Introduction to Nuclear Fusion as An Energy Source

Po-Yu Chang

Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 3

2024 spring semester

Wednesday 9:10-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=ma76b50f97b1c6d72db61de9eaa9f0b27

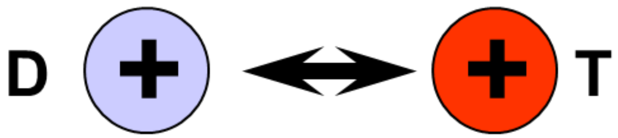
2024/3/21 updated

Course Outline

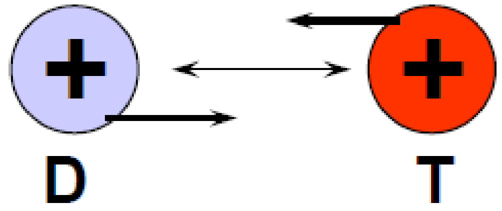
- Brief background reviews
  - Electromagnetics
  - Plasma physics
- Introduction to nuclear fusion
  - Nuclear binding energy (Fission vs Fusion)
  - Fusion reaction physics
  - Some important fusion reactions (Cross section)
    - Main controlled fusion fuels
    - Advanced fusion fuels
  - Maxwell-averaged fusion reactivities

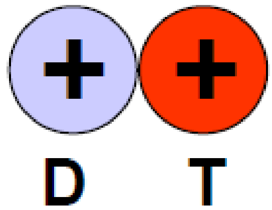
A “hot plasma” at 100M ⁰C is needed

- Probability for fusion reactions to occur is low at low temperatures due to the coulomb repulsion force.



- If the ions are sufficiently hot, i.e., large random velocity, they can collide by overcoming coulomb repulsion

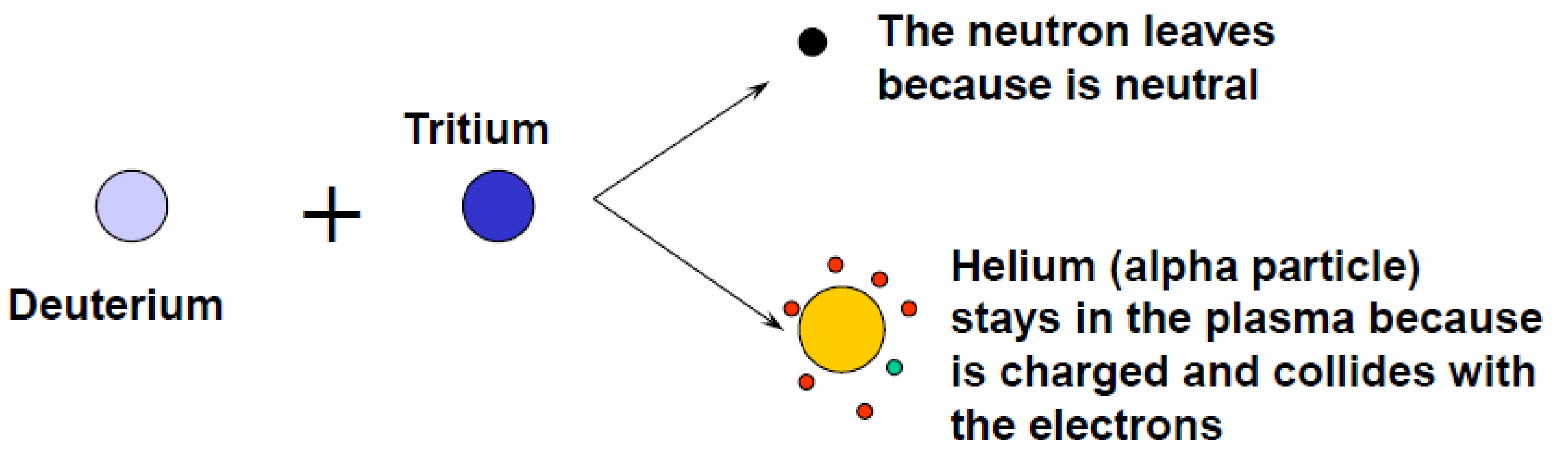




*R. Betti, HEDSA HEDP Summer School, 2015

It takes a lot of energy or power to keep the plasma at 100M ^{0}C

- **Let the plasma do it itself!**



- **The α-particles heat the plasma.**

*R. Betti, HEDSA HEDP Summer School, 2015

Under what conditions the plasma keeps itself hot?

- **Steady state 0-D power balance:**

  $S_\alpha + S_h = S_B + S_k$

  S_α: α particle heating

  S_h: external heating

  S_B: Bremsstrahlung radiation

  S_k: heat conduction lost

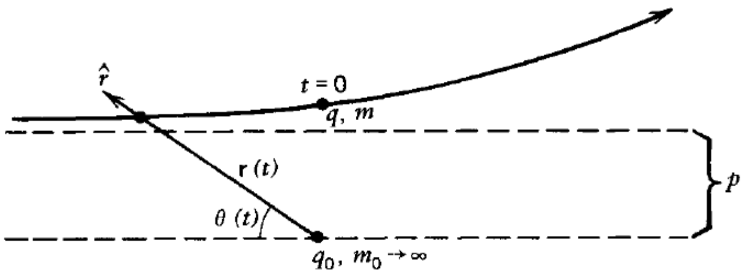
Ignition condition: $P\tau$ > 10 atm-s = 10 Gbar - ns

- **P: pressure, or called energy density**
- **τ is confinement time**

Course Outline

- **Introduction to nuclear fusion (cont.)**
  - **Collisions (Bremsstrahlung radiation)**
  - **Columb scattering. Cross section of the Columb scattering**
  - **Beam-target fusion vs thermonuclear fusion**
  - **Lawson criteria, ignition conditions**
  - **Magnetic confinement fusion (MCF) vs Inertial confinement fusion (ICF)**

Charged particles collide with each other through coulomb collisions



$$\mathrm{m}v_\perp = \int_{-\infty}^{\infty} dt\, \mathrm{F}_\perp(t)$$

- Coulomb force:

$$m\ddot{\vec{r}} = \frac{qq_0}{r^2}\hat{r}$$

$$F_\perp = \frac{qq_0}{p^2}\sin^3\theta$$

- Relation between θ and t is

$$x = -r\cos\theta = -\frac{p\cos\theta}{\sin\theta} = v_0 t$$

- Therefore,

$$v_\perp = \frac{qq_0}{mv_0 p}\int_0^{\pi} d\theta \sin\theta = \frac{2qq_0}{mv_0 p} \equiv \frac{v_0 p_0}{p}$$

where $p_0 \equiv \frac{2qq_0}{m{v_0}^2}$

- Note that this is valid only when $v_\perp << v_0$, i.e., $p >> p_0$.

Cumulative effect of many small angle collisions is more important than large angle collisions

- **Consider a variable Δx that is the sum of many small random variables Δx_i, i=1,2,3,…,N,**

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_N = \sum_{i=1}^{N} \Delta x_i$$

- **Suppose** $< \Delta x_i > = < \Delta x_i \Delta x_j >_{i \neq j} = 0$

$$\left\langle (\Delta x)^2 \right\rangle = \left\langle \left(\sum_{i=1}^{N} \Delta x_i \right)^2 \right\rangle = \sum_{i=1}^{N} \left\langle (\Delta x_i)^2 \right\rangle = N \left\langle (\Delta x_i)^2 \right\rangle$$

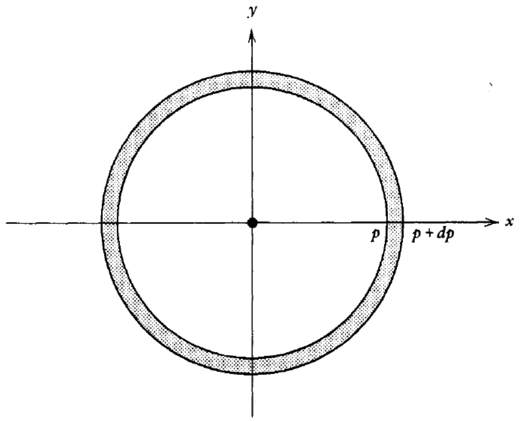
- **For one collision:**

$$\left\langle {v_\perp}^2 \right\rangle = \left\langle (\Delta v_x)^2 \right\rangle + \left\langle \left(\Delta v_y\right)^2 \right\rangle = \frac{{v_0}^2 {p_0}^2}{p^2} \qquad \left\langle (\Delta v_x)^2 \right\rangle = \left\langle \left(\Delta v_y\right)^2 \right\rangle = \frac{1}{2} \frac{{v_0}^2 {p_0}^2}{p^2}$$

- **The total velocity in** $\hat{x}$

$$\left\langle \left(\Delta {v_x}^{tot}\right)^2 \right\rangle = N \left\langle (\Delta v_x)^2 \right\rangle = \frac{N}{2} \frac{{v_0}^2 {p_0}^2}{p^2}$$

The collision frequency can be obtained by integrating all the possible impact parameter



