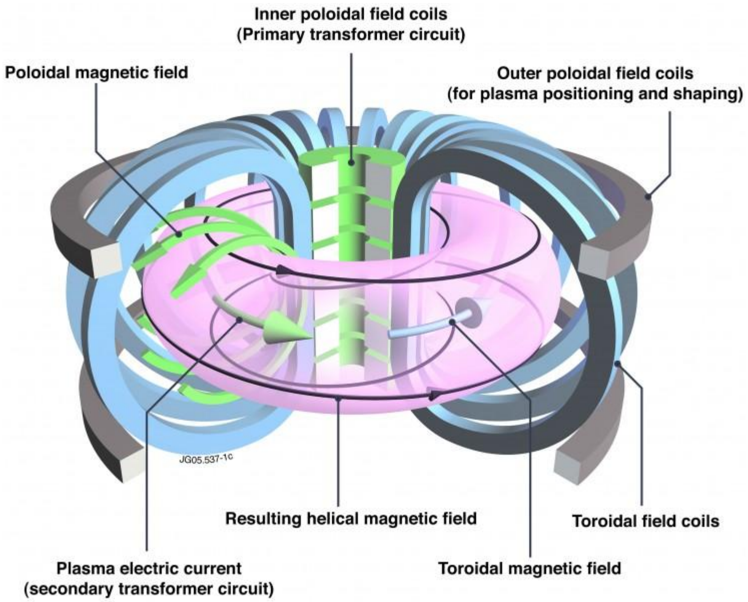
A poloidal magnetic field is required to reduce the drift across field lines



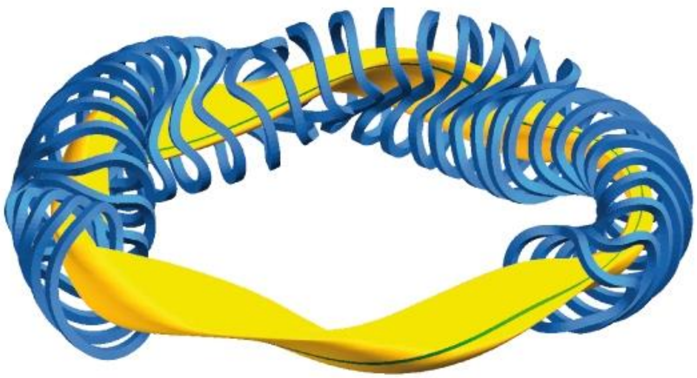


Stellarator uses twisted coil to generate poloidal magnetic field

Tokamak



Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Magnetohydrodynamics description of plasma

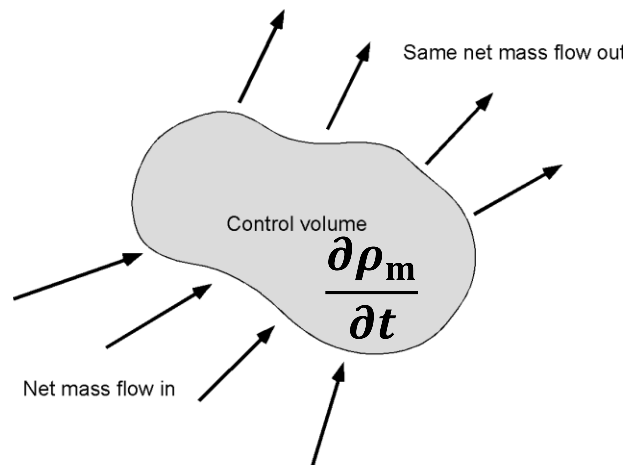
- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \rho_{\mathrm{q}} \vec{E} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{\rho_{\mathrm{m}}{}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

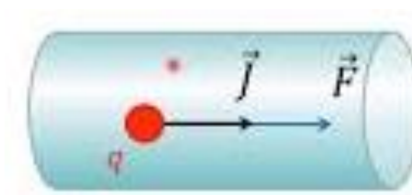
$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{q}}}{\epsilon_o}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j} + \epsilon_{\mathrm{o}} \mu_{\mathrm{o}} \frac{\partial \vec{E}}{\partial t}$$





https://eaglepubs.erau.edu/introductiontoaerospaceflightvehicles/chapter/conservation-of-mass-continuity-equation/
https://www.youtube.com/watch?v=Iu0Ep8_Gp8U

Magnetohydrodynamics (MHD) description of plasma w/ low-freq. and long-wavelength approximation

- **Continuity eq:** $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ **w/ long wavelength ($\lambda >> \lambda_d$)**
- **Momentum eq:** $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \cancel{\rho_q \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$
- **Ohm's law:** $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- **Equation of state:** $\frac{d}{dt}\left(\frac{P}{\rho_m{}^\gamma}\right) = 0$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_o} \approx 0$$

w/ long wavelength ($\lambda >> \lambda_d$) => quasi neutral

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \cancel{\epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega << \omega_{pe}$)

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt}\left(\frac{P}{\rho_m{}^{\gamma}}\right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

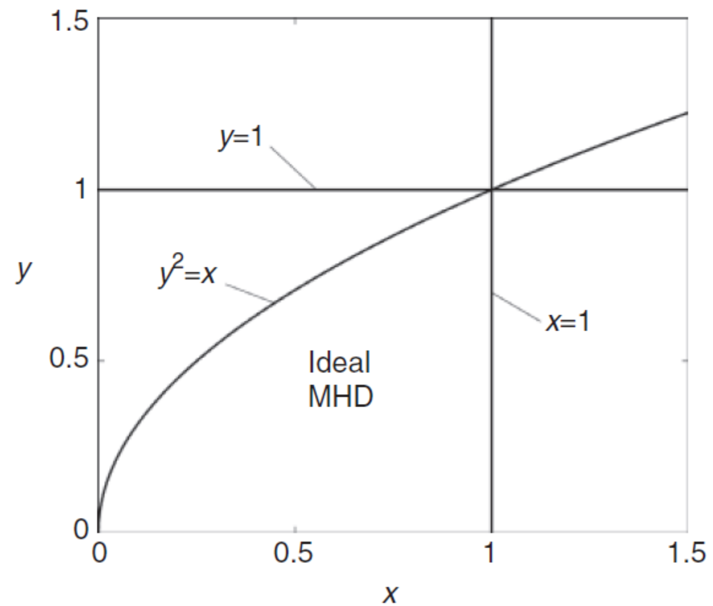
- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Conflict!

$$\omega \sim \frac{\partial}{\partial t} \sim \frac{v_{Ti}}{a} \qquad \omega_{ci} = \frac{v_{Ti}}{r_{Li}} \qquad \frac{\omega}{\omega_{ci}} \sim \frac{v_{Ti}}{a}\frac{r_{Li}}{v_{Ti}} = \frac{r_{Li}}{a} << 1$$

Scale length of nonuniformity.

