

Introduction to Nuclear Fusion as An Energy Source



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Tuesday 9:00-12:00

Materials:

<https://capst.ncku.edu.tw/PGS/index.php/teaching/>

Online courses:

<https://reurl.cc/MMnkOL>



Course Outline



- **Brief background reviews**
 - Electromagnetics
 - Plasma physics
- **Introduction to nuclear fusion**
 - **Nuclear binding energy (Fission vs Fusion)**
 - **Fusion reaction physics**
 - **Some important fusion reactions (Cross section)**
 - **Main controlled fusion fuels**
 - **Advanced fusion fuels**
 - **Maxwell-averaged fusion reactivities**

Magnetohydrodynamics description of plasma w/ low-freq. and long-wavelength approximation



- Continuity eq: $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ w/ long wavelength ($\lambda \gg \lambda_d$)
- Momentum eq: $\rho_m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = \cancel{\rho_q \vec{E}} + \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$
- Ohm's law: $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$
- Equation of state: $\frac{d}{dt} (P \rho_m^{-\gamma}) = 0$

- Maxwell's eqs:

$$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0} \approx 0 \quad \text{w/ long wavelength (} \lambda \gg \lambda_d \text{)} \Rightarrow \text{quasi neutral}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \cancel{\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

w/ low freq. ($\omega \ll \omega_{pe}$)

Magnetohydrodynamics description of plasma w/ low-freq. and long-wavelength approximation

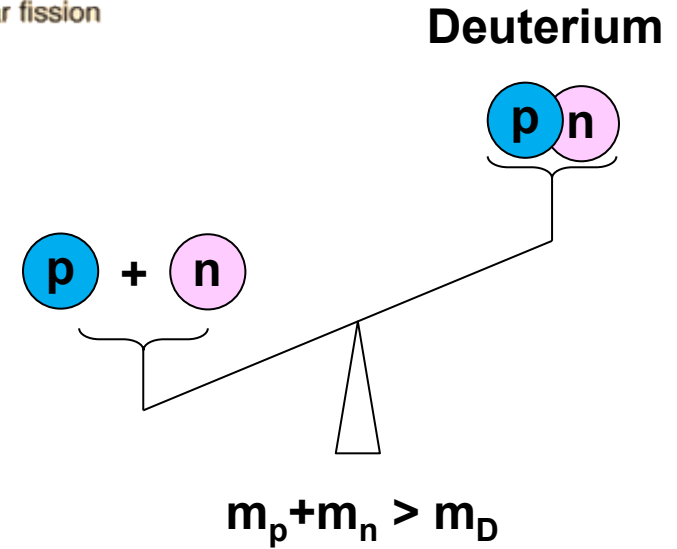
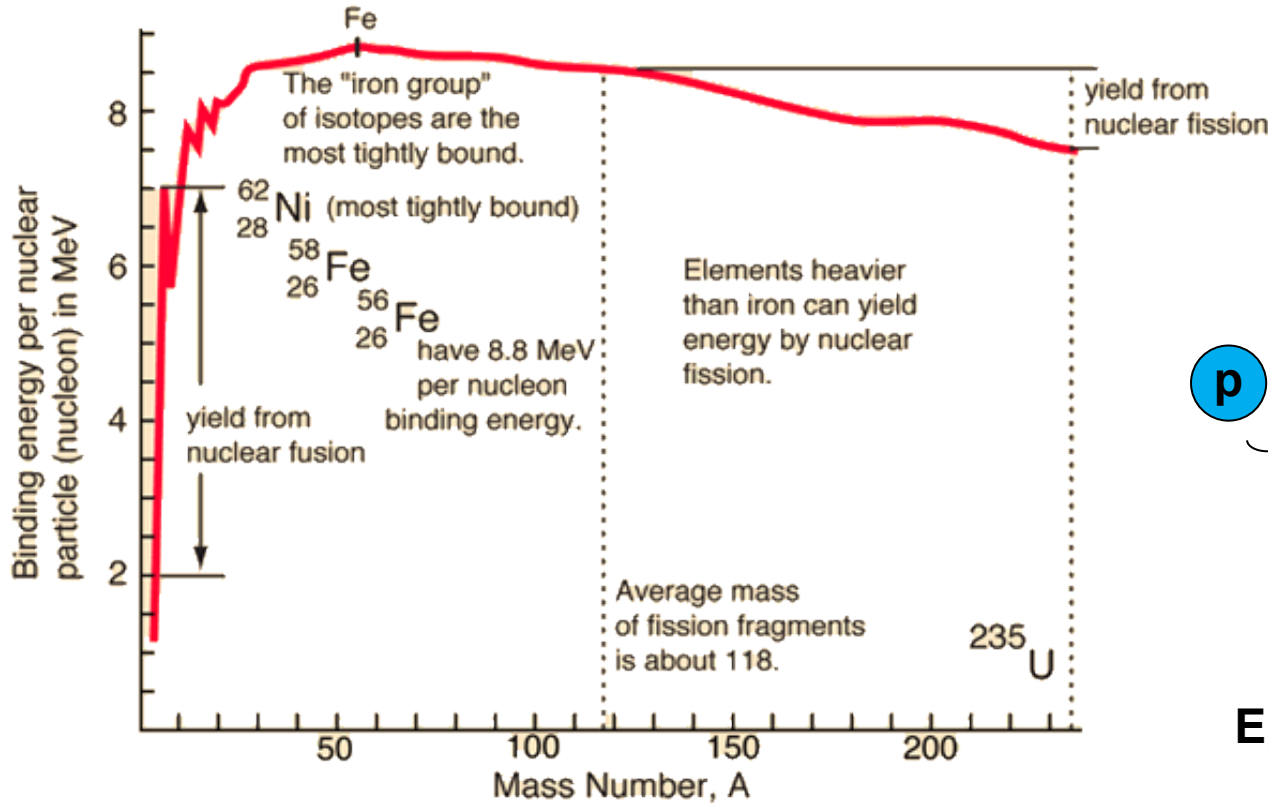


- Continuity eq: $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$
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- Equation of state: $\frac{d}{dt} (P \rho_m^{-\gamma}) = 0$
- Maxwell's eqs:

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \mu_0 \vec{j} \quad \Rightarrow \quad \nabla \cdot \vec{j} = 0 \end{aligned}$$

- Force balance condition: $0 = \vec{j} \times \vec{B} - \nabla \cdot \vec{P}$

The “iron group” of isotopes are the most tightly bound



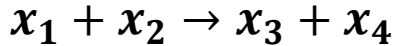
$$E_{\text{binding}} = [(m_p + m_n) - m_D]c^2$$

$$Q = \left(\sum_i m_i - \sum_f m_f \right) c^2$$

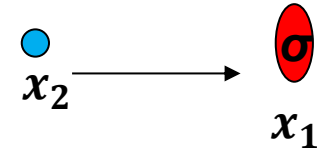
$$\Delta m = z m_p + (A - z) m_n - m$$

- Binding energy: $B = \Delta m c^2$
- Output energy: $Q = \sum_f B_f - \sum_i B_i$

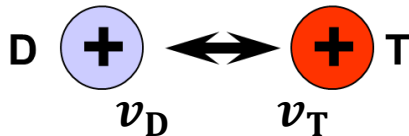
Cross section measures the probability per pair of particles for the occurrence of the reaction



- The hard sphere cross section:
 $\sigma \approx \pi R^2$ where $R \approx 5 \times 10^{-15}$ m is the nuclear radius, i.e.,
 $\sigma = 0.8 \times 10^{-28} \text{ m}^2 \approx 1 \text{ barn}$. (barn $\equiv 10^{-28} \text{ m}^2$)



- Classical cross section:

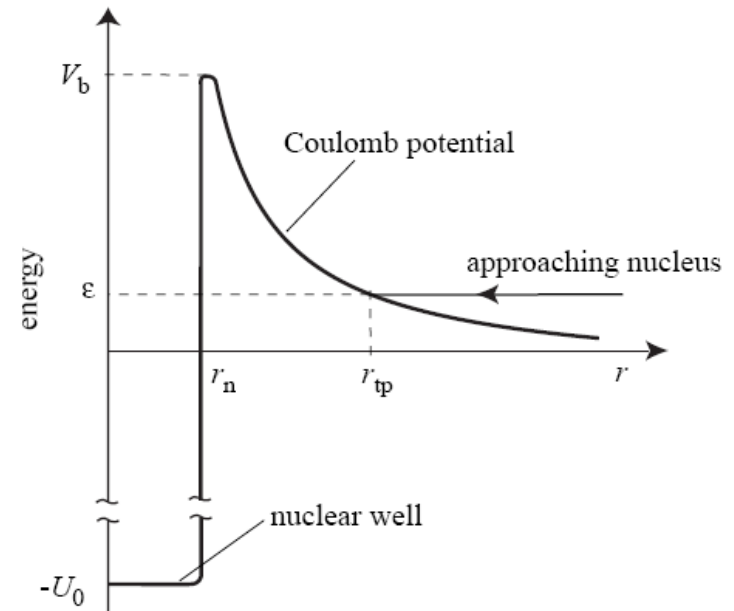


$$\frac{m_D}{2} v_D^2 + \frac{m_T}{2} v_T^2 \geq \frac{e^2}{4\pi\epsilon_0 R}$$

- Let $v = |\vec{v}_D - \vec{v}_T|$

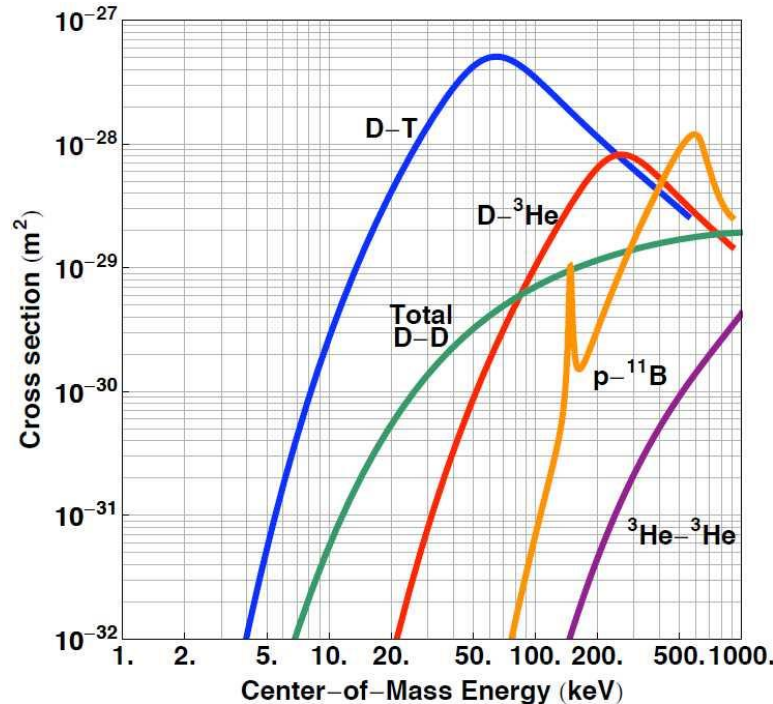
$$v_D = \frac{m_T}{m_D + m_T} v \quad v_T = \frac{m_D}{m_D + m_T} v$$

$$\frac{m_T}{2} v^2 \geq \frac{e^2}{4\pi\epsilon_0 R} \quad m_T = \frac{m_D m_T}{m_D + m_T}$$