- Number of collisions in a time interval:

$$dN = n_0 2\pi p\, dp\, v_0\, dt$$

i.e., $$\frac{dN}{dt} = 2\pi p\, dp\, n_0 v_0$$

- Therefore

$$\frac{d}{dt}\left\langle \left(\Delta v_x^{\mathrm{tot}}\right)^2 \right\rangle = \frac{1}{2}\frac{v_0^2 p_0^2}{p^2}\frac{dN}{dt}$$

$$= \pi\, n_0 v_0^3 p_0^2 \frac{dp}{p}$$

$$\frac{d}{dt}\left\langle \left(\Delta_\perp^{\mathrm{tot}}\right)^2 \right\rangle = 2\frac{d}{dt}\left\langle \left(\Delta v_x^{\mathrm{tot}}\right)^2 \right\rangle$$

$$= 2\pi n_0 v_0^3 p_0^2 \int_{p_{\min}}^{p_{\max}} \frac{dp}{p}$$

$$= 2\pi n_0 v_0^3 p_0^2 \ln\left(\frac{p_{\max}}{p_{\min}}\right)$$

$$\approx 2\pi n_0 v_0^3 p_0^2 \ln\left(\frac{\lambda_D}{|p_0|}\right)$$

$$\approx 2\pi n_0 v_0^3 p_0^2 \ln\Lambda$$

- Note that

$$\lambda_D \approx \left(\frac{KT_e}{4\pi n_0 e^2}\right)^{1/2}$$

$$\frac{\lambda_D}{|p_0|} \approx \frac{\lambda_D m_e v_e^2}{2e^2} \approx \frac{\lambda_D K T_e}{e^2} \approx 4\pi n_0 \lambda_D^3$$

$$\approx \Lambda$$

Comparison between the mean free path and the system size L determines the regime of the plasma

- **A reasonable definition for the scattering time due to small angle collisions is the time it takes $\left\langle \left(\Delta v_\perp^{\text{tot}}\right)^2 \right\rangle$ to equal v_0^2. The collision frequency ν_c due to small-angle collisions:**

$$\frac{d}{dt}\left\langle \left(\Delta_\perp^{\text{tot}}\right)^2 \right\rangle \approx 2\pi n_0 v_0^3 p_0^2 \ln\Lambda \approx v_0^2 \nu_c, \quad p_0 \equiv \frac{2qq_0}{m_e v_0^2} \Rightarrow \nu_c = \frac{8\pi n_0 e^4 \ln\Lambda}{m_e^2 v_0^3}$$

- **With more careful derivation, the collisional time is obtained:**

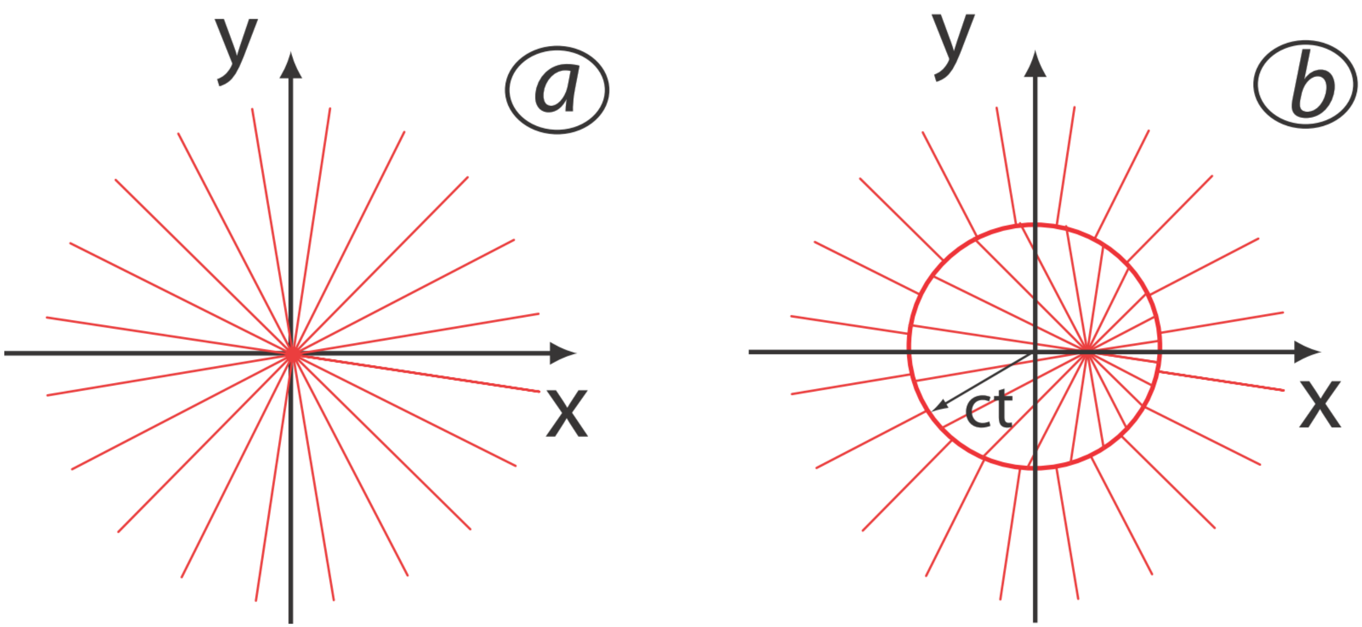
$$\tau_e^{-1} = \nu_c = \frac{4\sqrt{2\pi} n e^4 \ln\Lambda}{3\sqrt{m_e}(KT_e)^{3/2}}$$

- **Mean free path:** $l_{\text{mfp}} = v_e \tau_e$

$$\begin{cases} l_{\text{mfp}} < L & \textbf{Fluid Theory} \\ l_{\text{mfp}} > L & \textbf{Kinetic Theory} \end{cases}$$

Electromagnetic wave is radiated when a charge particle is accelerated

- **The retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.**

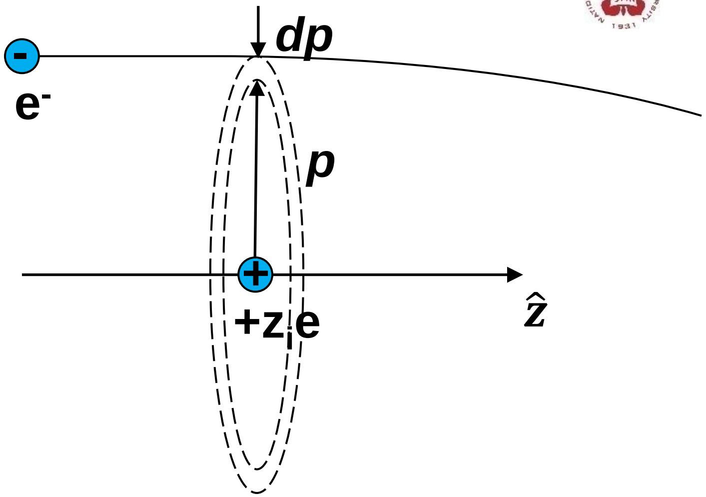


https://jseldredphysics.files.wordpress.com/2018/04/lecture18.pdf

Bremsstrahlung emission

- **When an electron collides with a nucleus via coulomb interaction, the electron is accelerated and thus radiates, called Bremsstrahlung radiation.**
- **In the non-relativistic limit, the Bremsstrahlung radiation power is:**

$$P_{\mathrm{B,e1,i1}} = \frac{e^2}{6\pi\epsilon_o}\frac{\dot{v}^2}{c^3}$$



p: **Impact parameter**

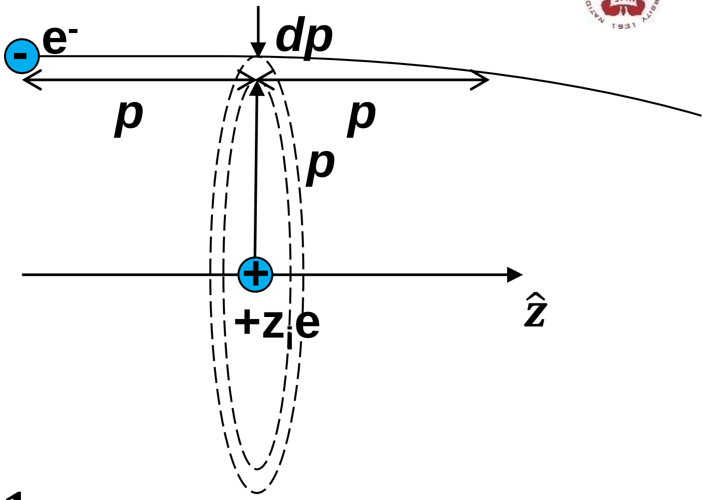
- **The coulomb force experienced by the electron is:**

$$\dot{v} = \frac{F}{m_e} = \frac{ze^2}{4\pi\epsilon_o m_e r^2} = \frac{ze^2}{4\pi\epsilon_o m_e[p^2 + (vt)^2]} \approx \frac{ze^2}{4\pi\epsilon_o m_e p^2}$$

=>

$$P_{\mathrm{B,e1,i1}} = \frac{z^2 e^6}{96\pi^3 {\epsilon_o}^3 c^3 {m_e}^2}\frac{1}{p^4} \quad \mathrm{(W)}$$