Region of validity for ideal MHD



Mean free path: $\lambda_i \sim v_{Ti}\tau_{ii}$

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a}$$

$$y = \frac{r_{Li}}{a}$$

- **Requirement:**
  - **High collisionality** $x \ll 1$
  - **Small gyro radius** $y \ll 1$
  - **Small resistivity** $y^2/x \ll 1$

Low resistivity requirement (small η)

$$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \eta\,\vec{j} = \vec{E} + \vec{v} \times \vec{B} \qquad \frac{|\eta j|}{|v \times B|} \sim ?$$

$$|j \times B| \sim |\nabla p| \qquad j \sim \frac{|\nabla p|}{B} \sim \frac{1}{a}\frac{nT}{B} \sim \frac{1}{a}\frac{n m_{\mathrm{i}} {v_{\mathrm{Ti}}}^2}{B} \qquad \omega \sim \frac{\partial}{\partial t} \sim \frac{v_{\mathrm{Ti}}}{a} \qquad \omega_{\mathrm{ci}} = \frac{v_{\mathrm{Ti}}}{r_{\mathrm{Li}}}$$

$$\eta \sim \frac{m_{\mathrm{e}}}{n e^2 \tau_{\mathrm{ei}}} \qquad \tau_{\mathrm{ei}} \sim \tau_{\mathrm{ee}} \sim \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \tau_{\mathrm{ii}} \qquad k \sim \nabla \sim \frac{1}{a} \qquad \omega_{\mathrm{ci}} = \frac{eB}{m_{\mathrm{i}}}$$

$$x = \left(\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\right)^{1/2} \frac{v_{\mathrm{Ti}}\tau_{\mathrm{ii}}}{a} \qquad y = \frac{r_{\mathrm{Li}}}{a}$$

$$\frac{|\eta j|}{|v \times B|} \sim \frac{\eta j}{v_{\mathrm{Ti}} B} \sim \frac{m_{\mathrm{e}}}{n e^2 \tau_{\mathrm{ei}}}\frac{1}{a}\frac{n m_{\mathrm{i}} {v_{\mathrm{Ti}}}^2}{B}\frac{1}{v_{\mathrm{Ti}} B} = \frac{m_{\mathrm{e}}}{\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a}\frac{m_{\mathrm{i}}}{e^2 B^2} = \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a}\frac{{m_{\mathrm{i}}}^2}{e^2 B^2} = \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ei}}}\frac{v_{\mathrm{Ti}}}{a {\omega_{\mathrm{ci}}}^2}$$

$$\sim \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}\tau_{\mathrm{ii}}}\left(\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\right)^{1/2}\frac{v_{\mathrm{Ti}}}{a {\omega_{ci}}^2} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{v_{\mathrm{Ti}}}{\tau_{\mathrm{ii}} a}\frac{{r_{\mathrm{Li}}}^2}{{v_{\mathrm{Ti}}}^2} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{1}{\tau_{\mathrm{ii}} a}\frac{{r_{\mathrm{Li}}}^2}{v_{\mathrm{Ti}}} \sim \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2}\frac{1}{\omega\tau_{\mathrm{ii}}}\left(\frac{r_{\mathrm{Li}}}{a}\right)^2$$

$$= \frac{y^2}{x} \ll 1$$

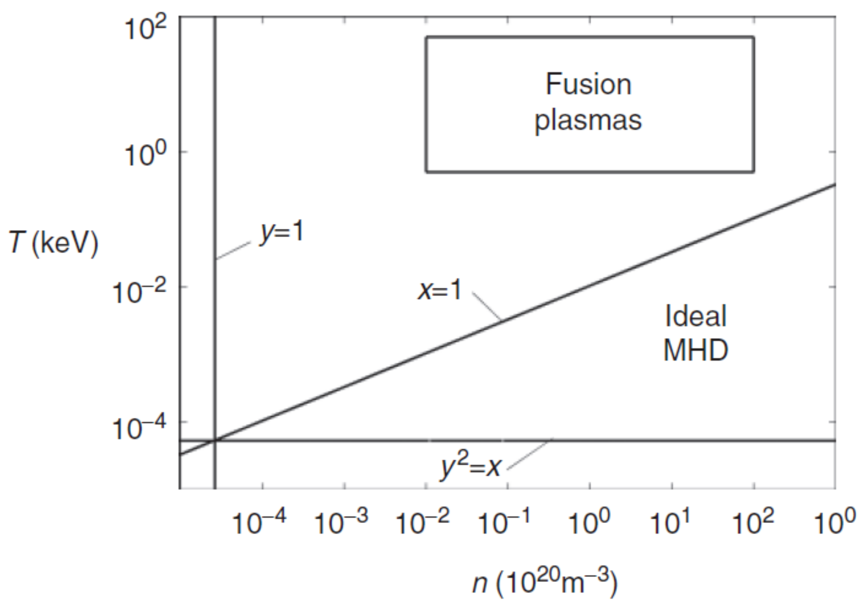
Fusion plasma is not in the ideal MHD region!

$$x = \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{Ti}\tau_{ii}}{a} \qquad y = \frac{r_{Li}}{a}$$

$$10^{18}\ \mathrm{m}^{-3} < n < 10^{22}\ \mathrm{m}^{-3}$$

$$0.5\ \mathrm{keV} < T < 50\ \mathrm{keV}$$

$$\beta \equiv \frac{2\mu_o nT}{B^2}$$



- **Requirement:**
  - **High collisionality** $x = 3 \times 10^3 \frac{T^2}{an} << 1$
  - **Small gyro radius** $y = 2.3 \times 10^{-2} \left(\frac{\beta}{na^2}\right)^{1/2} << 1$
  - **Small resistivity** $\frac{y^2}{x} = 1.8 \times 10^{-7} \frac{\beta}{aT^2} << 1$

- **With strong B, the gyromotion mimic the collisional characteristics.**

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla \cdot \overleftrightarrow{P}$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt} \left(\frac{P}{{\rho_{\mathrm{m}}}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_{\mathrm{o}}\, \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

Additional simplification of the momentum equation

- **Momentum eq:** $\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] = \vec{j}\times\vec{B} - \nabla\cdot\overleftrightarrow{P}$

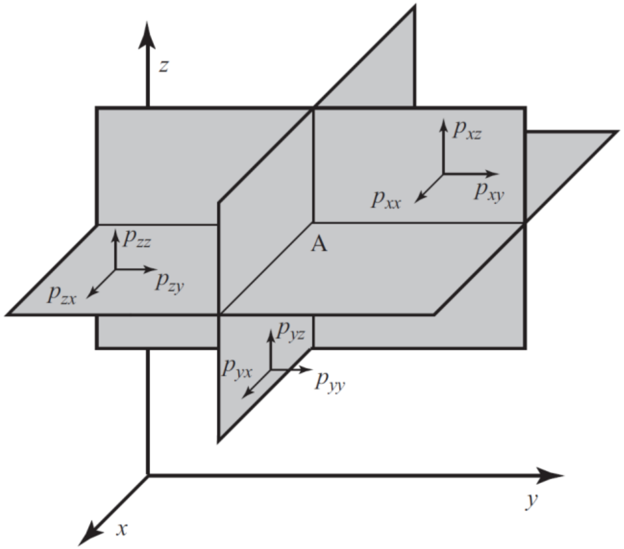
$$\nabla\cdot\overleftrightarrow{P} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix} p_{\mathrm{xx}} & p_{\mathrm{xy}} & p_{\mathrm{xz}} \\ p_{\mathrm{yx}} & p_{\mathrm{yy}} & p_{\mathrm{yz}} \\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & p_{\mathrm{zz}} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{\mathrm{xx}}}{\partial x} + \frac{\partial p_{\mathrm{yx}}}{\partial y} + \frac{\partial p_{\mathrm{zx}}}{\partial z} \\ \frac{\partial p_{\mathrm{xy}}}{\partial x} + \frac{\partial p_{\mathrm{yy}}}{\partial y} + \frac{\partial p_{\mathrm{zy}}}{\partial z} \\ \frac{\partial p_{\mathrm{xz}}}{\partial x} + \frac{\partial p_{\mathrm{yz}}}{\partial y} + \frac{\partial p_{\mathrm{zz}}}{\partial z} \end{pmatrix}$$

$$\nabla\cdot\overleftrightarrow{P} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix} p_{\mathrm{xx}} & 0 & 0 \\ 0 & p_{\mathrm{yy}} & 0 \\ 0 & 0 & p_{\mathrm{zz}} \end{pmatrix} + \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix} 0 & p_{\mathrm{xy}} & p_{\mathrm{xz}} \\ p_{\mathrm{yx}} & 0 & p_{\mathrm{yz}} \\ p_{\mathrm{zx}} & p_{\mathrm{zy}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial p_{\mathrm{xx}}}{\partial x} \\ \frac{\partial p_{\mathrm{yy}}}{\partial y} \\ \frac{\partial p_{\mathrm{zz}}}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial p_{\mathrm{yx}}}{\partial y} + \frac{\partial p_{\mathrm{zx}}}{\partial z} \\ \frac{\partial p_{\mathrm{xy}}}{\partial x} + \frac{\partial p_{\mathrm{zy}}}{\partial z} \\ \frac{\partial p_{\mathrm{xz}}}{\partial x} + \frac{\partial p_{\mathrm{yz}}}{\partial y} \end{pmatrix}$$