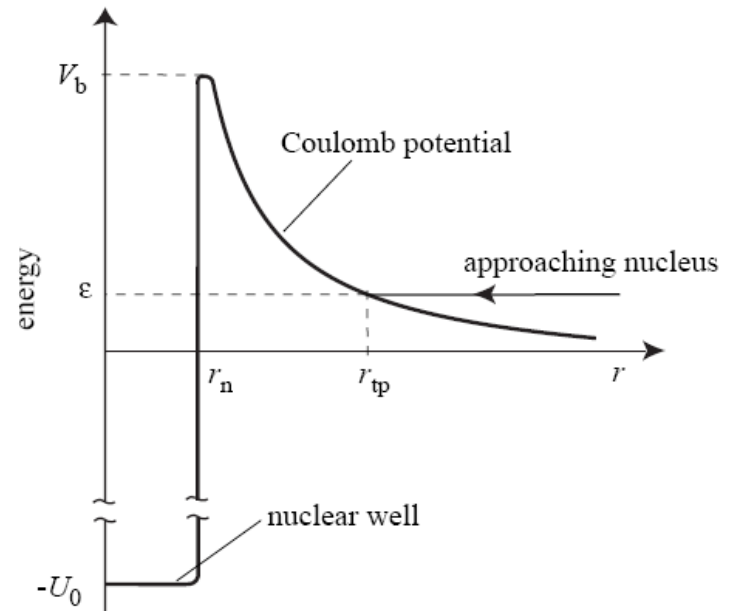
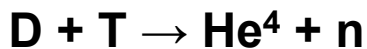
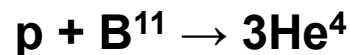
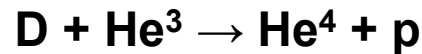
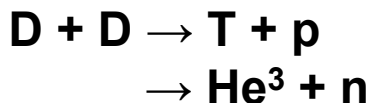


- Classical kinetic energy required for fusion is
 $K_{c.m.} > 288 \text{ keV} \quad !!!$

Cross section of fusion reaction is much larger than the classical approach



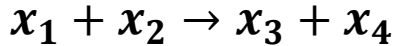
- Classical kinetic energy required for fusion is $K_{c.m.} > 288 \text{ keV} !!!$
- DT cross section has a peak of ~ 5 barns at $E_{neD} = 120 \text{ keV}$.
- $\sigma_{DT} \approx 100\sigma_{DD} @ 20 \text{ keV}$.



<https://i.stack.imgur.com/wXQD5.jpg>

Santarius, J. F., "Fusion Space Propulsion – A Shorter Time Frame Than You Think", JANNAF, Monterey, 5-8 December 2005.

Flux of incident particles reduces after collisions

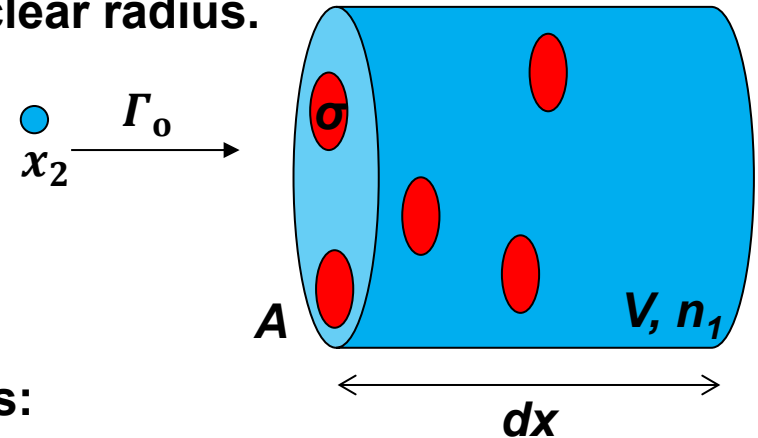


- **Cross section:** $\sigma \approx \pi R^2$ where R is the nuclear radius.

$$V = A dx$$

$$N_1 = n_1 V = n_1 A dx$$

$$A_{\text{Target}} = N_1 \sigma = \sigma n_1 A dx$$



- **Fraction of total area blocked by targets is:**

$$dF = \frac{\sigma N_1}{A} = \frac{\sigma n_1 A dx}{A} = \sigma n_1 dx$$

$$\frac{dF}{dx} = \sigma n_1$$

- **Flux of incident particles (x_1) is Γ_0**

$$-d\Gamma = dF\Gamma = \sigma n_1 \Gamma dx$$

$$\frac{-d\Gamma}{\Gamma} = \sigma n_1 dx \quad \Gamma = \Gamma_0 \exp\left(-\frac{x}{\lambda_{\text{mfp}}}\right)$$

- **Mean free path:**

$$\lambda_{\text{mfp}} = \frac{1}{n_1 \sigma}$$

- **Collision frequency:**

$$\nu = \frac{1}{\tau}, \tau = \frac{\lambda_{\text{mfp}}}{v} = \frac{1}{n_1 \sigma v}$$

Reactions happen when collision happen



- Reaction rate R_{12} : number of fusion collisions/reactions per unit volume per unit time.
- In the time $dt=dx/v$, $n_2 A dx$ incident particles will pass through the target volume.
- The number having a collisions is: $dF(n_2 A dx)$
- The volumetric reaction rate R_{12} , i.e., the number of reaction per unit time and per unit volume is:

$$R_{12} = \frac{dF(n_2 A dx)}{A dx dt} = \sigma n_1 n_2 \frac{dx}{dt} = n_1 n_2 \sigma v \quad \left(\frac{dF}{dx} = \sigma n_1 \right)$$

- The fusion power density (W/m^3) is: $S_f = E_f n_1 n_2 \sigma v$ (W/m^3)
- For DT fusion, $E_f=17.6$ MeV.
- For a particle population with a distribution function in velocity space:

$$n = \int d\vec{v} f(\vec{r}, \vec{v}, t)$$

- Therefore,

$$n_1 \rightarrow d\vec{v}_1 f_1(\vec{r}, \vec{v}_1, t) \quad n_2 \rightarrow d\vec{v}_2 f_2(\vec{r}, \vec{v}_2, t) \quad v \rightarrow |\vec{v}_1 - \vec{v}_2|$$

$$R_{12} = \int f_1(\vec{v}_1) f_2(\vec{v}_2) \sigma \left(|\vec{v}_1 - \vec{v}_2| \right) |\vec{v}_1 - \vec{v}_2| d\vec{v}_1 d\vec{v}_2$$

The fusion power density needs to consider the distribution function of particles



$$R_{12} = \int f_1(\vec{v}_1) f_2(\vec{v}_2) \sigma(|\vec{v}_1 - \vec{v}_2|) |\vec{v}_1 - \vec{v}_2| d\vec{v}_1 d\vec{v}_2 = n_1 n_2 \langle \sigma v \rangle$$

$$\begin{aligned} \langle \sigma v \rangle &\equiv \frac{\int f_1(\vec{v}_1) f_2(\vec{v}_2) \sigma(|\vec{v}_1 - \vec{v}_2|) |\vec{v}_1 - \vec{v}_2| d\vec{v}_1 d\vec{v}_2}{\int f_1(\vec{v}_1) f_2(\vec{v}_2) d\vec{v}_1 d\vec{v}_2} \\ &= \frac{\int f_1(\vec{v}_1) f_2(\vec{v}_2) \sigma(|\vec{v}_1 - \vec{v}_2|) |\vec{v}_1 - \vec{v}_2| d\vec{v}_1 d\vec{v}_2}{n_1 n_2} \end{aligned}$$

- The fusion power density (W/m^3) is:

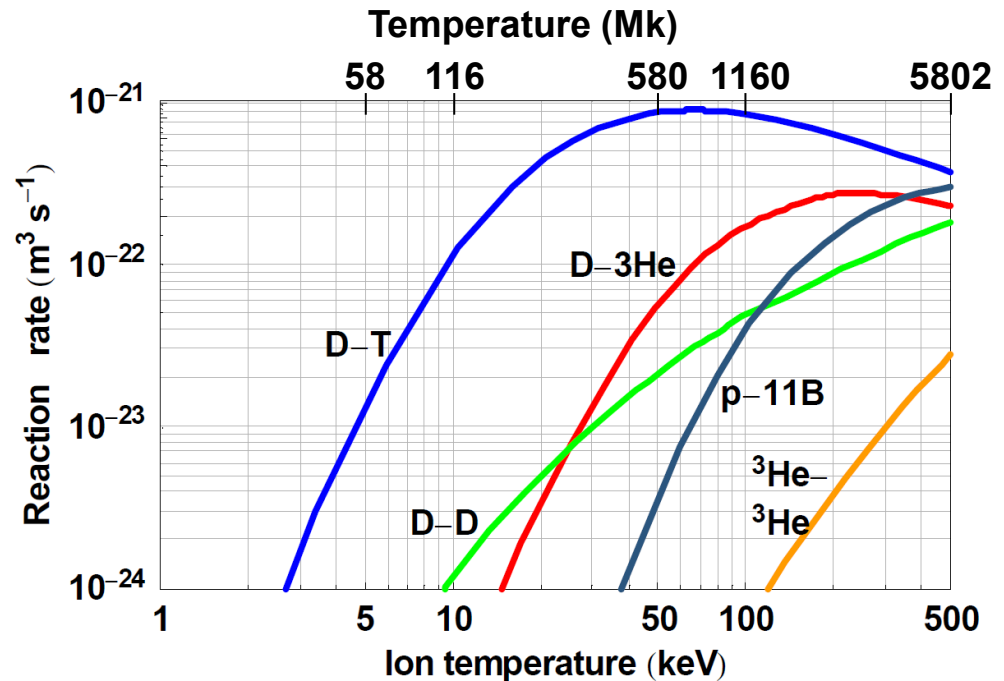
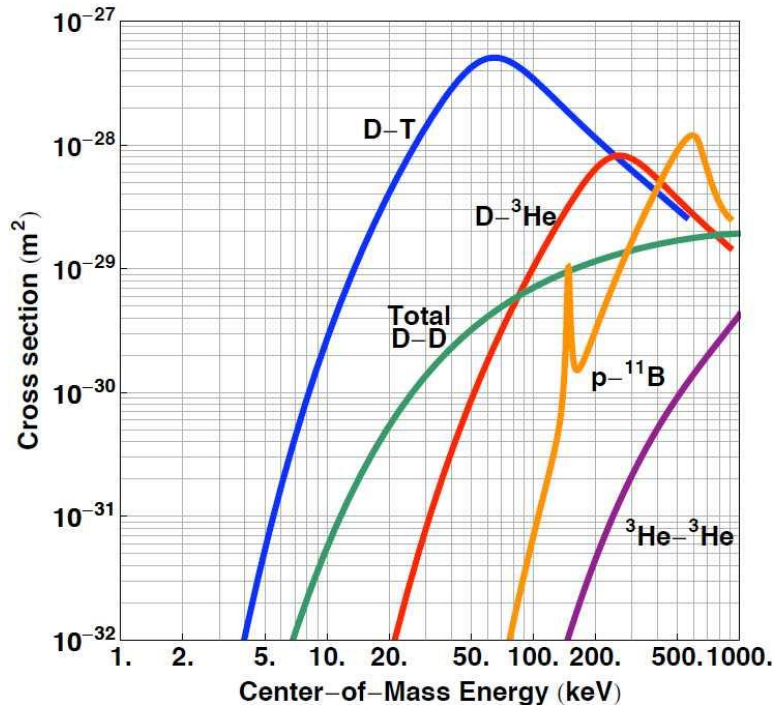
$$S_f = E_f n_1 n_2 \langle \sigma v \rangle (\text{W/m}^3)$$

- Optimum concentration of DT fusion is 50-50.

$$S_f = E_f n_D n_T \langle \sigma v \rangle \quad n_D = k n_0 \quad n_T = (1 - k) n_0$$

$$S_f = E_f k(1 - k) n_0^2 \text{ which peak at } k = 0.5 .$$

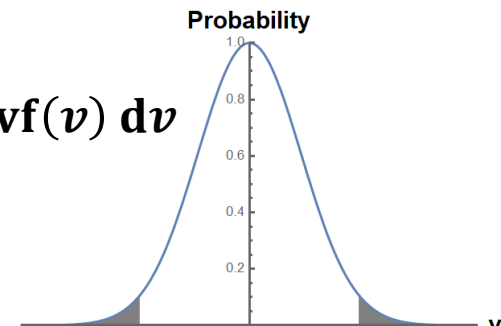
Fusion doesn't come easy



- The DT fusion reactivity is maximum at $T \approx 64$ keV
- @ $T = 10$ keV, $\langle \sigma v \rangle_{DT} \approx 100 \langle \sigma v \rangle_{DD}$

- Reaction rate:

$$\langle \sigma v \rangle = \int \sigma(v) v f(v) dv$$



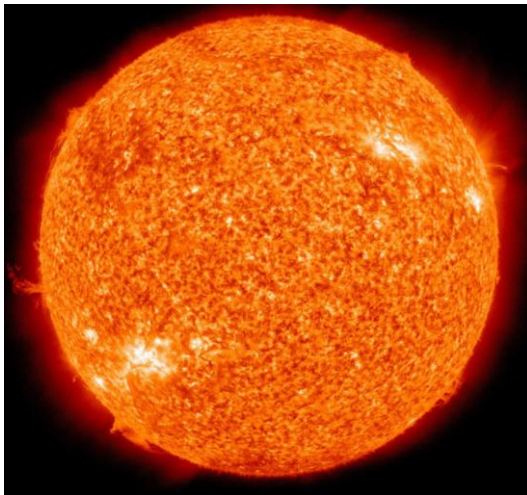
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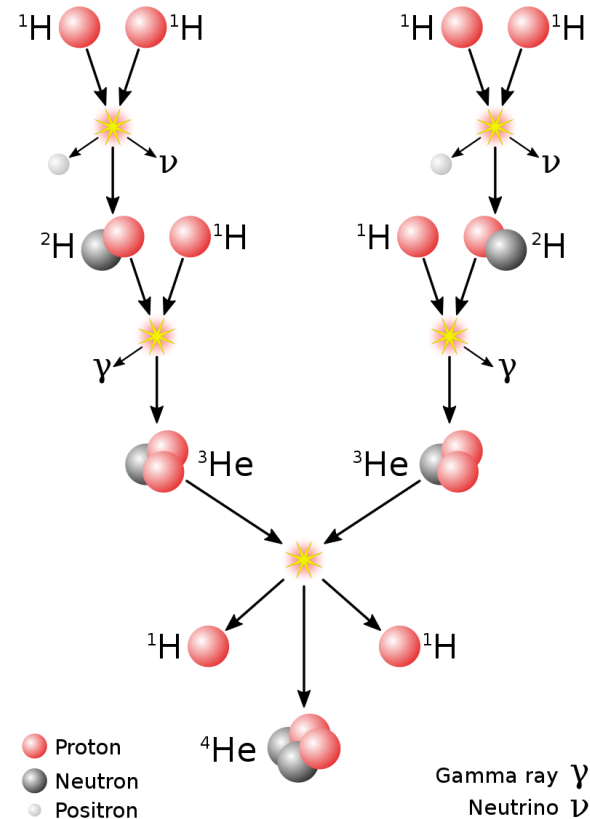
Fusion in the sun provides the energy



- Proton-proton chain in sun or smaller



- Particles are confined by the gravity.



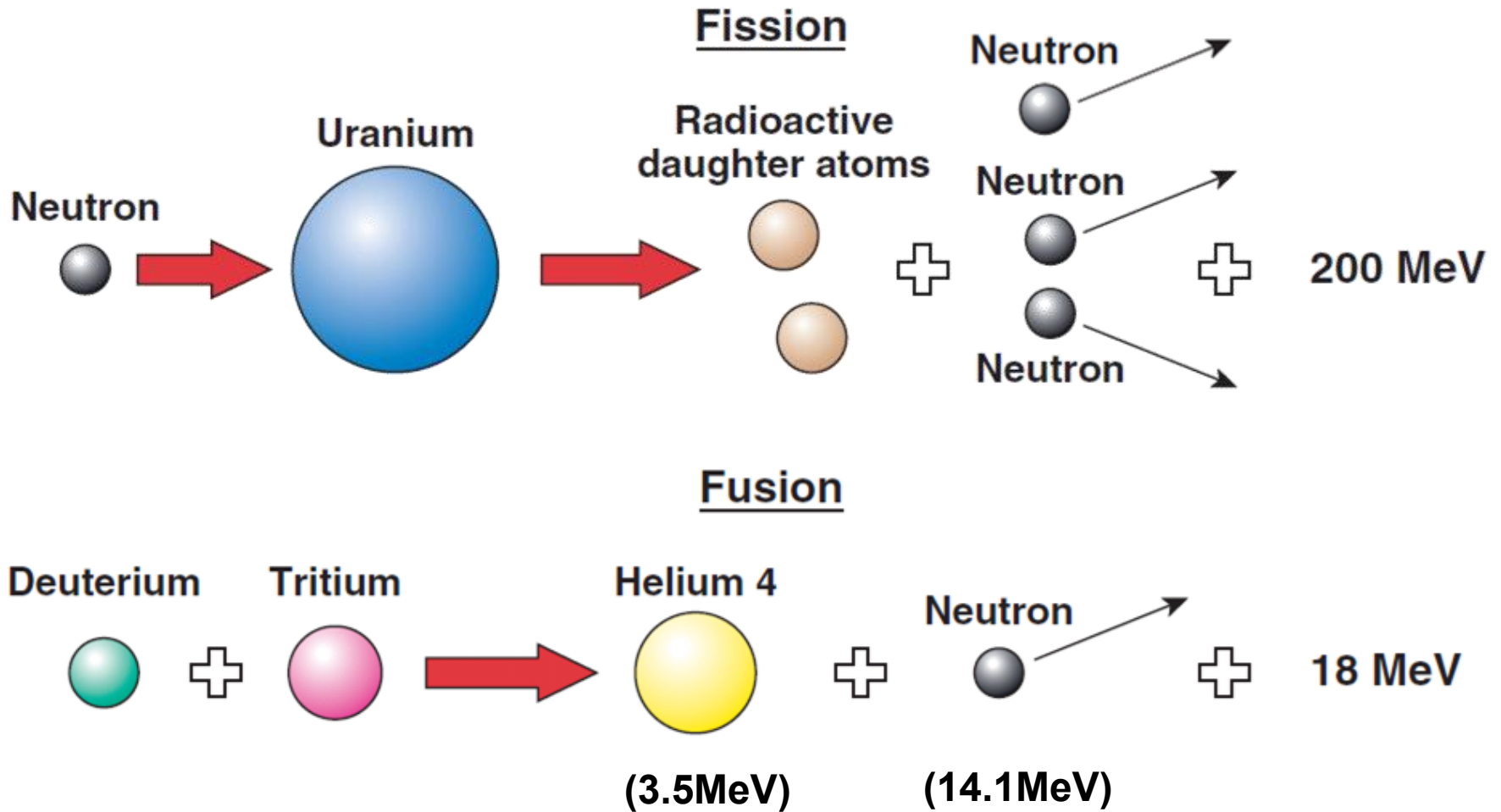
The cross section of proton-proton chain is much smaller than D T fusion