Bremsstrahlung emission



- The electron begins to accelerate when it is about a distance *p* from the ion. It continuous to accelerate until it travels a distance *p* away from the ion.

$$\Delta t = \frac{2p}{v}$$

- Therefore, the energy loss by one electron is:

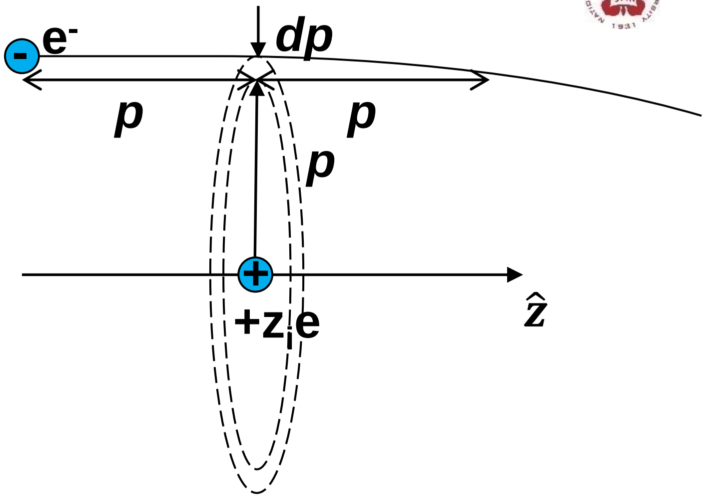
$$E_{\mathrm{B,e1,i1}} \approx P_{\mathrm{B,e1,i1}} \Delta t = \frac{z^2 e^6}{48\pi^3 \epsilon_{\mathrm{o}}{}^3 c^3 m_{\mathrm{e}}{}^2} \frac{1}{v p^3} \quad (\mathrm{J})$$

- With careful integration:

$$E_{\mathrm{B,e1,i1}} = \int_{-\infty}^{\infty} P_{\mathrm{B,e1,i1}}\, \mathrm{d}t = \frac{2 z^2 e^6}{3(4\pi\epsilon_{\mathrm{o}})^3 m_{\mathrm{e}}{}^2 c^3} \int_{-\infty}^{\infty} \frac{1}{[p^2 + (vt)^2]^2}\, dt$$

$$= \frac{\pi z^2 e^6}{3(4\pi\epsilon_{\mathrm{o}})^3 m_{\mathrm{e}}{}^2 c^3} \frac{1}{v p^3}$$

Bremsstrahlung emission



- To consider the electron colliding with all ions with impact parameter *p* from 0 to ∞ and include the distribution function of ions $f_i(v_i)$.

$$\overline{P}_{B,e1} = \int d\vec{v}_{\mathrm{i}} \int_0^{\infty} \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_{\mathrm{e}} - \vec{v}_{\mathrm{i}}| f_{\mathrm{i}}(\vec{v}_{\mathrm{i}}) 2\pi p\, dp$$

- In addition, we need to consider the distribution function of electrons $f_e(v_e)$. The total power loss is:

$$\overline{P}_{B} = \int d\vec{v}_{\mathrm{i}} \int d\vec{v}_{\mathrm{e}} \int_0^{\infty} \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_{\mathrm{e}} - \vec{v}_{\mathrm{i}}| f_{\mathrm{i}}(\vec{v}_{\mathrm{i}}) f_{\mathrm{e}}(\vec{v}_{\mathrm{e}}) 2\pi p\, dp$$

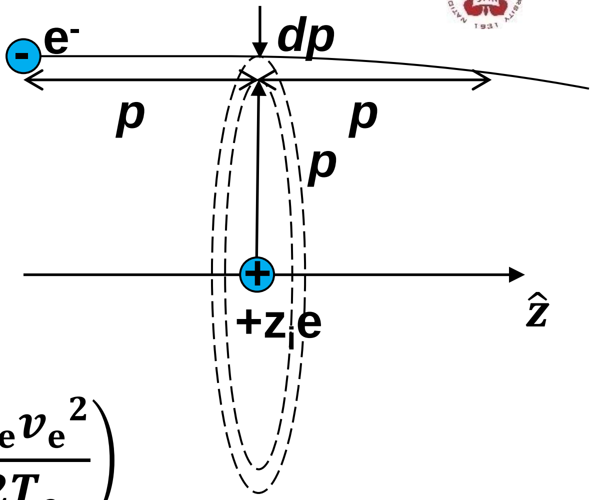
- Since $|v_e| >> |v_i|$, $|\vec{v}_{\mathrm{e}} - \vec{v}_{\mathrm{i}}| \approx v_{\mathrm{e}}$.
- In addition:

$$\int f_i(\vec{v}_i)\, \mathrm{d}\,\vec{v}_i \equiv n_i$$

$$d\,\vec{v}_e = dv_x dv_y dv_z = v_{\mathrm{e}}^2 \sin\theta\, dv d\theta d\phi \rightarrow 4\pi v_{\mathrm{e}}^2 dv_{\mathrm{e}}$$

$$f_{\mathrm{e}} = n_{\mathrm{e}} \left(\frac{m_{\mathrm{e}}}{2\pi \mathrm{T_e}}\right)^{3/2} \exp\left(-\frac{m_{\mathrm{e}} v_{\mathrm{e}}^2}{2T_{\mathrm{e}}}\right)$$

Bremsstrahlung emission



$$\overline{P}_B = \int d\vec{v}_\mathrm{i} \int d\vec{v}_\mathrm{e} \int_0^\infty \overline{E}_{\mathrm{B,e1,i1}} |\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}| f_\mathrm{i}(\vec{v}_\mathrm{i}) f_\mathrm{e}(\vec{v}_\mathrm{e}) 2\pi p \, dp$$

$$= 2\pi \int f_\mathrm{i}(\vec{v}_\mathrm{i}) \, d\vec{v}_i \int 4\pi {v_\mathrm{e}}^2 dv_\mathrm{e} \int_0^\infty p \, dp E_{\mathrm{B,e1,i1}} v_\mathrm{e} f_\mathrm{e}(\vec{v}_\mathrm{e})$$

$$= 8\pi^2 n_\mathrm{i} \int {v_\mathrm{e}}^3 dv_\mathrm{e} \int_0^\infty p \, dp E_{\mathrm{B,e1,i1}} n_\mathrm{e} \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right)$$

$$= 8\pi^2 n_\mathrm{i} n_\mathrm{e} \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \int_0^\infty {v_\mathrm{e}}^3 dv_\mathrm{e} \int_0^\infty p \, dp \left(\frac{z^2 e^6}{48\pi^3 {\epsilon_\mathrm{o}}^3 c^3 {m_\mathrm{e}}^2} \frac{1}{v_\mathrm{e} p^3}\right) \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right)$$

$$= 8\pi^2 n_\mathrm{i} n_\mathrm{e} \left(\frac{z^2 e^6}{48\pi^3 {\epsilon_\mathrm{o}}^3 c^3 {m_\mathrm{e}}^2}\right) \left(\frac{m_\mathrm{e}}{2\pi \mathrm{T_e}}\right)^{3/2} \int_0^\infty {v_\mathrm{e}}^2 \exp\left(-\frac{m_\mathrm{e} {v_\mathrm{e}}^2}{2T_\mathrm{e}}\right) dv_\mathrm{e} \int_0^\infty \frac{dp}{p^2}$$

Bremsstrahlung emission

- **Notice that we are using classical physics. We are not taking account of quantum effects which happen on a length scale of deBroglie wavelength $\Delta x = \hbar/(m_e v)$. Therefore, we have $p_{\min} = \hbar/(m_e v)$.**

$$\int_0^\infty \frac{dp}{p^2} \rightarrow \int_{p_{\min}}^\infty \frac{dp}{p^2} = \frac{1}{p_{min}} = \frac{m_e v_e}{\hbar} = \frac{2\pi m_e v_e}{h}$$

$$\bar{P}_B = 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_o{}^3 c^3 m_e{}^2}\right)\left(\frac{m_e}{2\pi T_e}\right)^{3/2} \int_0^\infty v_e{}^2 \exp\left(-\frac{m_e v_e{}^2}{2T_e}\right) dv_e \int_0^\infty \frac{dp}{p^2}$$

$$= 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_o{}^3 c^3 m_e{}^2}\right)\left(\frac{m_e}{2\pi T_e}\right)^{3/2} \frac{2\pi m_e}{h} \int_0^\infty v_e{}^3 \exp\left(-\frac{m_e v_e{}^2}{2T_e}\right) dv_e$$

- **With** $\int_0^\infty x^3 e^{-x^2}\, dx = \frac{1}{2}$, **a better value**: $\left(\frac{2^{1/2}}{3\pi^{5/2}}\right)$

$$\bar{P}_B = \left(\frac{2^{1/2}}{6\pi^{3/2}}\right)\left(\frac{e^6}{\epsilon_o{}^3 c^3 h m_e{}^{3/2}}\right) z^2 n_i n_e T_e{}^{1/2} \quad \left(\frac{W}{m^3}\right)$$