Additional simplification of the momentum equation

$$\nabla \cdot \overleftrightarrow{P} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 0 & p_{xy} & p_{xz} \\ p_{yx} & 0 & p_{yz} \\ p_{zx} & p_{zy} & 0 \end{pmatrix}$$



$$= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{zy}}{\partial z} \\ \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} \end{pmatrix}}_{\text{Viscosity } \nabla \cdot \overleftrightarrow{\Pi}}$$

- **Isotropic plasma:** $p_{xx} = p_{yy} = p_{zz} \equiv p$

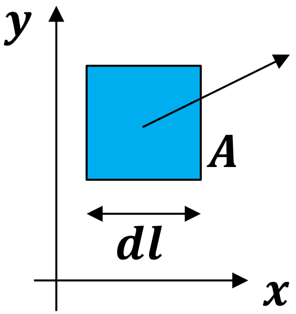
$$\begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix} \equiv p\overleftrightarrow{1}$$

$$\nabla \cdot \overleftrightarrow{P} = \nabla p + \nabla \cdot \overleftrightarrow{\Pi}$$

$$\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

Viscosity is negligible in a collision-dominated plasma

- Y component of momentum transfer through the surface A.



$$\pi_{\mathrm{xy}}^{\mathrm{ii}} \sim (m v_{\mathrm{y}} n) \frac{d v_{\mathrm{x}}}{dx} dl \sim \mu \frac{d v_x}{dx}$$

$$\mu \sim mnvdl \sim mnv(v\tau_{\mathrm{ii}}) \sim mn\left(\sqrt{\frac{T_{\mathrm{i}}}{m}}\right)^2 \tau_{\mathrm{ii}} \sim nT_{\mathrm{i}}\tau_{\mathrm{ii}}$$

$$\overleftrightarrow{\Pi} \sim \mu\left(2\nabla_{\parallel} \cdot \vec{v}_{\parallel} - \frac{2}{3}\nabla \cdot \vec{v}\right) \sim \mu \frac{v_{\mathrm{Ti}}}{a}$$

$$\left|\frac{\nabla \cdot \overleftrightarrow{\Pi}}{\nabla p}\right| \sim \frac{\overleftrightarrow{\Pi} a}{ap} \sim \frac{nT_{\mathrm{i}}\tau_{\mathrm{ii}}v_{\mathrm{Ti}}}{ap} \sim \frac{\tau_{\mathrm{ii}}v_{\mathrm{Ti}}}{a} \sim \frac{\lambda_i}{a} \ll 1 \qquad \lambda_i \sim v_{\mathrm{Ti}}\tau_{\mathrm{ii}}$$

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi}$$

⇩

$$\rho_{\mathrm{m}}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \vec{j} \times \vec{B} - \nabla p$$

Ideal MHD

- **Continuity eq:** $$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$$
- **Momentum eq:** $$\rho_{\mathrm{m}} \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \vec{j} \times \vec{B} - \nabla p$$
- **Ohm's law:** $$\vec{E} + \vec{v} \times \vec{B} \approx 0$$
- **Equation of state:** $$\frac{d}{dt} \left(\frac{P}{\rho_{\mathrm{m}}{}^{\gamma}} \right) = 0$$
- **Maxwell's eqs:**

$$\nabla \cdot \vec{E} \approx 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

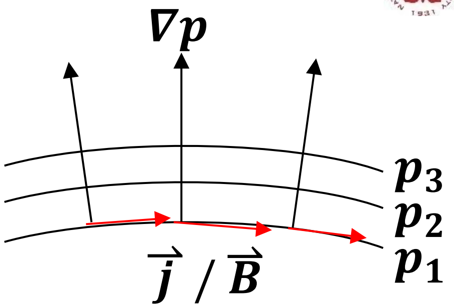
$$\nabla \times \vec{B} = \mu_{\mathrm{o}} \vec{j}$$

$$\nabla \cdot \vec{j} = 0$$

- **Requirement:**
  - **High collisionality – fluid model**
  - **Small gyro radius – low frequency**
  - **Small resistivity – a perfect conductor**

When forces are balances, the system is in the equilibrium state, or called "Magnetohydrostatics"

- **Equilibrium state:**



$$\rho_m\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right]=\vec{j}\times\vec{B}-\nabla p\equiv 0$$

$$\vec{j}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_o\,\vec{j}$$

$$\vec{j}\times\vec{B}=\frac{1}{\mu_o}(\vec{\nabla}\times\vec{B})\times\vec{B}=\frac{1}{\mu_o}\left[(\vec{B}\cdot\vec{\nabla})\,\vec{B}-\frac{1}{2}\,\vec{\nabla}\,B^2\right]=\nabla p$$

$$\nabla\left(P+\frac{B^2}{2\mu_o}\right)=\frac{1}{\mu_o}(\vec{B}\cdot\vec{\nabla})\,\vec{B}$$

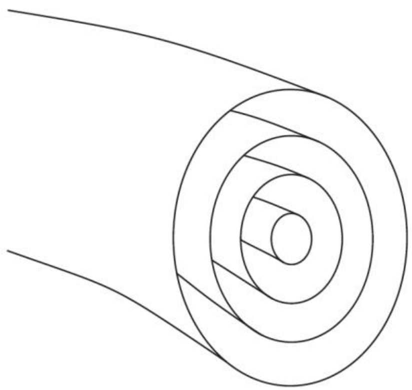
Magnetic pressure | **Magnetic tension** ← **Forces caused by curvature of the field lines**

$\vec{j}\perp\nabla p$ $\quad\vec{B}\perp\nabla p$ ⇒ $\vec{j}\cdot\nabla p=0$ $\quad\vec{B}\cdot\nabla p=0$

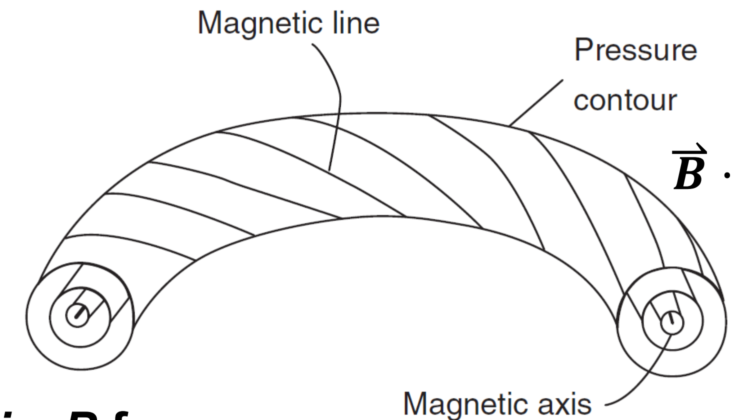
- **The surfaces with p = constant are both magnetic surfaces (i.e., they are made up of magnetic field lines) and current surfaces (i.e., they are made of current flow lines).**