Reaction	$\sigma_{10 \text{ keV}}$ (barn)	$\sigma_{100 \text{ keV}}$ (barn)	σ_{max} (barn)	ϵ_{max} (keV)
$\text{D}+\text{T}\rightarrow\alpha+\text{n}$	2.72×10^{-2}	3.43	5.0	64
$\text{D}+\text{D}\rightarrow\text{T}+\text{p}$	2.81×10^{-4}	3.3×10^{-2}	0.06	1250
$\text{D}+\text{D}\rightarrow{}^3\text{He}+\text{n}$	2.78×10^{-4}	3.7×10^{-2}	0.11	1750
$\text{T}+\text{T}\rightarrow\alpha+2\text{n}$	7.90×10^{-4}	3.4×10^{-2}	0.16	1000
$\text{D}+{}^3\text{He}\rightarrow\alpha+\text{p}$	2.2×10^{-7}	0.1	0.9	250
$\text{p}+{}^6\text{Li}\rightarrow\alpha+{}^3\text{He}$	6×10^{-10}	7×10^{-3}	0.22	1500
$\text{p}+{}^{11}\text{B}\rightarrow 3\alpha$	(4.6×10^{-17})	3×10^{-4}	1.2	550
$\text{p}+\text{p}\rightarrow\text{D}+\text{e}^++\text{v}$	(3.6×10^{-26})	(4.4×10^{-25})		
$\text{p}+{}^{12}\text{C}\rightarrow{}^{13}\text{N}+\gamma$	(1.9×10^{-26})	2.0×10^{-10}	1.0×10^4	400
${}^{12}\text{C}+{}^{12}\text{C}$ (all branches)		(5.0×10^{-103})		

- “()” are theoretical values while others are measured values.

Nuclear fusion and fission release energy through energetic neutrons



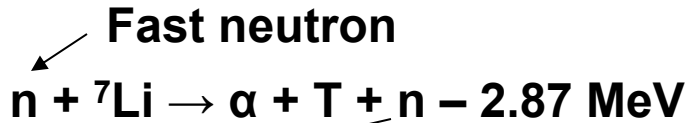
Nuclear fusion provides more energy per atomic mass unit (amu) than nuclear fission



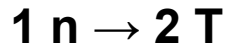
- Fission of ^{235}U : $\frac{Q}{A} = \frac{200 \text{ MeV}}{(235 + 1)\text{amu}} = 0.85 \text{ MeV/amu}$
- Fusion of D+T: $\frac{Q}{A} = \frac{17.6 \text{ MeV}}{(2 + 3)\text{amu}} = 3.5 \text{ MeV/amu}$

	Half-life (years)
U235	7.04×10^8
U238	4.47×10^9
...	
Tritium	12.3

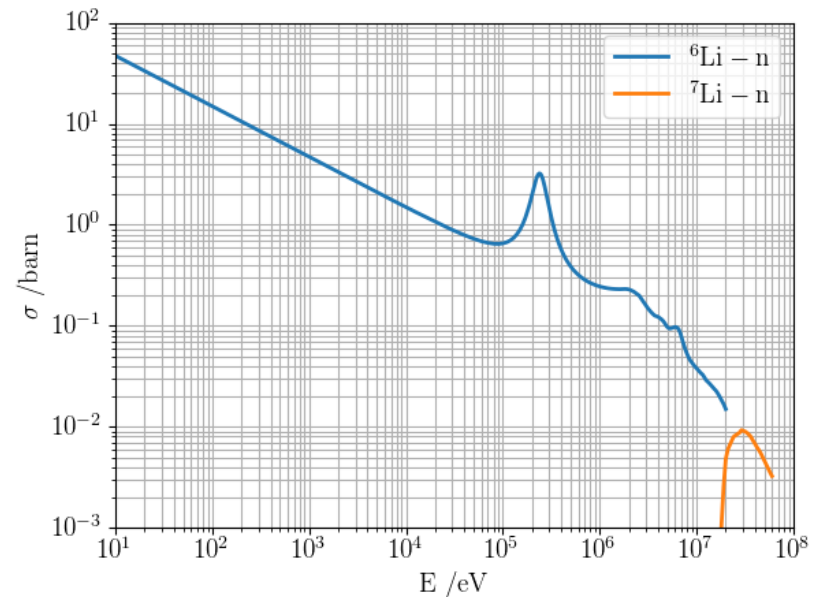
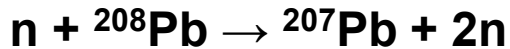
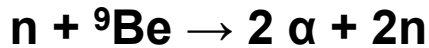
Tritium breeding