Bremsstrahlung emission

- **For multiple ion species: n_j, z_j**

$$\bar{P}_B = \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{{\epsilon_o}^3 c^3 h {m_e}^{3/2}}\right) n_e {T_e}^{1/2} \sum_j z_j^2 n_{i,j} \left(\frac{W}{m^3}\right)$$

$$= \left(\frac{2^{1/2}}{3\pi^{5/2}}\right)\left(\frac{e^6}{{\epsilon_o}^3 c^3 h {m_e}^{3/2}}\right) Z_{eff} n_e^2 {T_e}^{1/2} \left(\frac{W}{m^3}\right)$$

where $$Z_{eff} \equiv \frac{\sum_j {z_j}^2 n_j}{n_e} = \frac{\sum_j {z_j}^2 n_j}{\sum_j z_j n_j} \qquad n_e = \sum_j z_j n_j$$

$$\bar{P}_B = 5.35 \times 10^{-37} Z_{eff} n_{e\,(m^{-3})}^2 T_{e\,(keV)}^{1/2} \left(\frac{W}{m^3}\right)$$

$$\bar{P}_B \equiv C_B Z_{eff} n_{e\,(m^{-3})}^2 T_{e\,(keV)}^{1/2} \left(\frac{W}{m^3}\right)$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$S_\alpha + S_h = S_B + S_k$

S_h: external heating

S_α: α particle heating

$$D + T \rightarrow He^4\,(3.5\ \mathrm{MeV}) + n\,(14.1\ \mathrm{MeV})$$

$$S_\mathrm{f} = E_\mathrm{f} n_1 n_2 \langle \sigma v \rangle (\mathrm{W/m^3})$$

$$S_\alpha = \frac{1}{4} E_\alpha n^2 \langle \sigma v \rangle = \frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2}$$

E_α = 3.5 MeV

$$p = p_\mathrm{e} + p_\mathrm{i} = 2p_\mathrm{e} = 2n_e T \equiv 2nT$$

S_B: Bremsstrahlung radiation

$$S_B = C_\mathrm{B} Z_\mathrm{eff} n^2_{\mathrm{e}\,(\mathrm{m^{-3}})} T^{1/2}_{\mathrm{e}\,(\mathrm{keV})} \left(\frac{W}{m^3}\right)$$

$$= \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}}$$

S_k: heat conduction lost

$$S_\kappa = \frac{3}{2} \frac{p}{\tau}$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_\mathrm{B} \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

Ignition condition (Lawson criterion) revision

- **Steady state 0-D power balance:**

$$S_\alpha + S_h = S_B + S_k$$

$$\frac{1}{16} E_\alpha \frac{p^2 \langle \sigma v \rangle}{T^2} \geqslant \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

$$p\tau \geqslant \frac{6}{\frac{1}{4} E_\alpha \frac{\langle \sigma v \rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

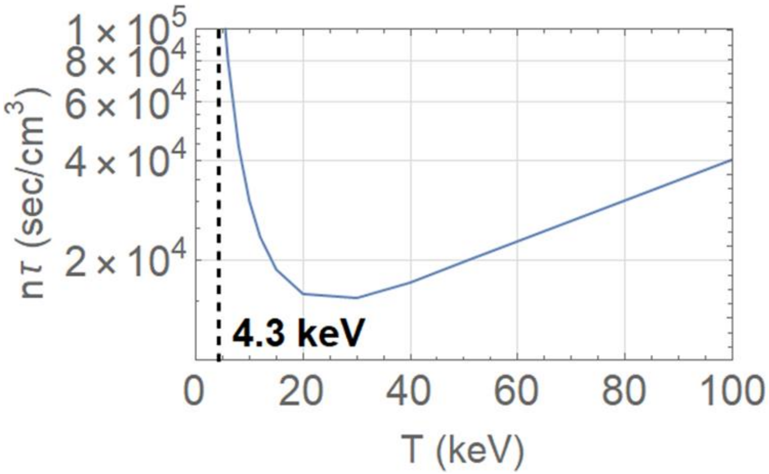
$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

$$n\tau > \frac{3T}{\frac{1}{4} \langle \sigma v \rangle \epsilon_\alpha - C_B \sqrt{T}}$$

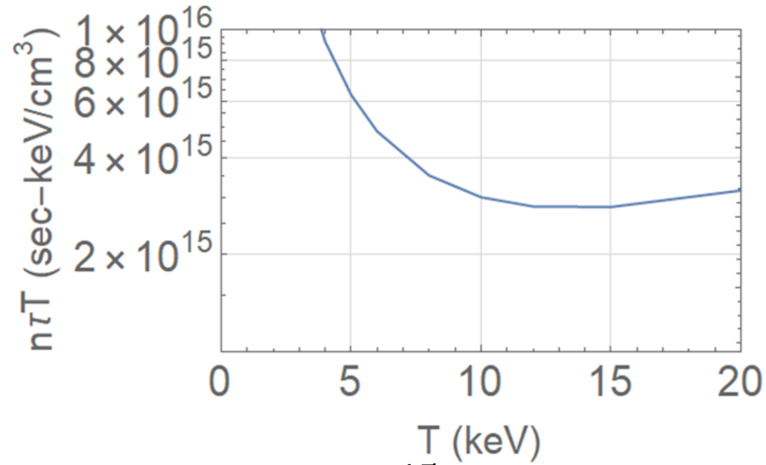
$$nT\tau > \frac{3T^2}{\frac{1}{4} \langle \sigma v \rangle \epsilon_\alpha - C_B \sqrt{T}}$$

Temperature needs to be greater than ~5 keV to ignite

$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle\epsilon_\alpha - C_B\sqrt{T}}$$



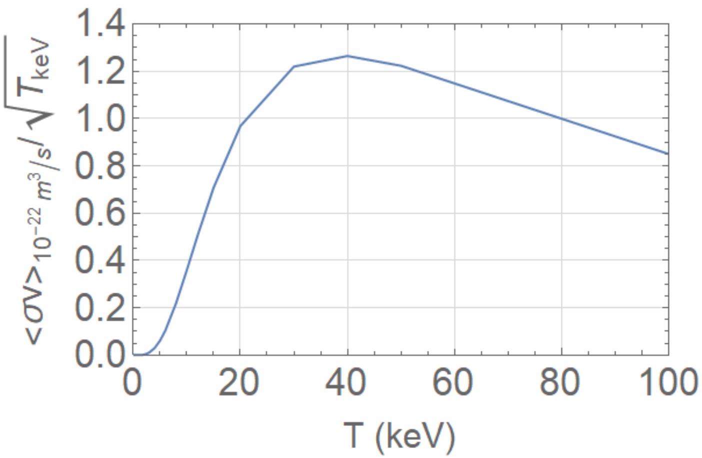
$$n\tau > 2\times 10^4\ \mathrm{sec/cm^3}$$



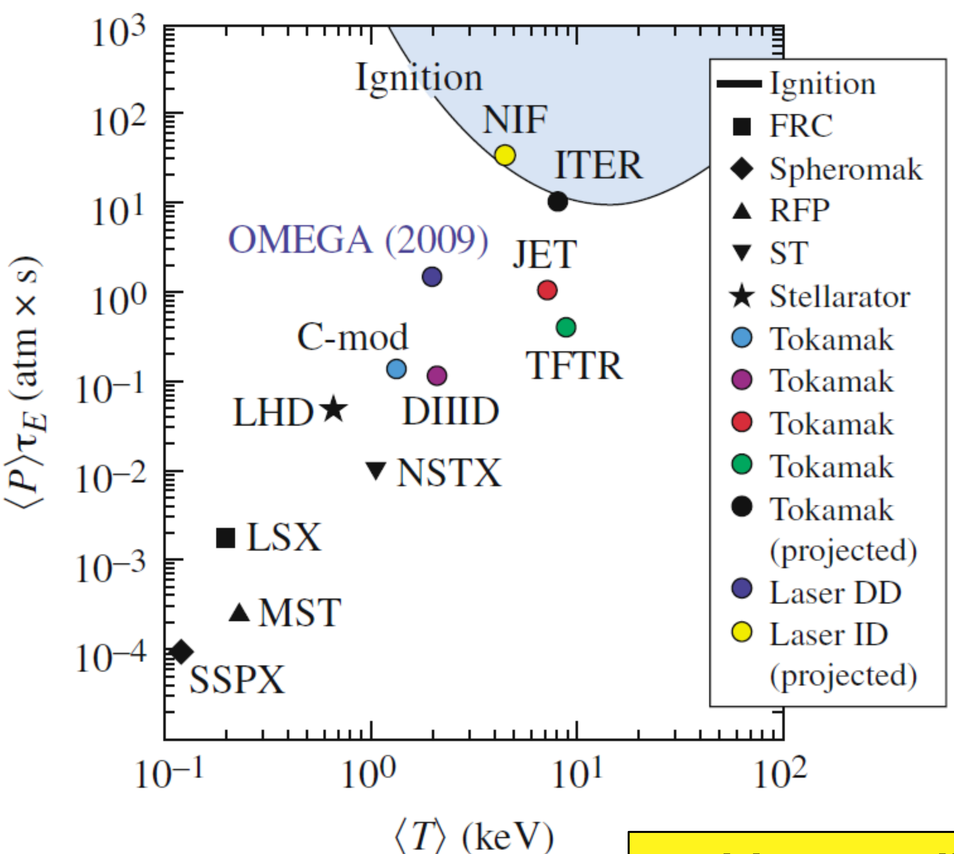
$$nT\tau > 3.5\times 10^{15}\ \mathrm{keV-sec/cm^3}$$