Magnetic lines lying on pressure contour

- **Contours of constant pressure**



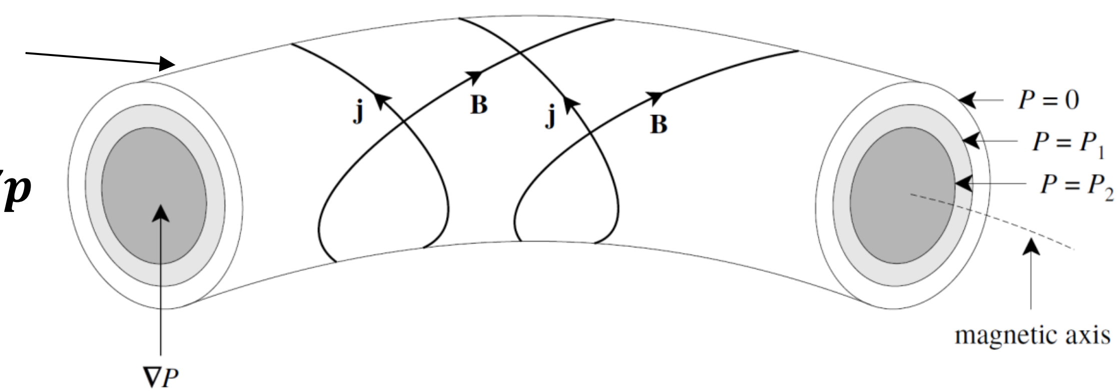
- **Magnetic lines lying on pressure contour**



$$\vec{B} \cdot \nabla p = 0$$

- **Pressure gradient is balanced by the *j* x *B* force**

$$\vec{j} \times \vec{B} = \nabla p$$



$$\vec{j} \cdot \nabla p = 0$$

$$\vec{B} \cdot \nabla p = 0$$

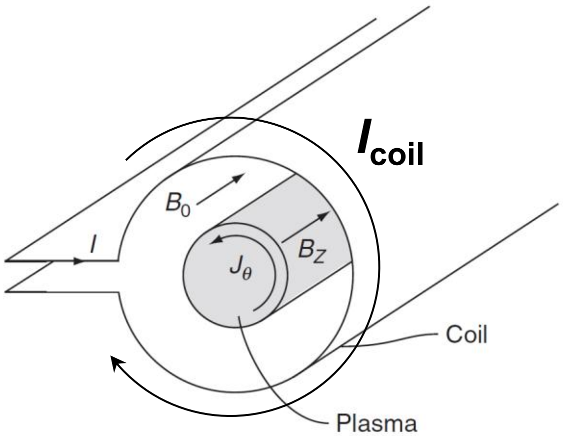
$$\vec{B} \cdot \nabla \Psi = 0$$

- **A magnetic (or flux) surface is one that is everywhere tangential to the field, i.e., the normal to the surface is everywhere perpendicular to B.**

Course Outline

- Magnetic confinement fusion (MCF)
  - Gyro motion, MHD
  - 1D equilibrium (z pinch, theta pinch)
  - Drift: ExB drift, grad B drift, and curvature B drift
  - Tokamak, Stellarator (toroidal field, poloidal field)
  - Magnetic flux surface
  - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
  - Stability (Kink instability, sausage instability, Safety factor Q)
  - Central-solenoid (CS) start-up (discharge) and current drive
  - CS-free current drive: electron cyclotron current drive, bootstrap current.
  - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Theta pinch – current in the azimuthal direction



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_z \hat{z}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r}$$

$$\nabla\left(P + \frac{B^2}{2\mu_o}\right) = \frac{1}{\mu_o}\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} = 0$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

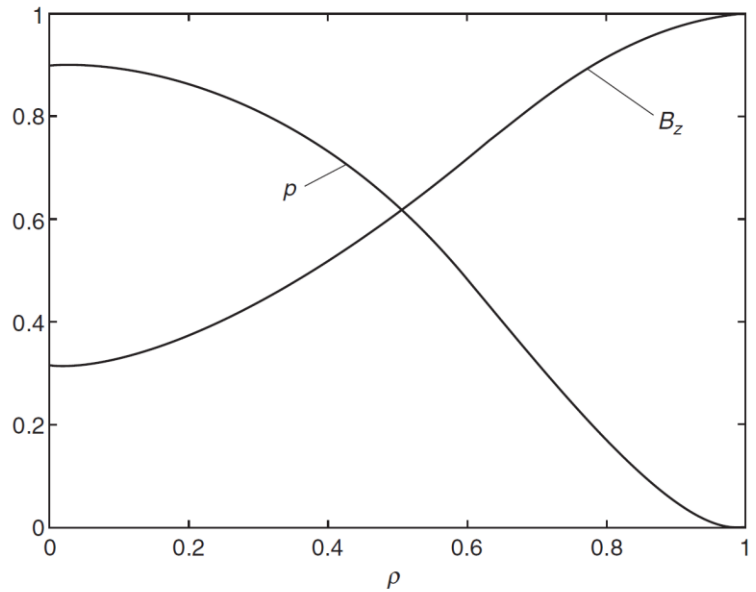
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z = \frac{dp}{dr}$$

Theta pinch is an excellent option for producing radial pressure balance in a fusion plasma

- **Example:**

$$\frac{2\mu_o p(r)}{B_o{}^2} = 1 - \left[1 - \hat{\beta}\left(1-\rho^2\right)^2\right]^2$$

$$\frac{B_z(r)}{B_o} = 1 - \hat{\beta}\left(1-\rho^2\right)^2$$



$$j_\theta B_z = \frac{dp}{dr} \;\Rightarrow\; \frac{a\mu_o j_\theta(r)}{B_o} = -4\hat{\beta}\rho\left(1-\rho^2\right)$$

$$\hat{\beta} = \frac{\beta_o}{1+\sqrt{(1-\beta_o)}} \qquad \beta_o = \frac{2\mu_o p_o}{B_o{}^2} \qquad \rho = \frac{r}{a}$$

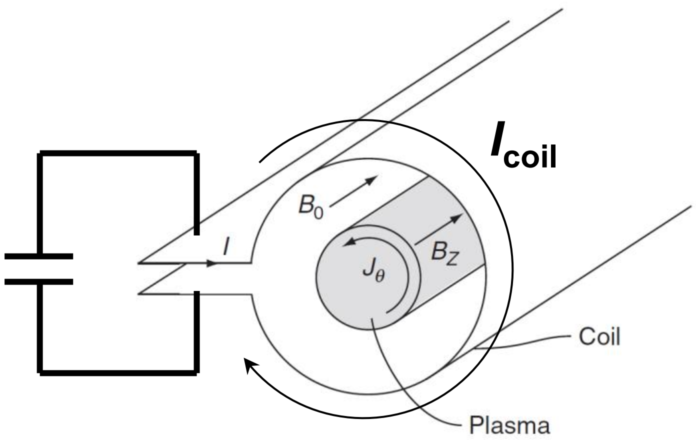
$$\beta \equiv \beta_t = \frac{2\mu_o\langle p\rangle}{B_o{}^2} = \frac{4\mu_o}{a^2 B_o{}^2}\int_0^a p r dr = 2\int_0^1\left(1-\frac{B_z{}^2}{B_o{}^2}\right)\rho d\rho = \hat{\beta}\left(\frac{2}{3}-\frac{\hat{\beta}}{5}\right)$$