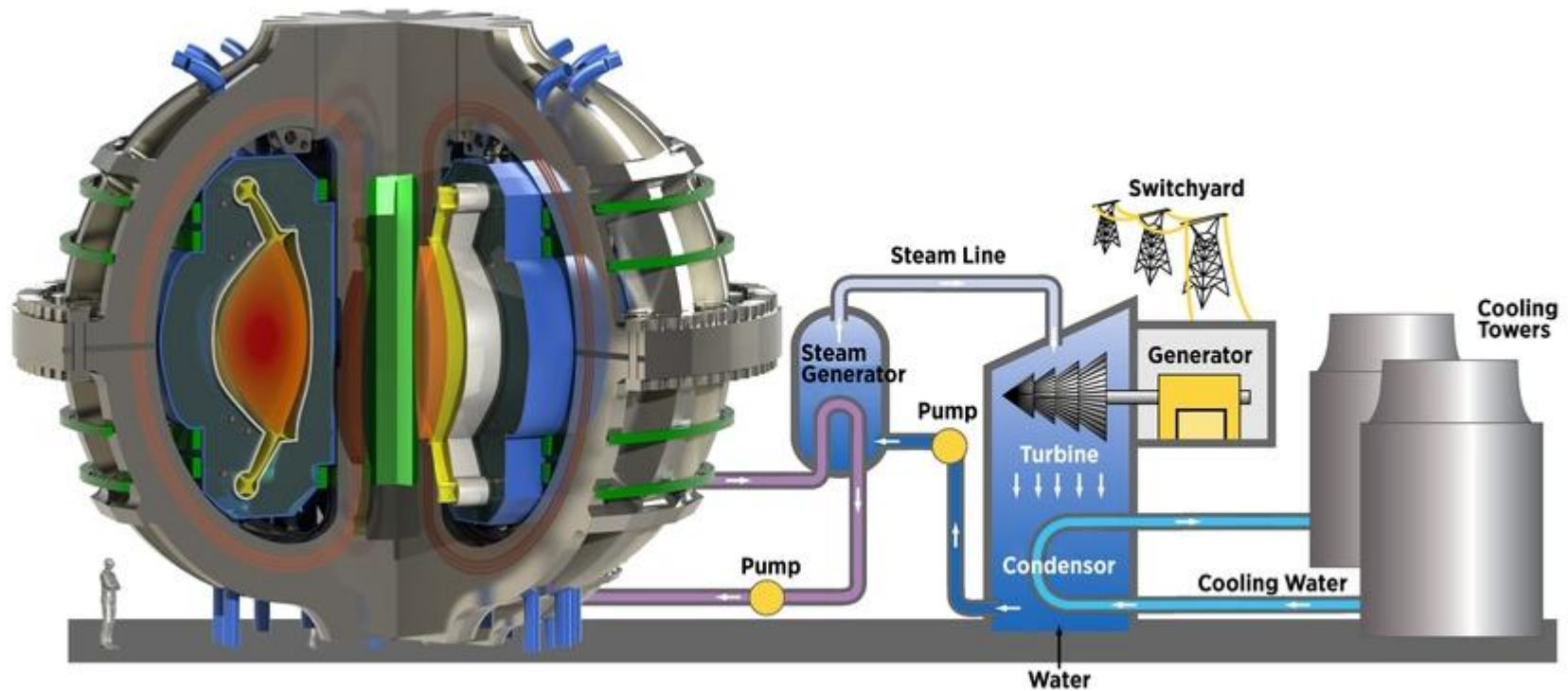
Slow neutron



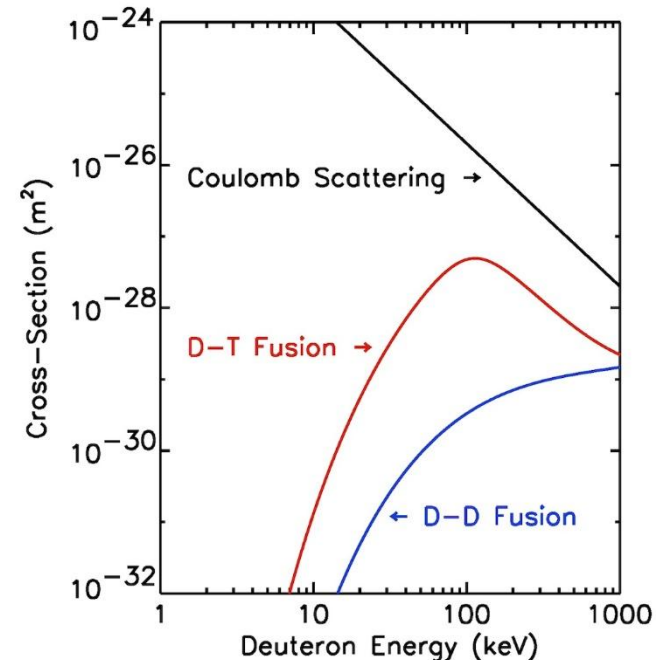
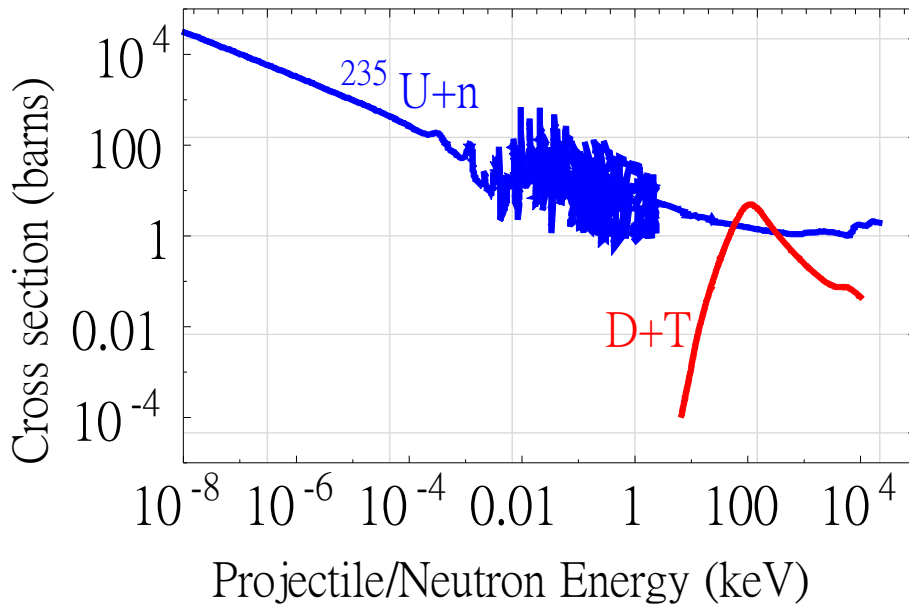
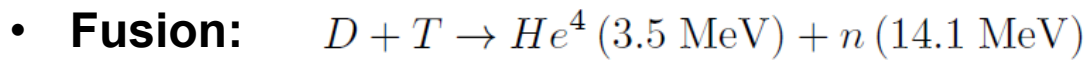
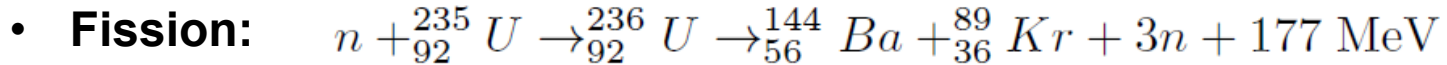
- Neutron multiplier



Nuclear fusion power plant proposed by Commonwealth Fusion Systems (CFS)



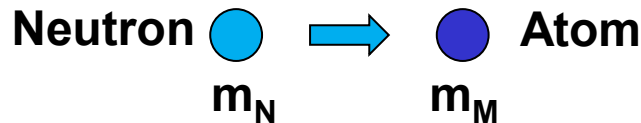
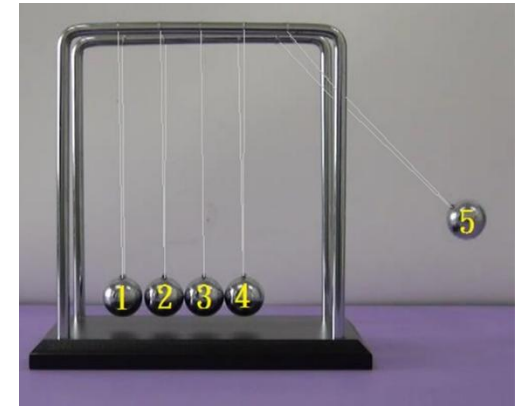
Fusion is much harder than fission



Fast neutrons are slowed down due to the collisions



- A moderator is used to slow down fast neutrons but not to absorb neutrons.
- For $m_M \sim m_N$, the energy decrement is higher. Therefore, H slows down neutron most efficiently.
- However, $H + n \rightarrow D$, i.e., H absorbs neutrons.



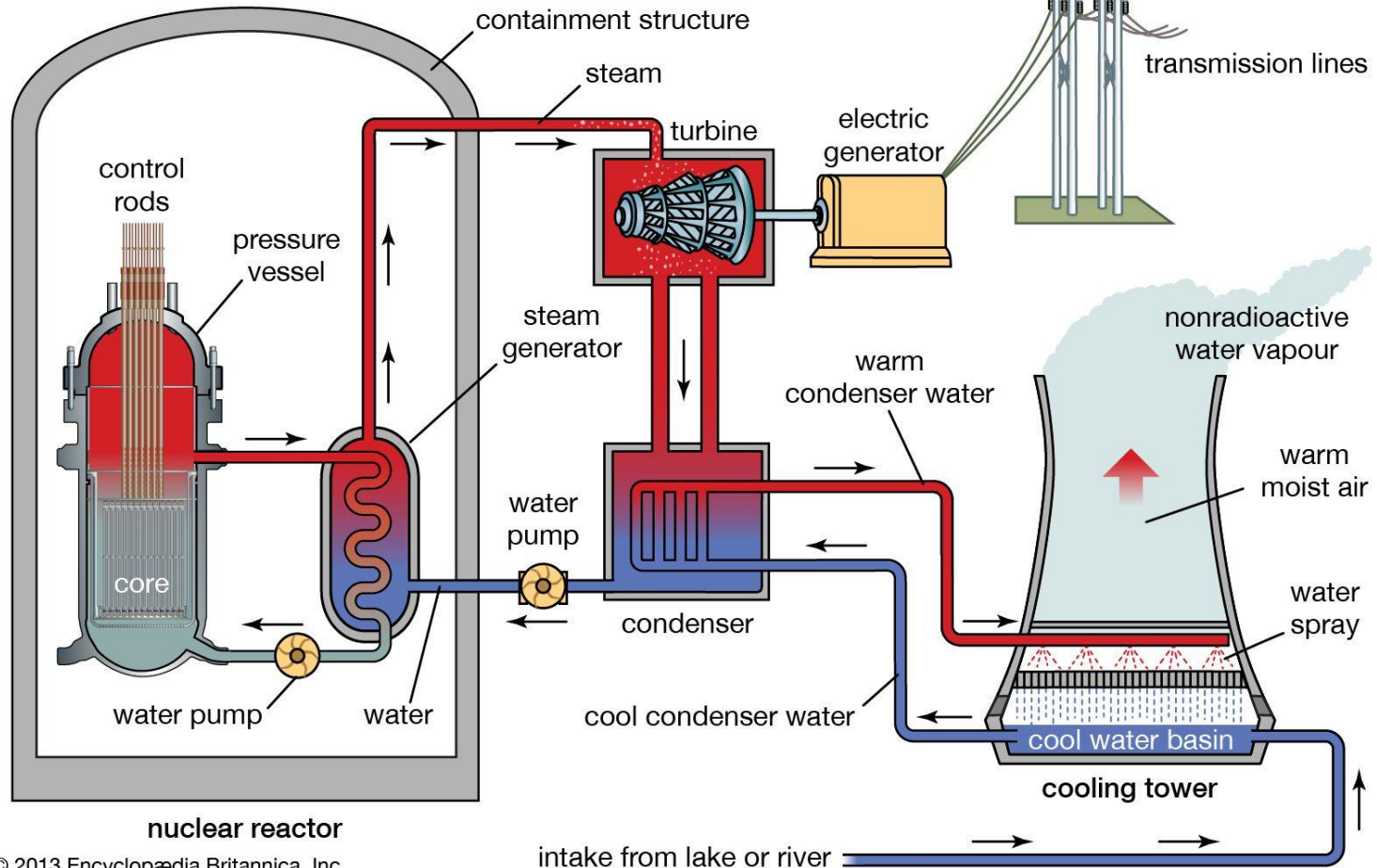
- The best option is the D in the heavy water (D_2O).

	Energy decrement	Neutron scattering cross section (σ_s) (Barns)	Neutron absorption cross section (σ_a) (Barns)
H	1	49 (H_2O)	0.66 (H_2O)
D	0.7261	10.6 (D_2O)	0.0013 (D_2O)
C	0.1589	4.7 (Graphite)	0.0035 (Graphite)

Nuclear power plant



Nuclear power plant



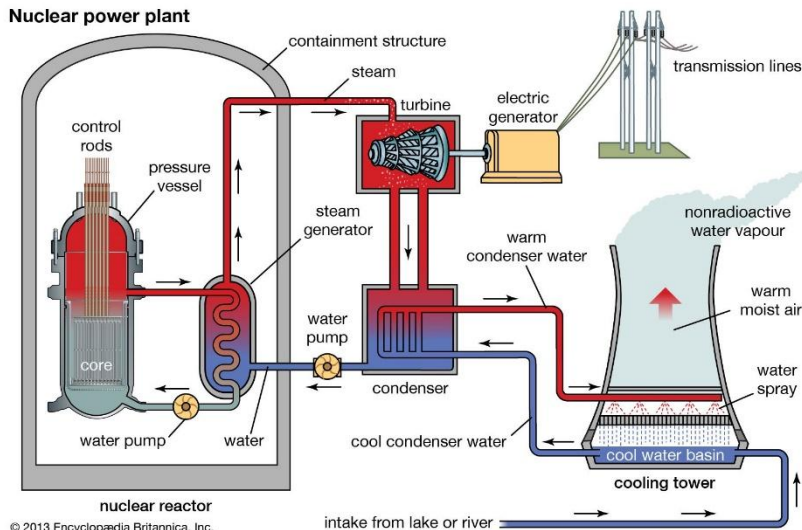
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Comparison between nuclear fission and nuclear fusion