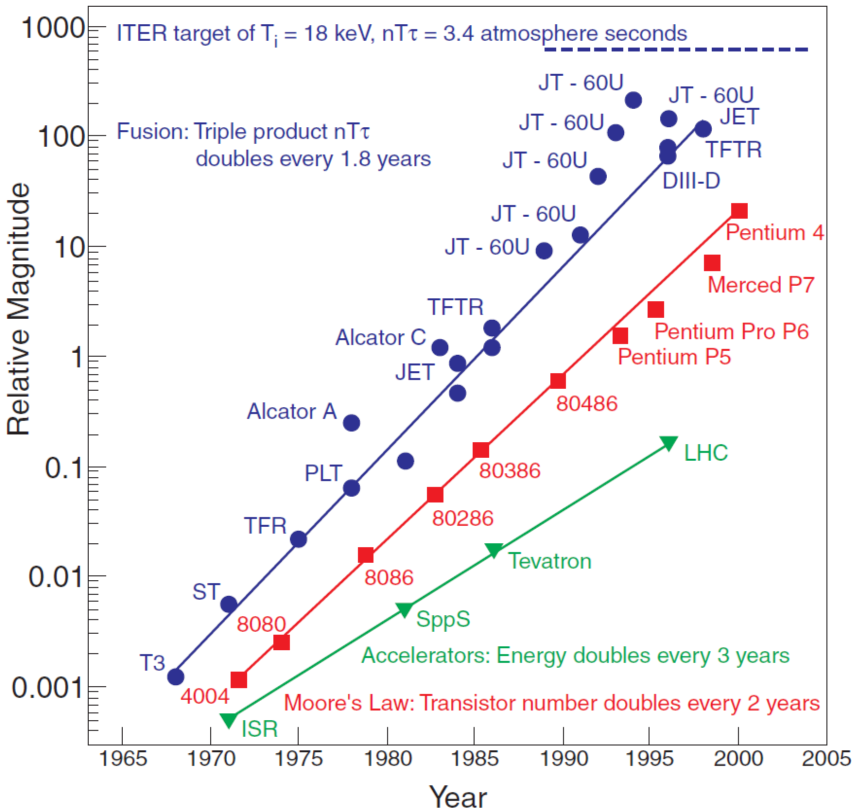
$S_\alpha > S_B$ $$\frac{1}{4}E_\alpha n^2\langle\sigma v\rangle > C_B n^2 T^{1/2}$$

$$\frac{\langle\sigma v\rangle}{T^{1/2}} > \frac{4C_B}{E_\alpha} \qquad T > 4.3\ \mathrm{keV}$$



We are closed to ignition!





Ignition condition: Pτ > 10 atm-s = 10 Gbar - ns

A. J. Webster, Phys. Educ. **38**, 135 (2003)
R. Betti, etc., Phys. Plasmas, **17**, 058102 (2010)

Under what conditions the plasma keeps itself hot?

- **Steady state 0-D power balance:**

  $S_\alpha + S_h = S_B + S_k$

  S_α: α particle heating

  S_h: external heating

  S_B: Bremsstrahlung radiation

  S_k: heat conduction lost

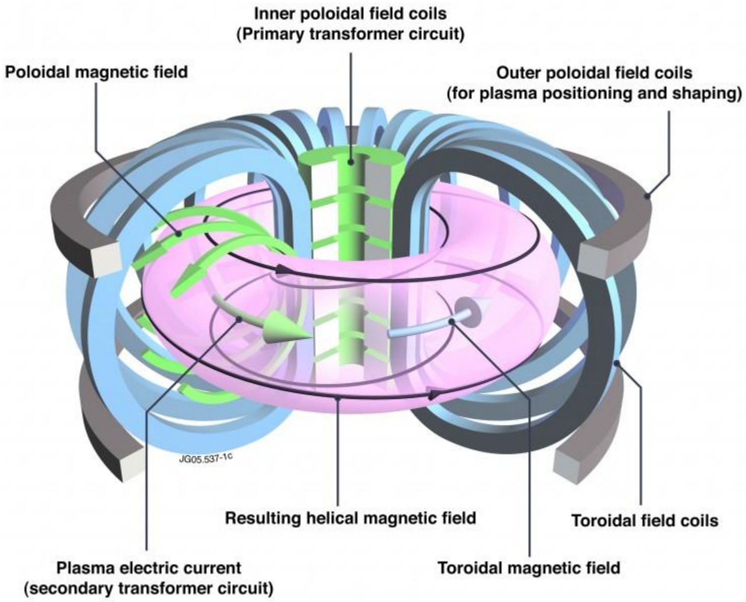
Ignition condition: $P\tau$ > 10 atm-s = 10 Gbar - ns

- **P: pressure, or called energy density**
- **τ is confinement time**

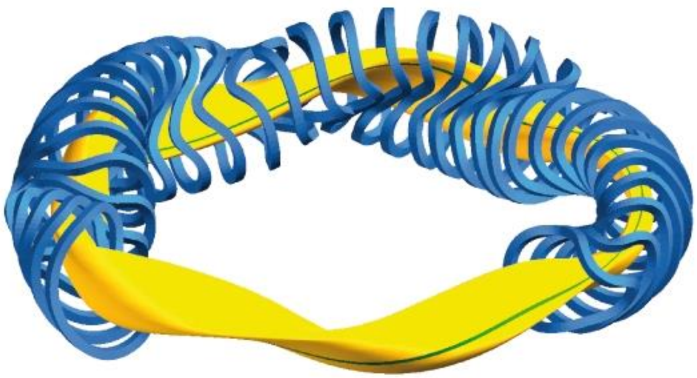
The plasma is too hot to be contained

- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. P~atm, τ~sec, T~10 keV (10^8 ºC)**

Tokamak



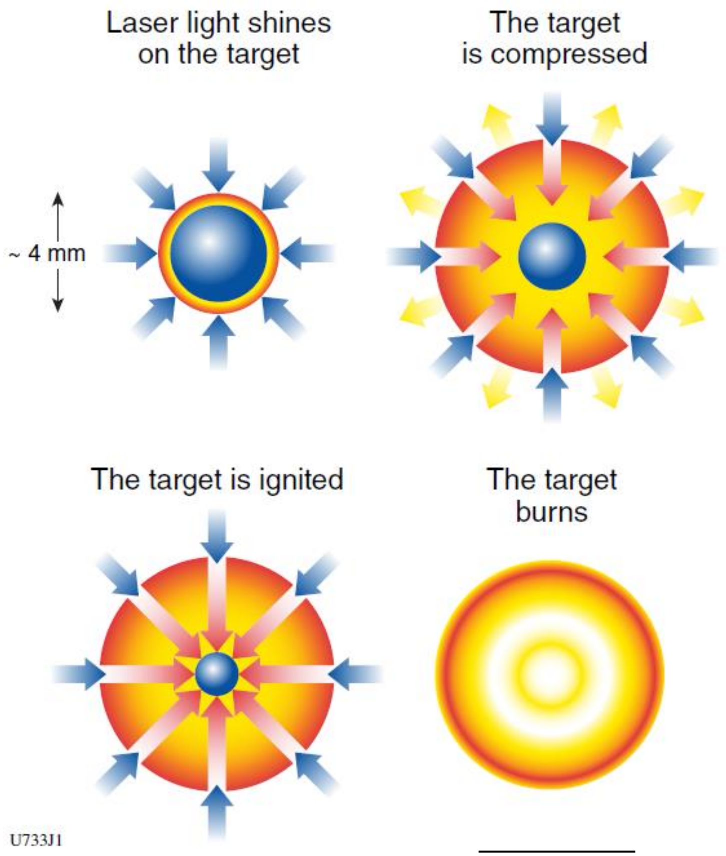
Stellarator



https://www.euro-fusion.org/2011/09/tokamak-principle-2/
https://en.wikipedia.org/wiki/Stellarator

Don’t confine it!