$$\beta_o \to 0 \quad\Rightarrow\quad \hat{\beta} \approx \frac{\beta_o}{2}, \beta \approx \frac{\beta_o}{3}$$

$$\beta_o \to 1 \quad\Rightarrow\quad \hat{\beta} \to 1, \beta \approx \frac{7}{15}$$

$$0 < \beta < 1$$

Theta pinches provide good radial confinement but NOT axially

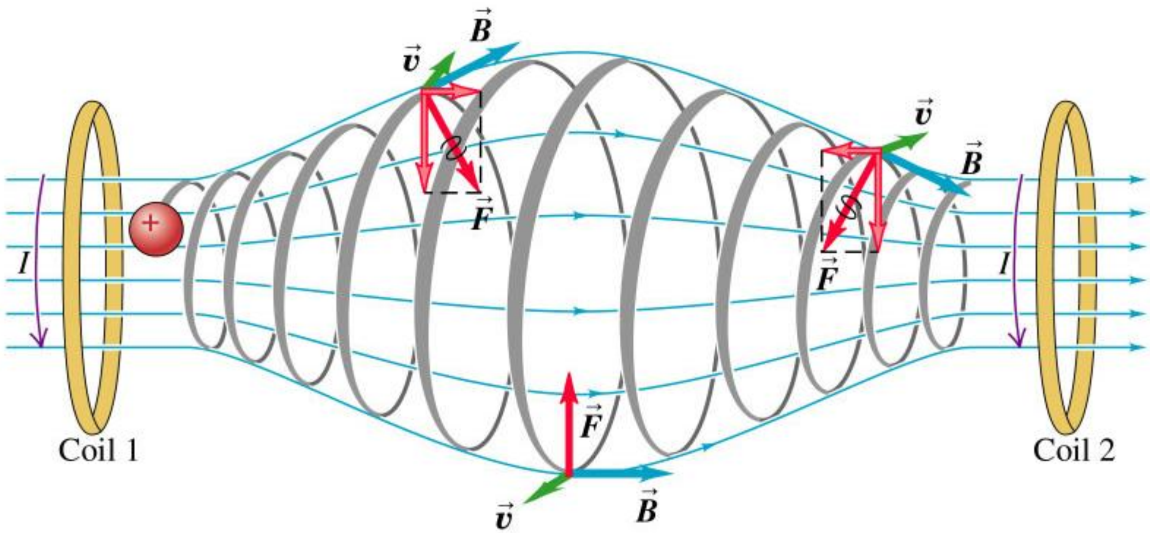


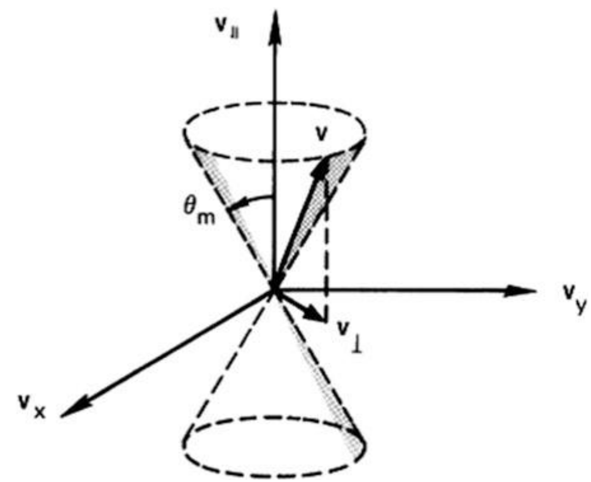
- The gas is initially preionized.
- The coil current is provided by a capacitor bank. The typical pulse length is 10-50 us.
- The rapidly rising magnetic field acts like a piston, imparting a large impulse of momentum and energy to the particles as they are reflected.
- This energy is ultimately converted to heat after repeated reflections off the converging piston.
- T_i ~ 1-4 keV, n ~ 1-2 x 10^{22} m^{-3}, βo ~ 0.7-0.9, β ~ 0.05.
- The plasma simply flowed out the end of the device along field lines in a characteristic time $\tau = L/V_{Ti} \sim 10\mu s$ for L = 5 m.

Main issue: end loss.

Charged particles can be partially confined by a magnetic mirror machine

- Charged particles with small $v_{||}$ eventually stop and are reflected while those with large $v_{||}$ escape.





$$v'_{\perp}{}^2 = v_{\perp o}{}^2 + v_{||o}{}^2 \equiv v_o{}^2$$

$$\frac{B_o}{B'} = \frac{v_{\perp o}{}^2}{v'_{\perp}{}^2} = \frac{v_{\perp o}{}^2}{v_o{}^2} \equiv \sin^2\theta$$

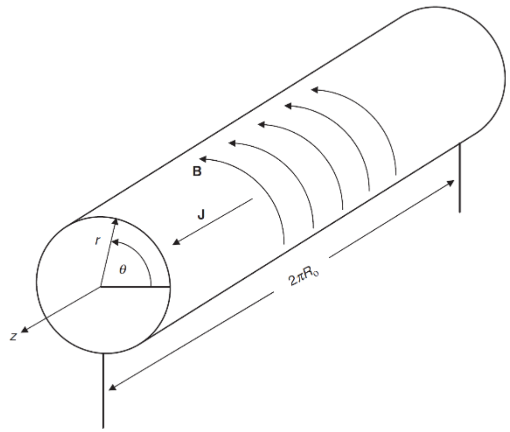
$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{||}{}^2 + \frac{1}{2}mv_{\perp}{}^2 \qquad \text{Invarient: } \mu \equiv \frac{1}{2}\frac{mv_{\perp}{}^2}{B}$$

$$\frac{B_o}{B_m} \equiv \frac{1}{R_m} = \sin^2\theta_m$$

- Large $v_{||}$ may occur from collisions between particles.
- Those confined charged particle are eventually lost due to collisions.

https://i.stack.imgur.com/GlzGZ.jpg

Z pinch – current in the axial direction. The radial confinement of the plasma is provided by the tension force



- **Symmetry:** $\partial_\theta = \partial_z = 0$

$$\vec{B} = B_\theta \hat{\theta}$$

- **All quantities are only functions of the radius *r*.**

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

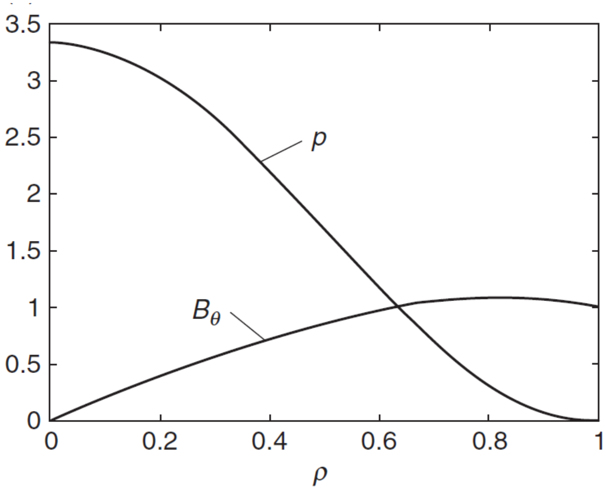
$$\vec{j} \times \vec{B} = \nabla p \qquad j_z B_\theta = -\frac{dp}{dr}$$