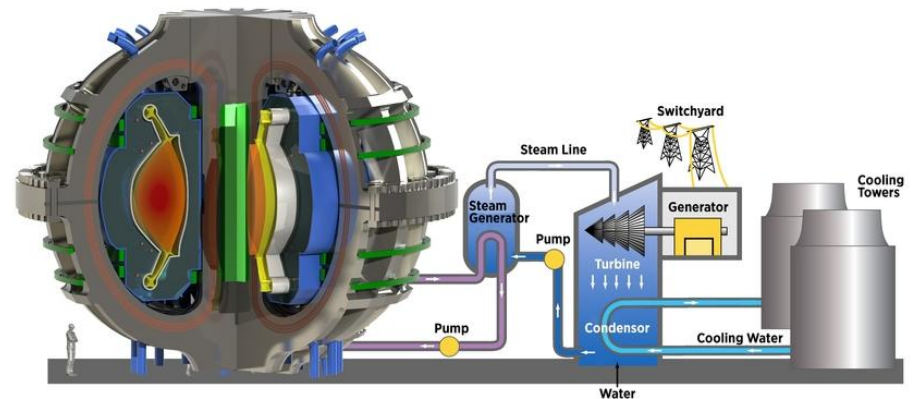
	Nuclear Fission	Nuclear Fusion
Chain reaction	Yes	No
Melt down	Possible	Impossible
Nuclear waste	High radiative	Low radiative / None

• Nuclear fission power plant



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• Nuclear fusion power plant



<https://www.britannica.com/technology/nuclear-power>

<https://news.mit.edu/2018/nas-report-right-path-fusion-energy-1221>

A “hot plasma” at 100M °C is needed



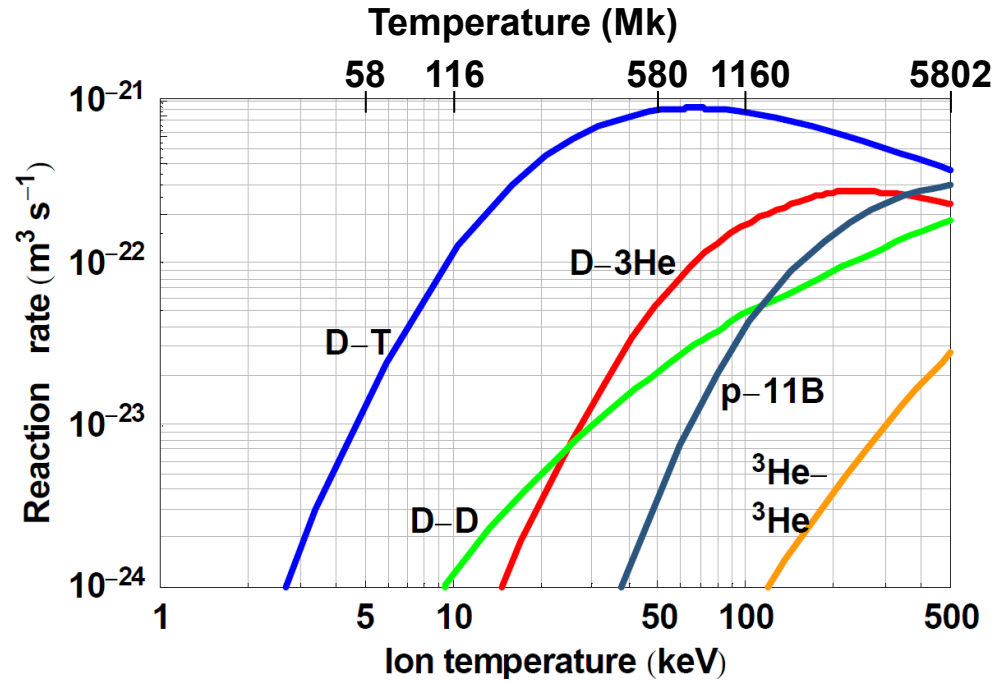
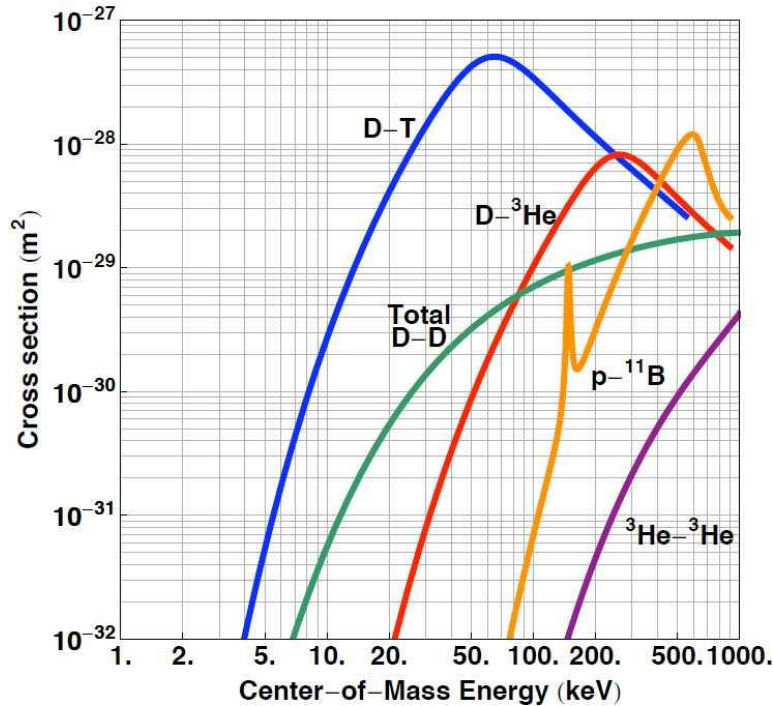
- Probability for fusion reactions to occur is low at low temperatures due to the coulomb repulsion force.



- If the ions are sufficiently hot, i.e., large random velocity, they can collide by overcoming coulomb repulsion



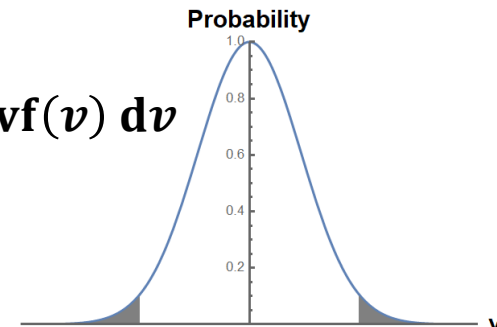
Fusion doesn't come easy



- The DT fusion reactivity is maximum at $T \approx 64$ keV
- @ $T = 10$ keV, $\langle \sigma v \rangle_{DT} \approx 100 \langle \sigma v \rangle_{DD}$

- Reaction rate:

$$\langle \sigma v \rangle = \int \sigma(v) v f(v) dv$$



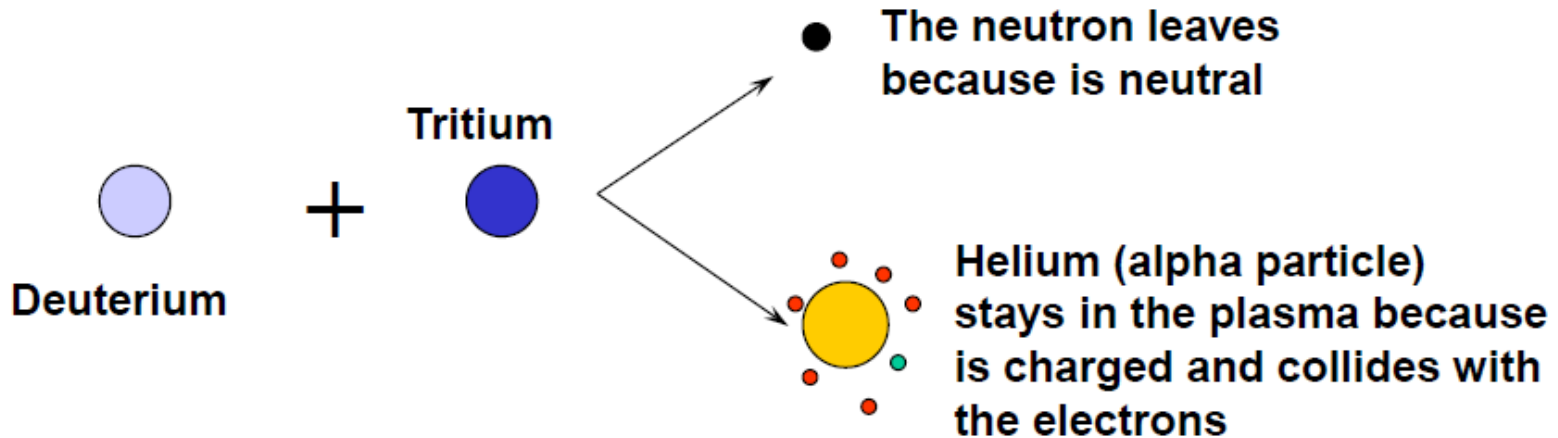
<https://i.stack.imgur.com/wXQD5.jpg>

Santarius, J. F., "Fusion Space Propulsion – A Shorter Time Frame Than You Think", JANNAF, Monterey, 5-8 December 2005.

It takes a lot of energy or power to keep the plasma at 100M °C



- Let the plasma do it itself!



- The α -particles heat the plasma.

Under what conditions the plasma keeps itself hot?