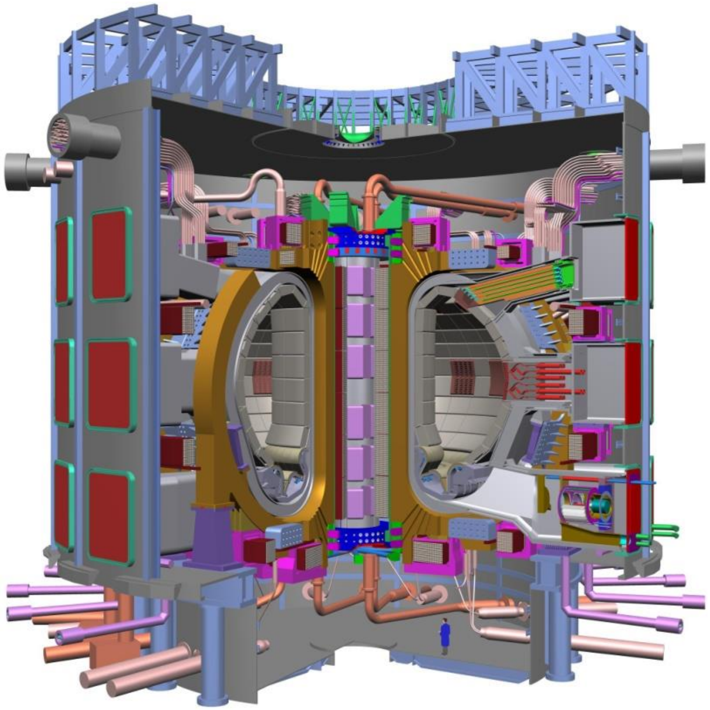
- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ~nsec, T~10 keV (10^8 ^{0}C)**



Inertial confinement fusion: an introduction, Laboratory for Laser Energetics, University of Rochester

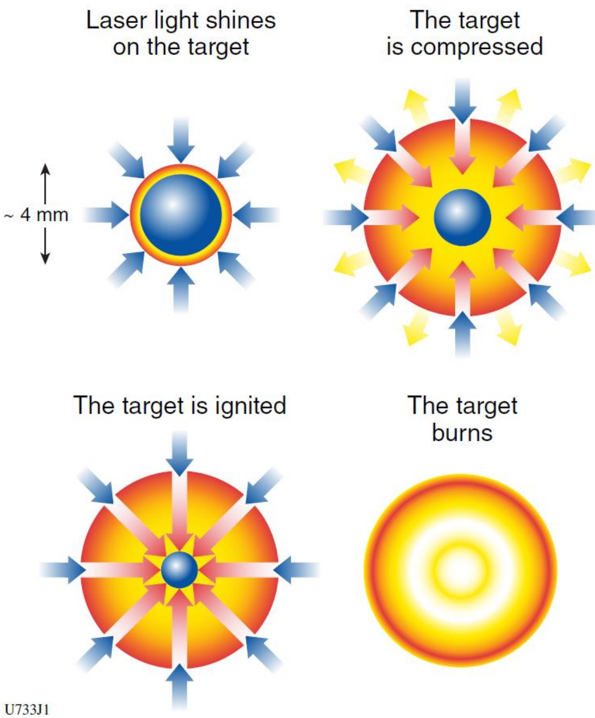
To control? Or not to control?

- **Magnetic confinement fusion (MCF)**



- **Plasma is confined by toroidal magnetic field.**

- **Inertial confinement fusion (ICF)**



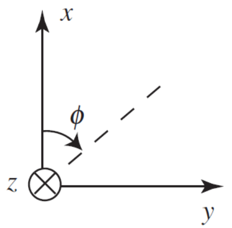
- **A DT ice capsule filled with DT gas is imploded by laser.**

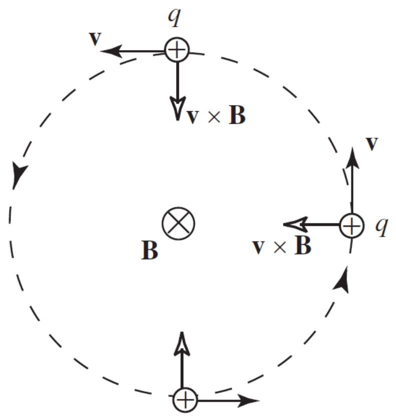
Laboratory for Laser Energetics, University of Rochester is a pioneer in laser fusion

Course Outline

- **Magnetic confinement fusion (MCF)**
  - **Gyro motion, MHD**
  - **1D equilibrium (z pinch, theta pinch)**
  - **Drift: ExB drift, grad B drift, and curvature B drift**
  - **Tokamak, Stellarator (toroidal field, poloidal field)**
  - **Magnetic flux surface**
  - **2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.**
  - **Stability (Kink instability, sausage instability, Safety factor Q)**
  - **Central-solenoid (CS) start-up (discharge) and current drive**
  - **CS-free current drive: electron cyclotron current drive, bootstrap current.**
  - **Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.**

Charged particles gyro around the magnetic field line





$$m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \qquad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

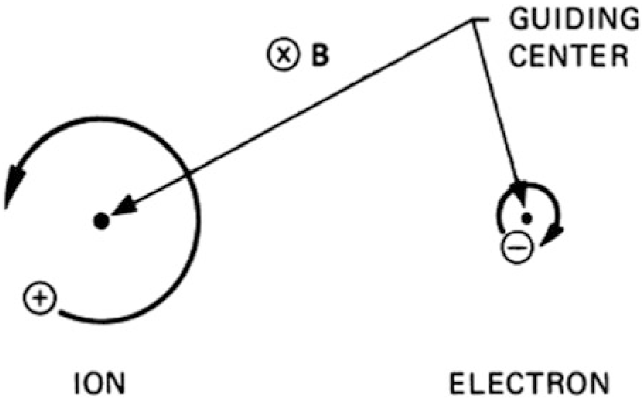
$$\omega_c \equiv \frac{|q|B}{m}$$ Cyclotron frequency or gyrofrequency

$$\ddot{v}_x + \omega_c{}^2 v_x = 0$$

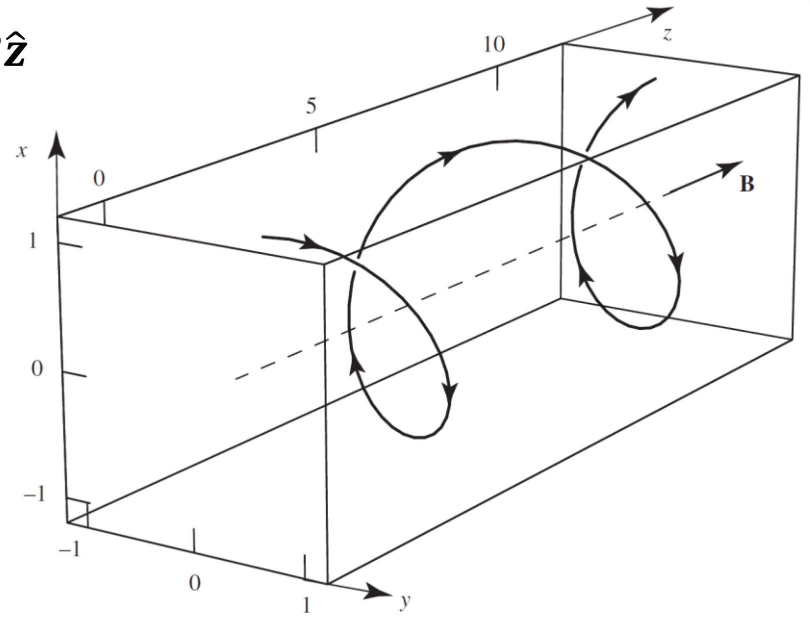
$$\ddot{v}_y + \omega_c{}^2 v_y = 0$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$\vec{B} = B\hat{z}$$



$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$
$$v_y = v_\perp \sin(\pm\omega_c t + \psi)$$
$$v_z = v_{||}$$

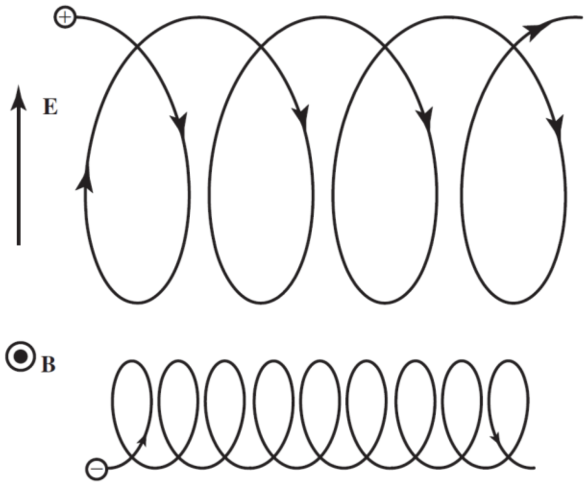
$$x = \mp r_c sin(\pm\omega_c t + \psi) + (x_o - r_c sin\psi)$$
$$y = \pm r_c cos(\pm\omega_c t + \psi) + (y_o + r_c cos\psi)$$
$$z = z_o + v_{||} t$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left|\frac{m{v_\perp}^2}{r}\right| = \left|q\,\vec{v} \times \vec{B}\right| = |qv_\perp B|$$

$$r_c = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}$$ **Larmor radius or gyroradius**

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$m\frac{d}{dt}\left(\vec{v}_E + \vec{v}_{ac}(t)\right) = q\left[\hat{x}E_\perp + \left(\vec{v}_E + \vec{v}_{ac}(t)\right) \times \hat{z}B\right]$$

$$m\frac{d\vec{v}_{ac}(t)}{dt} = q\left[\hat{x}E_\perp + \vec{v}_E \times \hat{z}B + \vec{v}_{ac}(t) \times \hat{z}B\right]$$