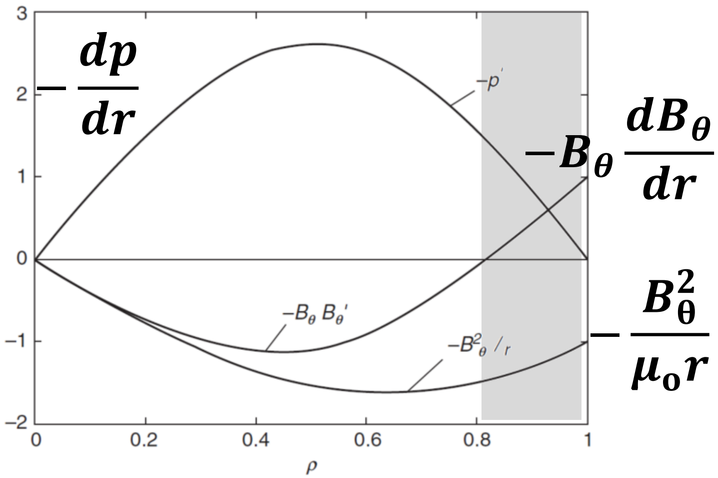
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Magnetic pressure **Magnetic tension**

Z pinch – there is no flexibility in achieving small to moderate β





$$\frac{d}{dr}\left(p+\frac{B_\theta^2}{2\mu_o}\right)+\frac{B_\theta^2}{\mu_o r}=0$$

- **Example:**

$$\frac{2\mu_o p(r)}{{B_{\theta a}}^2}=\frac{2}{3}(5-2\rho^2)(1-\rho^2)^2$$

$$\frac{B_\theta(r)}{B_{\theta a}}=2\rho\left(1-\frac{\rho^2}{2}\right)$$

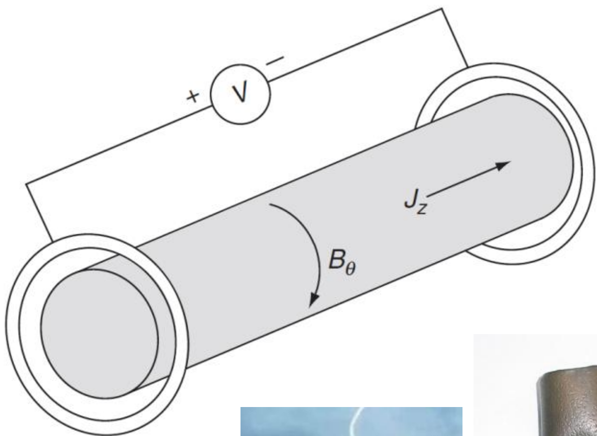
$$j_z=\frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) \Rightarrow \frac{a\mu_o j_z(r)}{B_{\theta a}}=4(1-\rho^2)$$

$$B_{\theta a}\equiv B_\theta(a)=\frac{\mu_o I}{2\pi a}$$

$$\beta\equiv\beta_p=\frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2}=\frac{4\mu_o}{a^2{B_{\theta a}}^2}\int_0^a prdr=1$$

Bennett pinch relation: $\beta=1$

Huge instabilities occur in a z pinch

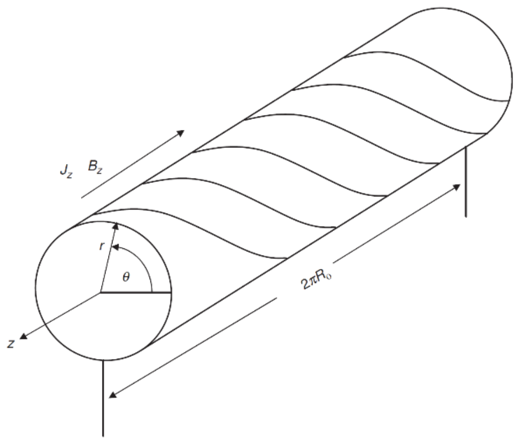


- A capacitor bank is discharged across two electrodes located at each end of a cylindrical quartz or Pyrex tube.
- The gas is ionized by the high voltage and produces a z current flowing along the plasma.
- Disastrous instabilities occurs often leading to a complete quenching of the plasma after 1-2 us.

Main issue: unstable.

Facebook@鄭明典 (https://www.facebook.com/share/p/18N8x8kFmu/)
https://en.wikipedia.org/wiki/Pinch_%28plasma_physics%29

General screw pinch – linear superposition of the theta pinch and the z pinch



- **Nonzero field:** $\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$

$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{j}$$

$$\left(\nabla \times \vec{B}\right)_r = \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\left(\nabla \times \vec{B}\right)_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial B_z}{\partial r}$$

$$\left(\nabla \times \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$$

$$j_\theta = -\frac{1}{\mu_o}\frac{\partial B_z}{\partial r} \qquad j_z = \frac{1}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta)$$

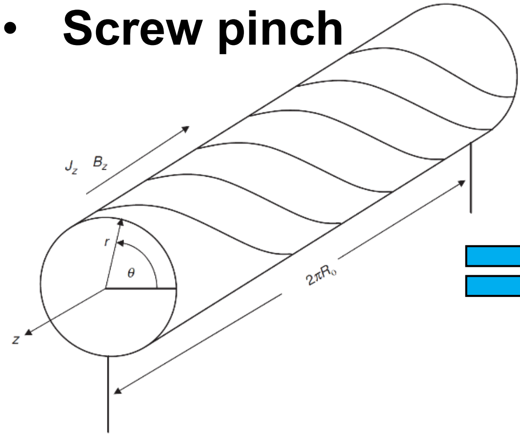
$$\vec{j} \times \vec{B} = \nabla p \qquad j_\theta B_z - j_z B_\theta = -\frac{dp}{dr}$$

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) = -\frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

General screw pinch is flexible with varies range of β

- **Screw pinch**



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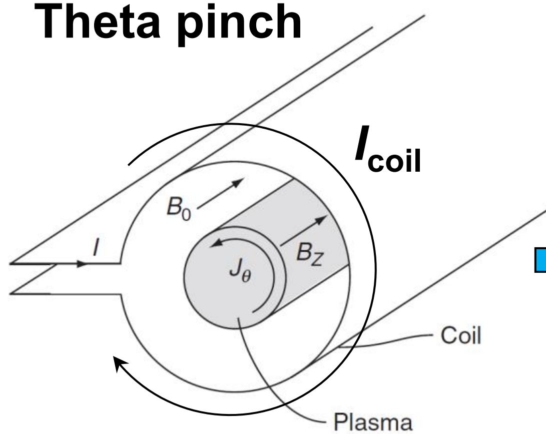
- **Theta pinch**



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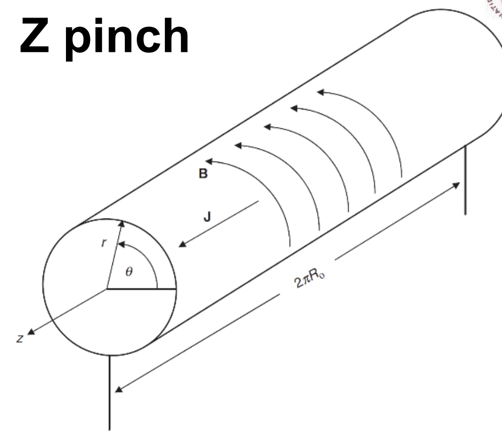
- **Z pinch**