- **Steady state 0-D power balance:**

$$S_{\alpha} + S_h = S_B + S_k$$

S_{α} : α particle heating

S_h : external heating

S_B : Bremsstrahlung radiation

S_k : heat conduction lost

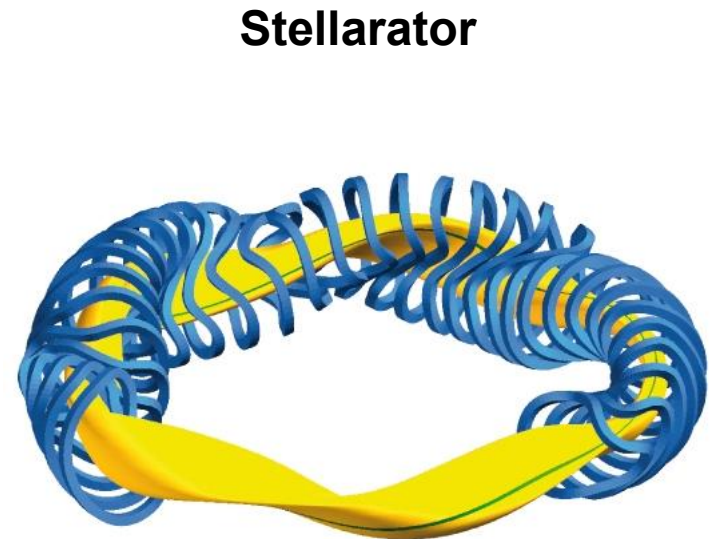
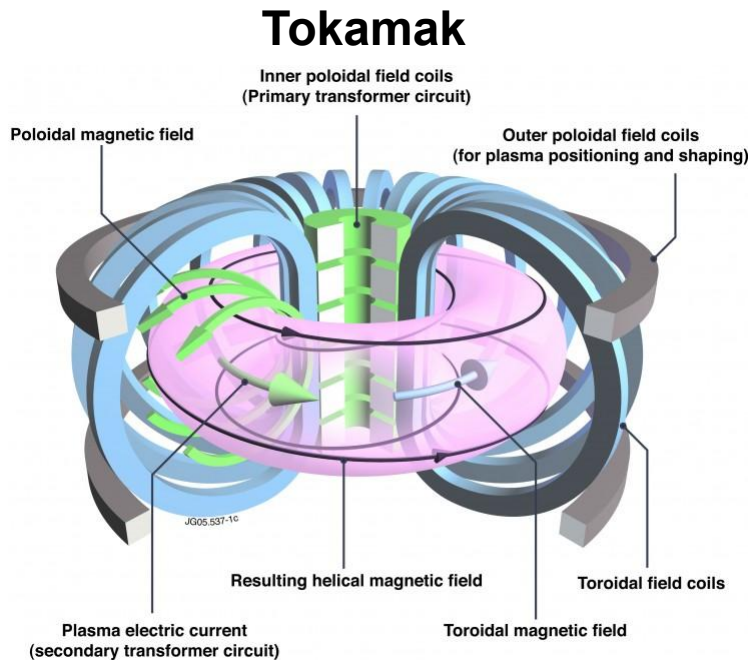
Ignition condition: $P\tau > 10 \text{ atm-s} = 10 \text{ Gbar} \cdot \text{ns}$

- **P: pressure, or called energy density**
- **τ is confinement time**

The plasma is too hot to be contained



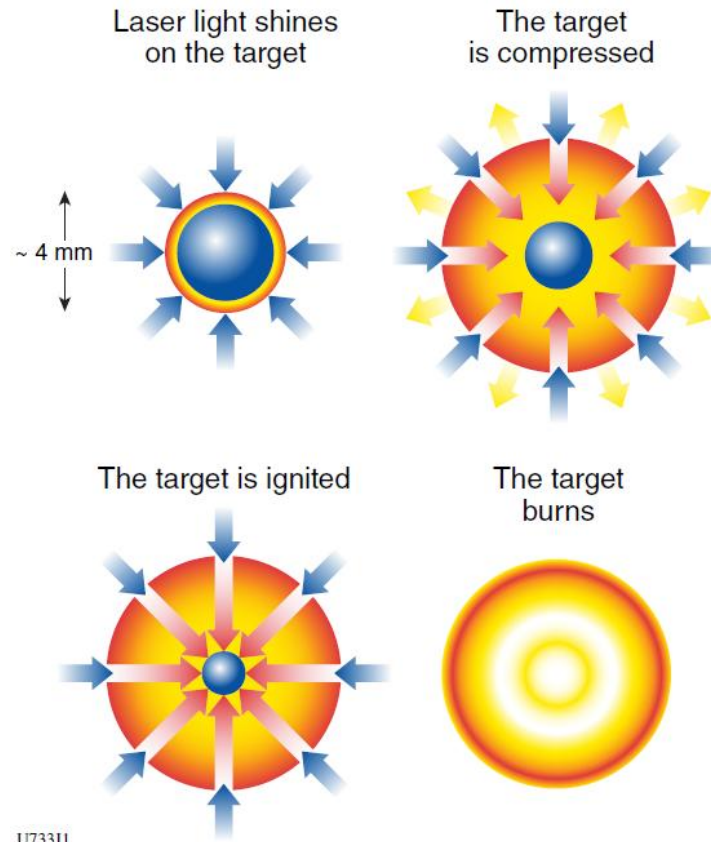
- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. $P \sim \text{atm}$, $\tau \sim \text{sec}$, $T \sim 10 \text{ keV}$ ($10^8 \text{ }^\circ\text{C}$)**



Don't confine it!



- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ ~nsec, T~10 keV (10^8 °C)**

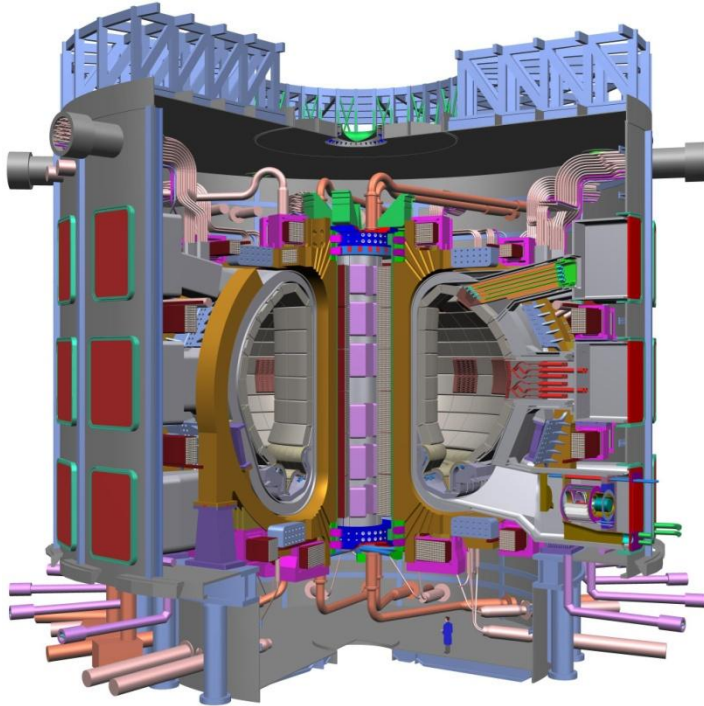


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To control? Or not to control?

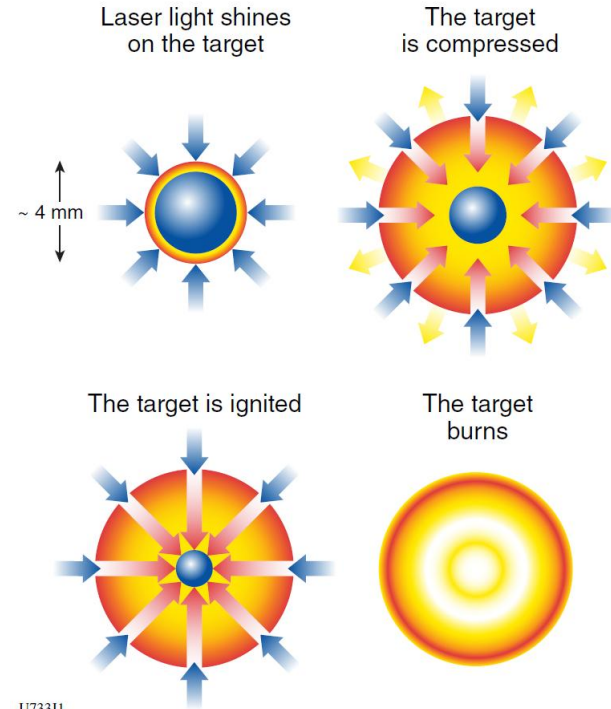


- **Magnetic confinement fusion (MCF)**



- **Plasma is confined by toroidal magnetic field.**

- **Inertial confinement fusion (ICF)**



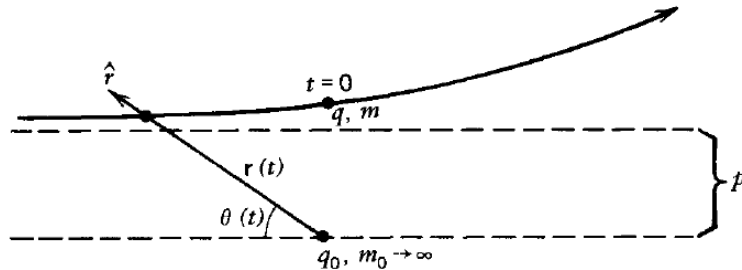
- **A DT ice capsule filled with DT gas is imploded by laser.**

Course Outline



- **Introduction to nuclear fusion (cont.)**
 - **Collisions (Bremsstrahlung radiation)**
 - **Columb scattering. Cross section of the Columb scattering**
 - **Beam-target fusion vs thermonuclear fusion**
 - **Lawson criteria, ignition conditions**
 - **Magnetic confinement fusion (MCF) vs Inertial confinement fusion (ICF)**

Charged particles collide with each other through coulomb collisions



$$m v_{\perp} = \int_{-\infty}^{\infty} dt F_{\perp}(t)$$

- Coulomb force:

$$m \ddot{\vec{r}} = \frac{qq_0}{r^2} \hat{r}$$

$$F_{\perp} = \frac{qq_0}{p^2} \sin^3 \theta$$

- Relation between θ and t is

$$x = -r \cos \theta = -\frac{b \cos \theta}{\sin \theta} = v_0 t$$

- Therefore,

$$v_{\perp} = \frac{qq_0}{m v_0 p} \int_0^{\pi} d\theta \sin \theta = \frac{2qq_0}{m v_0 p} \equiv \frac{v_0 p_0}{p}$$

where $p_0 \equiv \frac{2qq_0}{m v_0^2}$

- Note that this is valid only when $v_{\perp} \ll v_0$, i.e., $p \gg p_0$.