No E field case: $m\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$

$$\hat{x}E_\perp + \vec{v}_E \times \hat{z}B = 0$$

$$\vec{v}_E = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$m\frac{d\vec{v}_{ac}(t)}{dt} = q\,\vec{v}_{ac}(t) \times \hat{z}B$$

Gyro motion

$$\vec{v}(t) = \hat{z}v_{||}(t) + \vec{v}_E + \vec{v}_{ac}(t)$$

$$\langle\vec{v}(t)\rangle = \frac{1}{T_c}\int_0^{T_c}\vec{v}(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

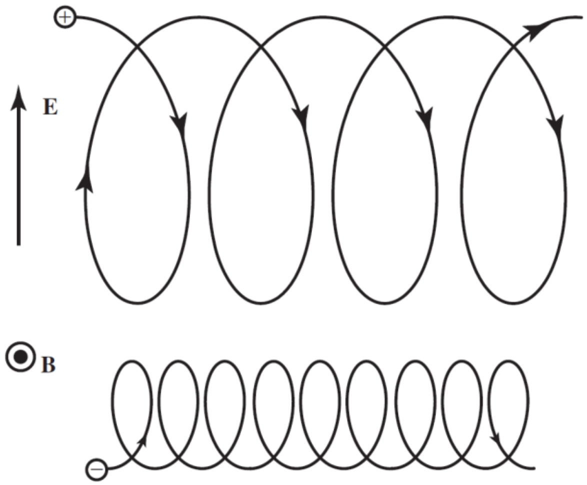
$$m\frac{d\vec{v}_\perp}{dt} = q\left(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B\right)$$

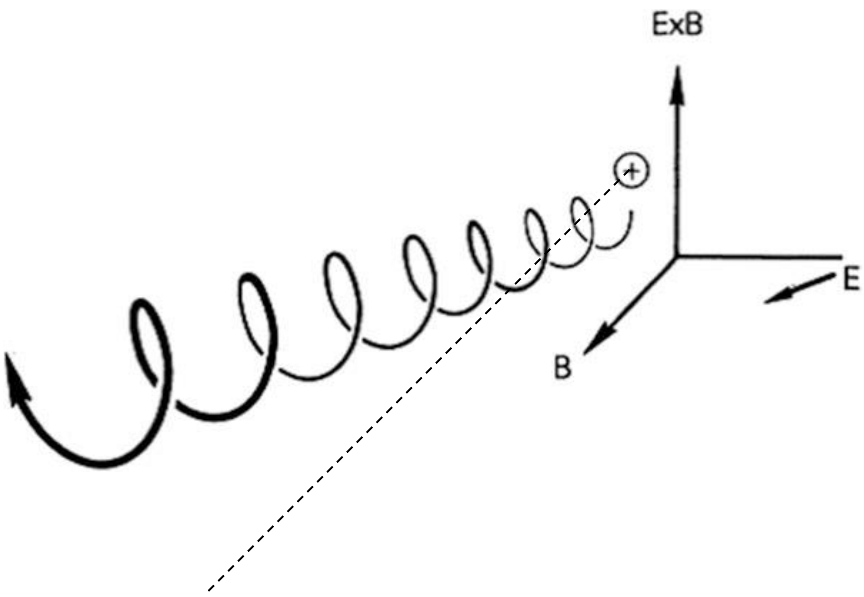
$$v_{||}(t) = \frac{qE_{||}}{m}t + v_{||,o}$$

$$\vec{v}_\perp(t) = \vec{v}_E + \vec{v}_{ac}(t)$$

- Electrons and ions drift in the same direction.

No current is generated in ExB drift





$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t)\, dt = \hat{z} v_{||}(t) + \vec{v}_E$$

$$\vec{v}_{\mathrm{E}} = \frac{\hat{x} E_{\perp} \times \hat{z} B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$ **ExB drift velocity**

- **Electrons and ions drift in the same direction.**

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs

$$\vec{E} = \vec{E}_\perp + \hat{z}E_{||} = \hat{x}E_\perp + \hat{z}E_{||}$$

$$m\frac{dv_{||}}{dt} = qE_{||}$$

$$m\frac{d\,\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_E$$

$$\vec{v}_\mathrm{E} = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_\perp + \hat{z}F_{||} = \hat{x}F_\perp + \hat{z}F_{||}$$

$$m\frac{dv_{||}}{dt} = F_{||}$$

$$m\frac{d\,\vec{v}_\perp}{dt} = q\left(\hat{x}\frac{F_\perp}{q} + \vec{v}_\perp \times \hat{z}B\right)$$

$$\langle \vec{v}\,(t)\rangle = \frac{1}{T_c}\int_0^{T_c} \vec{v}\,(t)\,dt = \hat{z}v_{||}(t) + \vec{v}_F$$

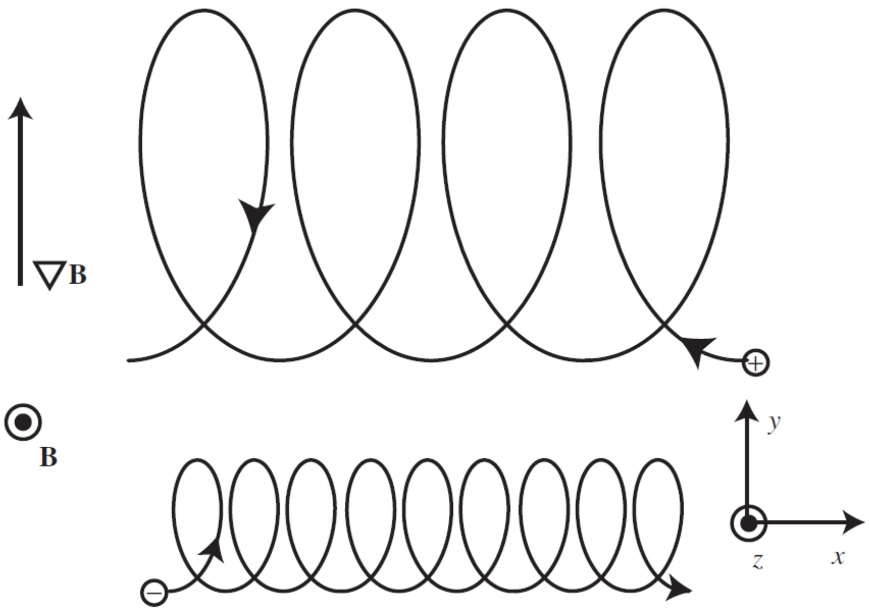
$$\vec{v}_\mathrm{F} = \frac{\hat{x}(F_\perp/q) \times \hat{z}B}{B^2} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

Gravitational drift velocity

- **Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.**

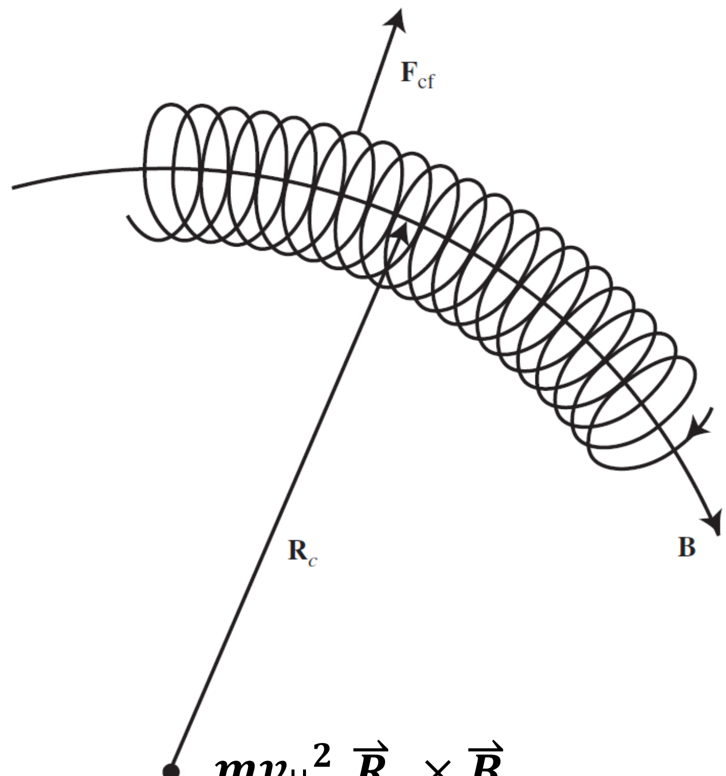
Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

- **Gradient-B drift**



$$\vec{v}_{\nabla} = \frac{m{v_{\perp}}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3}$$

- **Curvature drift**

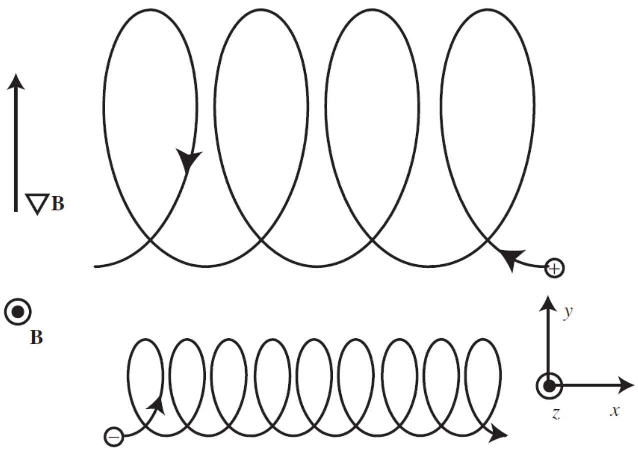


$$\vec{v}_{R} = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_{\text{R}} + \vec{v}_{\nabla} = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_{\perp}}^2\right)$$