Screw pinch:

$$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$$

$$\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

Theta pinch:

$$\vec{B} = B_z\hat{z}$$

$$\vec{j} = j_\theta\hat{\theta}$$

$$P + \frac{B_z^2}{2\mu_o} = \frac{B_o^2}{2\mu_o}$$

Z pinch:

$$\vec{B} = B_\theta\hat{\theta}$$

$$\vec{j} = j_z\hat{z}$$

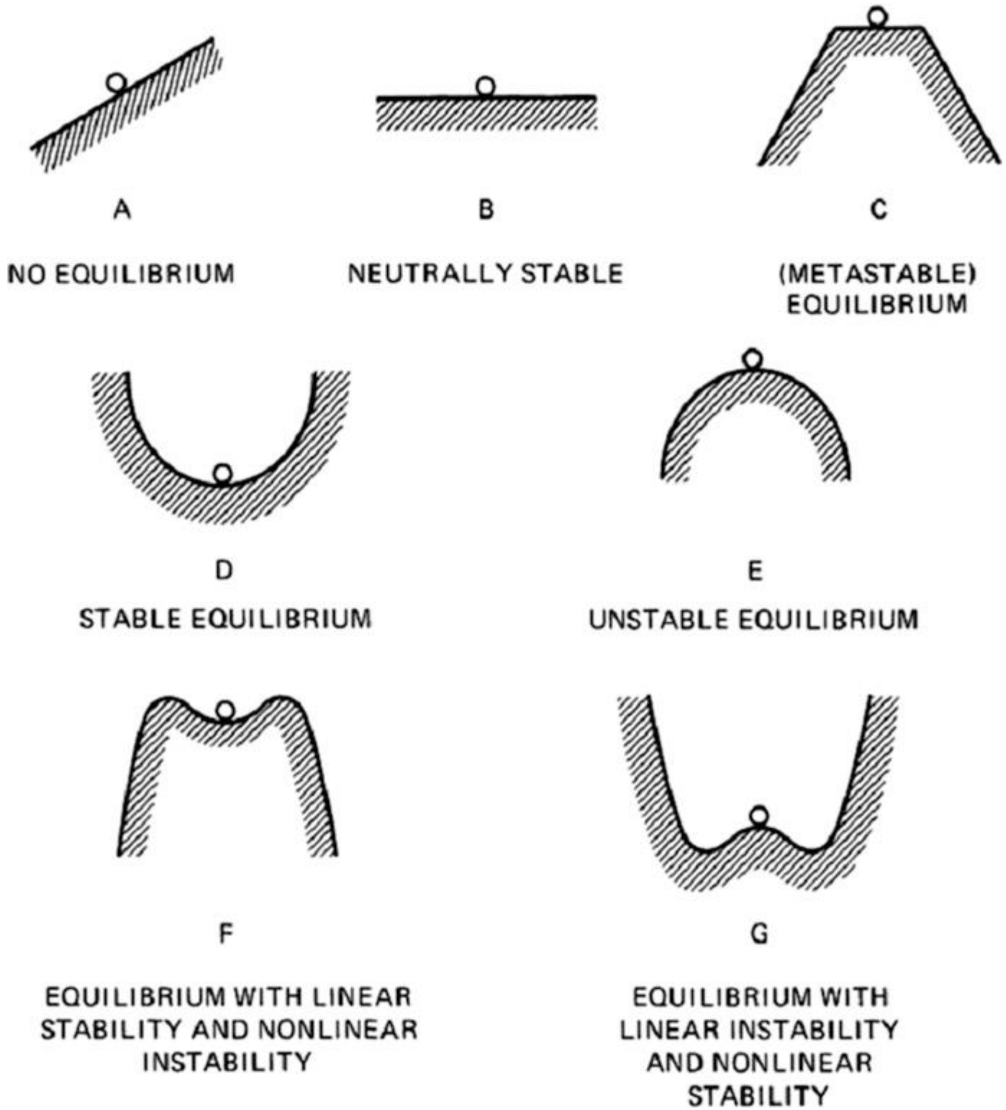
$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r} = 0$$

$$\int_0^a \pi r^2 dr\left[\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_o}\right) + \frac{B_\theta^2}{\mu_o r}\right] = 0 \qquad \langle p\rangle = \frac{B_{\theta a}^2}{2\mu_o} + \frac{1}{2\mu_o}\left(B_o^2 - \langle B_z^2\rangle\right)$$

$$\beta_t = \frac{2\mu_o\langle p\rangle}{{B_o}^2} \qquad \beta_p = \frac{2\mu_o\langle p\rangle}{{B_{\theta a}}^2} \qquad \beta = \frac{\beta_t\beta_p}{\beta_t + \beta_p} = \frac{2\mu_o\langle p\rangle}{B_o^2 + {B_{\theta a}}^2}$$

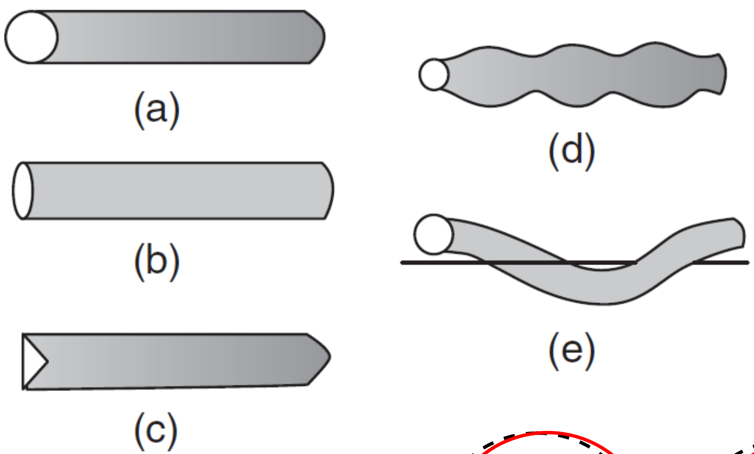
$$0 \le \langle\beta\rangle \le 1$$

An equilibrium state may not be stable

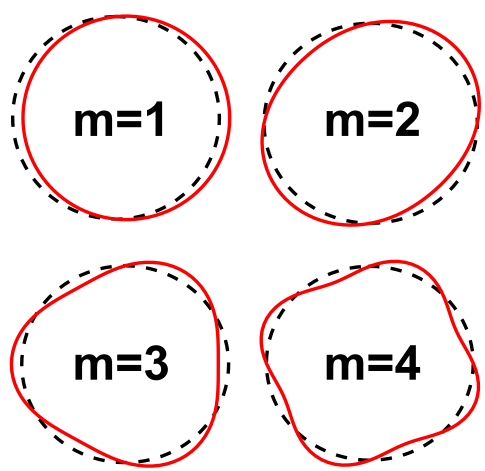


A cylindrical plasma column may not be stable

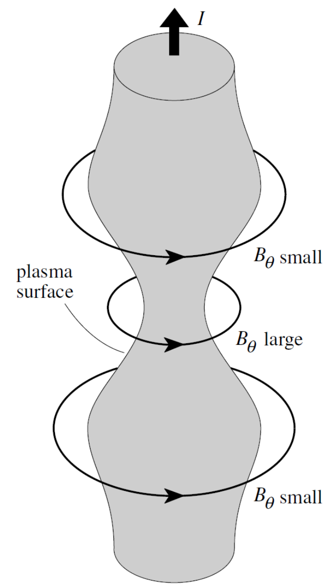
- **Instabilities of theta pinch**



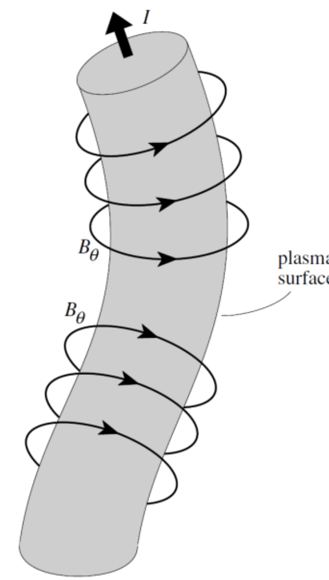
(a) Unperturbed
(b) m=2, k=0
(c) m=3, k=0
(d) m=0, k≠0
(e) m=1, k≠0