Gradient-B drift

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm\omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm\omega_c t + \psi)$$

$$v_x = v_\perp \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_\perp \sin(\pm\omega_c t + \psi)$$

$$\vec{F} = q(\vec{v} \times \vec{B}) = \hat{x} q v_y B_z - \hat{y} q v_x B_z$$

$$\simeq \hat{x} q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right) - \hat{y} q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$B_z(y) = B_o + y\frac{\partial B_z}{\partial y} + y^2 \frac{1}{2}\frac{\partial^2 B_z}{\partial y^2} + \ldots$$

$$F_x = q v_y \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

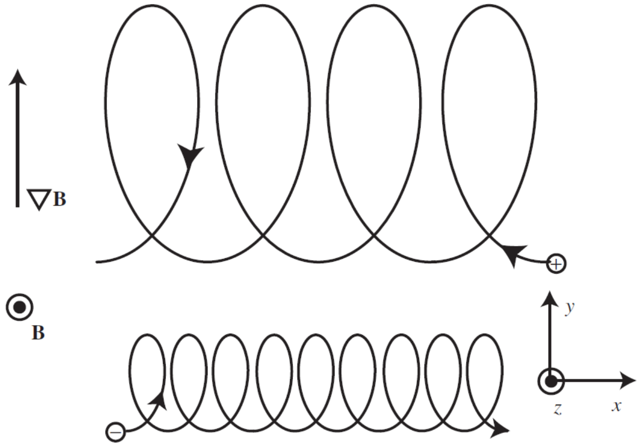
$$F_y = -q v_x \left(B_o + y\frac{\partial B_z}{\partial y}\right)$$

$$F_x \simeq -q v_\perp \sin(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$F_y = -q v_\perp \cos(\pm\omega_c t + \psi) \times \left(B_o \pm r_c \cos(\pm\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right)$$

$$\vec{v}_\nabla = \frac{m v_\perp{}^2}{2q}\frac{\vec{B} \times \nabla B}{B^3}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_{\mathrm{x}} \simeq -qv_{\perp}\sin(\pm\omega_{\mathrm{c}}t+\psi)\left(B_{\mathrm{o}} \pm r_{\mathrm{c}}\cos(\pm\omega_{\mathrm{c}}t+\psi)\frac{\partial B_{\mathrm{z}}}{\partial y}\right)$$

$$F_{\mathrm{y}} \simeq -qv_{\perp}\cos(\pm\omega_{\mathrm{c}}t+\psi)\left(B_{\mathrm{o}} \pm r_{\mathrm{c}}\cos(\pm\omega_{\mathrm{c}}t+\psi)\frac{\partial B_{\mathrm{z}}}{\partial y}\right)$$

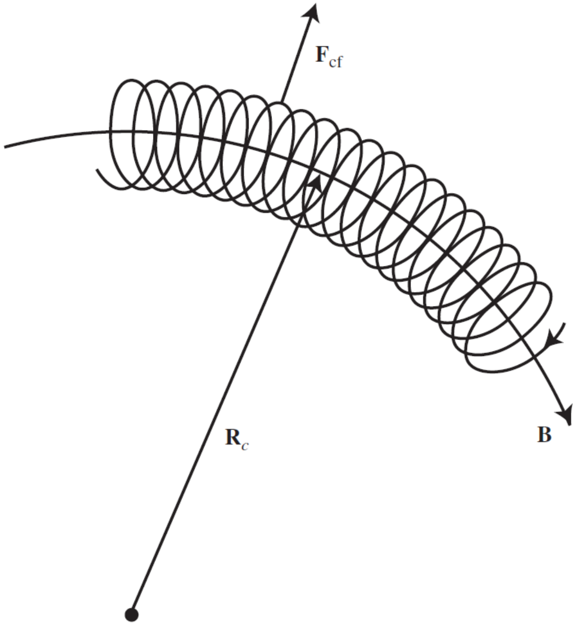
$$\langle F_{\mathrm{x}}\rangle = 0$$

$$\langle F_{\mathrm{y}}\rangle = \mp\frac{qv_{\perp}r_{\mathrm{c}}}{2}\frac{\partial B_{\mathrm{z}}}{\partial y} = -\frac{mv_{\perp}{}^{2}}{2B}\frac{\partial B_{\mathrm{z}}}{\partial y} \qquad r_c = \frac{v_{\perp}}{\omega_c} \quad \omega_{\mathrm{c}} \equiv \frac{|q|B}{m}$$

$$\vec{v}_{\mathrm{F}} = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \qquad \vec{v}_{\nabla} = \frac{1}{q}\frac{\langle F_{\mathrm{y}}\rangle\hat{y}\times\hat{z}B_{\mathrm{z}}}{B_{\mathrm{z}}{}^{2}} = -\frac{mv_{\perp}{}^{2}}{2qB_{\mathrm{z}}}\frac{\partial B_{\mathrm{z}}}{\partial y}\hat{x}$$

- More general: $\vec{v}_{\nabla} = \dfrac{mv_{\perp}{}^{2}}{2q}\dfrac{\vec{B}\times\nabla B}{B^3}$

Charge particles drift across magnetic field lines when the magnetic field line is curved

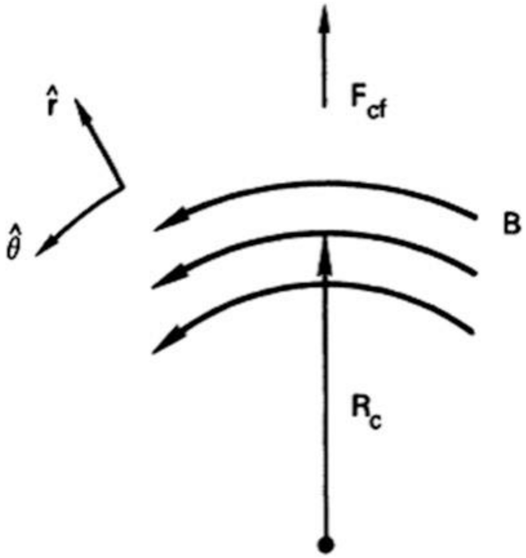


$$\vec{F}_{cf} = m v_{||}^{2} \frac{\vec{R}_c}{R_c^{2}}$$

$$\vec{v}_{\mathrm{F}} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^{2}}$$

$$\vec{v}_{R} = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^{2}} = \frac{m v_{||}^{2}}{2q} \frac{\vec{R}_{c} \times \vec{B}}{R_c B^{2}}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B\hat{r}$$

Cylindrical coordinate

$$\vec{v}_\nabla = \frac{m{v_\perp}^2}{2q}\frac{\vec{B}\times\nabla B}{B^3} \qquad \vec{v}_R = \frac{m{v_{||}}^2}{2q}\frac{\vec{R}_c\times\vec{B}}{R_c B^2}$$

$$\nabla\times\vec{B} = 0$$

$$\left(\nabla\times\vec{B}\right)_r = \left(\nabla\times\vec{B}\right)_\theta = 0$$

$$\nabla\times\vec{B} = \left(\nabla\times\vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \qquad B_\theta \propto \frac{1}{r}$$

$$\frac{\nabla|B|}{|B|} = -\frac{\vec{R}_c}{{R_c}^2}$$

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_\nabla = \frac{\vec{B}\times\nabla B}{\omega_c B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right) = \frac{m}{q}\frac{\vec{R}_c\times\vec{B}}{{R_c}^2 B^2}\left({v_{||}}^2 + \frac{1}{2}{v_\perp}^2\right)$$

- **Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.**

Quick summary of different drifts

- **ExB drift:** $\vec{v}_{\mathrm{E}} = \frac{\vec{E} \times \vec{B}}{B^2}$ **Independent to charge**
- **Gravitational drift:** $\vec{v}_{\mathrm{F}} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$ **Depended on charge**
- **Grad-B drift:** $\vec{v}_{\nabla} = \frac{m {v_{\perp}}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ **Depended on charge**
- **Curvature drift:** $\vec{v}_{R} = \frac{m {v_{||}}^2}{2q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{R_{\mathrm{c}} B^2}$ **Depended on charge**
- **Non-uniform B drift:**

$$\vec{v}_{\mathrm{total}} = \vec{v}_{\mathrm{R}} + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_{\mathrm{c}} B^2} \left({v_{||}}^2 + \frac{1}{2} {v_{\perp}}^2 \right) = \frac{m}{q} \frac{\vec{R}_{\mathrm{c}} \times \vec{B}}{{R_{\mathrm{c}}}^2 B^2} \left({v_{||}}^2 + \frac{1}{2} {v_{\perp}}^2 \right)$$