- **Instabilities of z pinch**



Sausage instability (m=0)

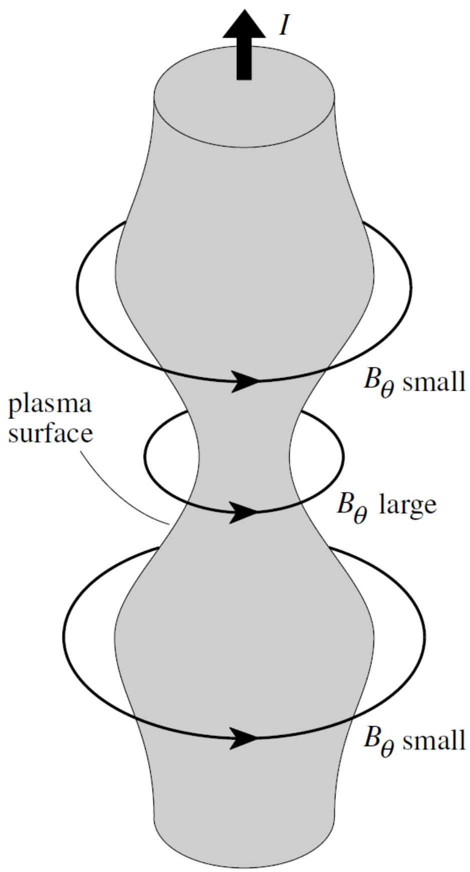


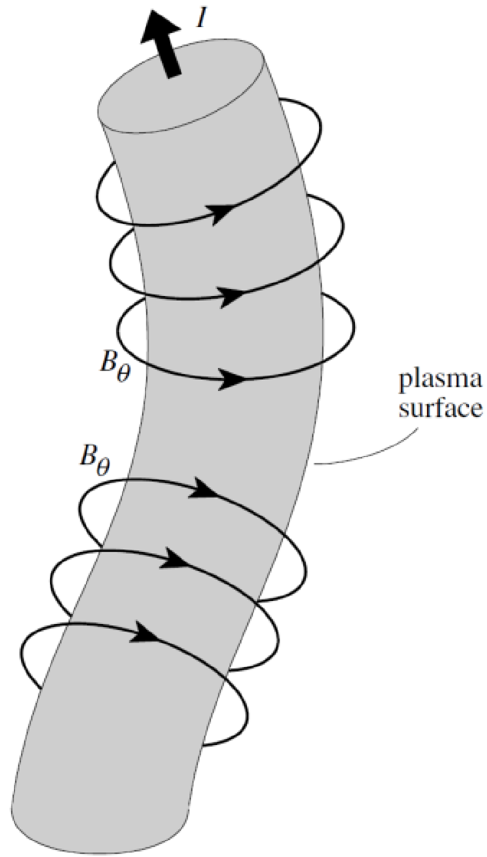
Kink instability (m=1)

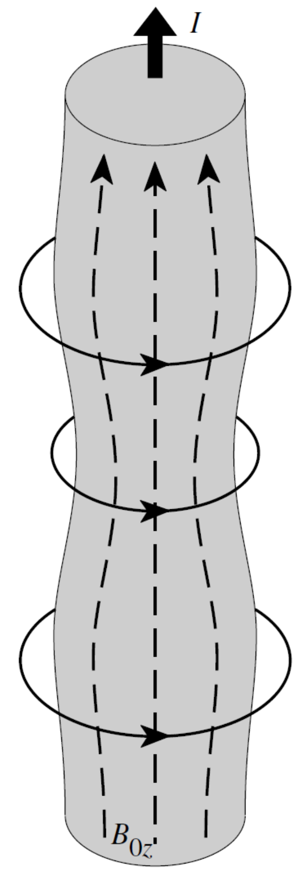
$$\zeta(\vec{r}) = \zeta(r) exp(im\theta + ikz)$$

A cylindrical plasma column is stable when the safety factor is greater than unity

- **Sausage instability (m=0)**
- **Kink instability**



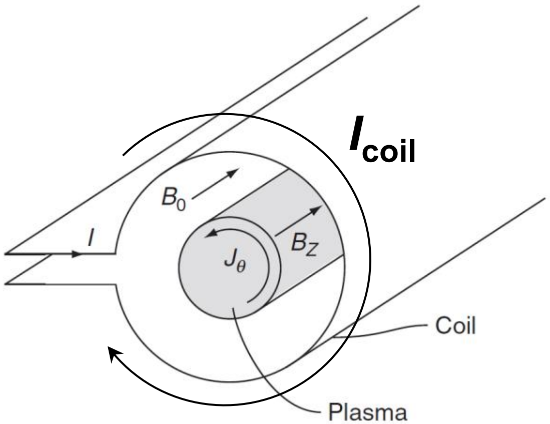


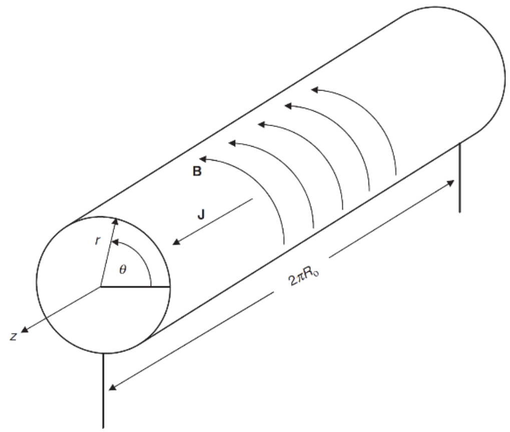


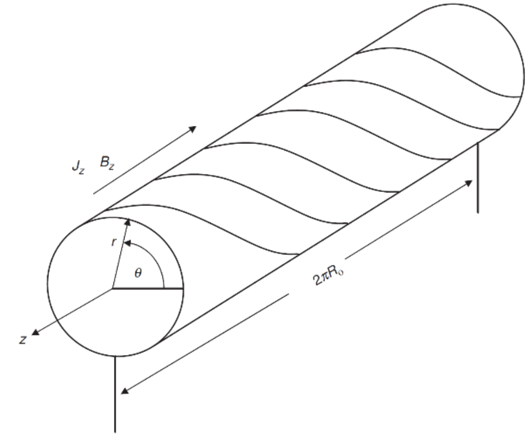
- **MHD Safety factor:** $q(r) = \dfrac{rB_z(r)}{R_o B_\theta(r)}$ **Kruskal–Shafranov limit**

Theta pinch is stable while z pinch is unstable

- **Theta pinch**
- **Z pinch**
- **Screw pinch**







Theta pinch	Z pinch	Screw pinch
$\vec{B} = B_z\hat{z}$	$\vec{B} = B_\theta\hat{\theta}$	$\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$ $\vec{j} = j_\theta\hat{\theta} + j_z\hat{z}$
$q_\theta = \infty$	$q_z = 0$	q can be controlled.
Stable	**Unstable**	**Stable/Unstable**

$$q(r) = \frac{rB_z(r)}{R_o B_\theta(r)}$$