Cumulative effect of many small angle collisions is more important than large angle collisions



- Consider a variable Δx that is the sum of many small random variables Δx_i , $i=1,2,3,\dots,N$,

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 + \dots + \Delta x_N = \sum_{i=1}^N \Delta x_i$$

- Suppose $\langle \Delta x_i \rangle = \langle \Delta x_i \Delta x_j \rangle_{i \neq j} = 0$

$$\langle (\Delta x)^2 \rangle = \left\langle \left(\sum_{i=1}^N \Delta x_i \right)^2 \right\rangle = \sum_{i=1}^N \langle (\Delta x_i)^2 \rangle = N \langle (\Delta x_i)^2 \rangle$$

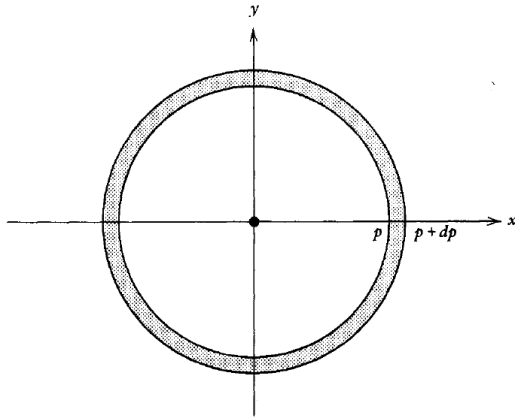
- For one collision:

$$\langle v_{\perp}^2 \rangle = \langle (\Delta v_x)^2 \rangle + \langle (\Delta v_y)^2 \rangle = \frac{v_0^2 p_0^2}{p^2} \quad \langle (\Delta v_x)^2 \rangle = \langle (\Delta v_y)^2 \rangle = \frac{1}{2} \frac{v_0^2 p_0^2}{p^2}$$

- The total velocity in \hat{x}

$$\langle (\Delta v_x^{\text{tot}})^2 \rangle = N \langle (\Delta v_x)^2 \rangle = \frac{N}{2} \frac{v_0^2 p_0^2}{p^2}$$

The collision frequency can be obtained by integrating all the possible impact parameter



- Number of collisions in a time interval:

$$dN = n_0 2\pi p dp v_0 dt$$

i.e., $\frac{dN}{dt} = 2\pi p dp n_0 v_0$

- Therefore

$$\begin{aligned} \frac{d}{dt} \langle (\Delta v_x^{\text{tot}})^2 \rangle &= \frac{1}{2} \frac{v_0^2 p_0^2}{p^2} \frac{dN}{dt} \\ &= \pi n_0 v_0^3 p_0^2 \frac{dp}{p} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \langle (\Delta_{\perp}^{\text{tot}})^2 \rangle &= 2 \frac{d}{dt} \langle (\Delta v_x^{\text{tot}})^2 \rangle \\ &= 2\pi n_0 v_0^3 p_0^2 \int_{p_{\min}}^{p_{\max}} \frac{dp}{p} \\ &= 2\pi n_0 v_0^3 p_0^2 \ln \left(\frac{p_{\max}}{p_{\min}} \right) \\ &\approx 2\pi n_0 v_0^3 p_0^2 \ln \left(\frac{\lambda_D}{|p_0|} \right) \\ &\approx 2\pi n_0 v_0^3 p_0^2 \ln \Lambda \end{aligned}$$

- Note that

$$\begin{aligned} \lambda_D &\approx \left(\frac{KT_e}{4\pi n_0 e^2} \right)^{1/2} \\ \frac{\lambda_D}{|p_0|} &\approx \frac{\lambda_D m_e v_e^2}{2e^2} \approx \frac{\lambda_D KT_e}{e^2} \approx 4\pi n_0 \lambda_D^3 \\ &\approx \Lambda \end{aligned}$$

Comparison between the mean free path and the system size L determines the regime of the plasma



- A reasonable definition for the scattering time due to small angle collisions is the time it takes $\langle (\Delta v_{\perp}^{\text{tot}})^2 \rangle$ to equal v_0^2 . The collision frequency ν_c due to small-angle collisions:

$$\frac{d}{dt} \langle (\Delta v_{\perp}^{\text{tot}})^2 \rangle \approx 2\pi n_0 v_0^3 p_0^2 \ln \Lambda \approx v_0^2 \nu_c, \quad p_0 \equiv \frac{2qq_0}{m_e v_0^2} \Rightarrow \nu_c = \frac{8\pi n_0 e^4 \ln \Lambda}{m_e^2 v_0^3}$$

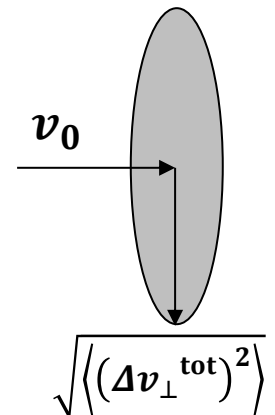
- With more careful derivation, the collisional time is obtained:

$$\tau_e^{-1} = \nu_c = \frac{4\sqrt{2\pi} n e^4 \ln \Lambda}{3\sqrt{m_e} (KT_e)^{3/2}}$$

- Mean free path:

$$l_{\text{mfp}} = v_e \tau_e$$

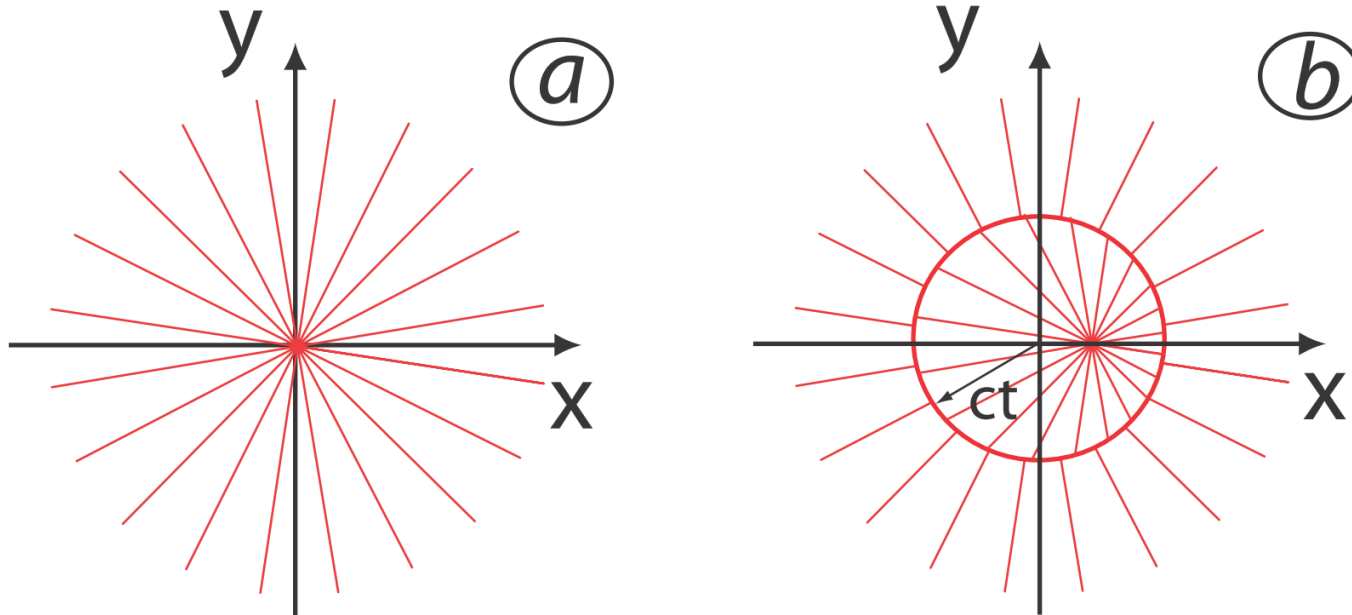
$$\left\{ \begin{array}{ll} l_{\text{mfp}} < L & \text{Fluid Theory} \\ l_{\text{mfp}} > L & \text{Kinetic Theory} \end{array} \right.$$



Electromagnetic wave is radiated when a charge particle is accelerated



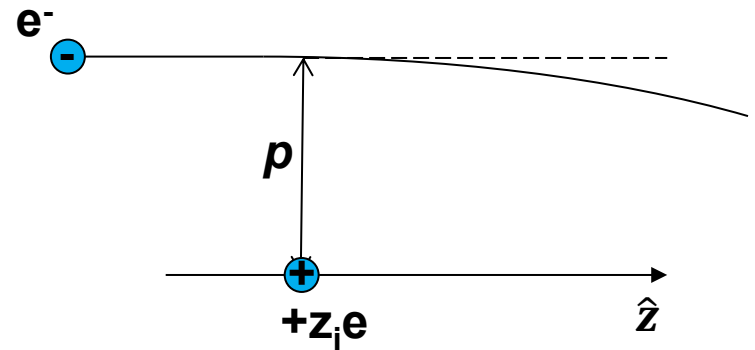
- The retarded potentials are the electromagnetic potentials for the electromagnetic field generated by time-varying electric current or charge distributions in the past.



Bremsstrahlung emission



- When an electron collides with a nucleus via coulomb interaction, the electron is accelerated and thus radiates, called Bremsstrahlung radiation.
- In the non-relativistic limit, the Bremsstrahlung radiation power is:



$$P_{B,e1,i1} = \frac{e^2}{6\pi\epsilon_0} \frac{\dot{v}^2}{c^3}$$

p : Impact parameter

- The coulomb force experienced by the electron is:

$$\dot{v} = \frac{F}{m_e} = \frac{ze^2}{4\pi\epsilon_0 m_e r^2} = \frac{ze^2}{4\pi\epsilon_0 m_e [p^2 + (vt)^2]} \approx \frac{ze^2}{4\pi\epsilon_0 m_e p^2}$$

$$\Rightarrow P_{B,e1,i1} = \frac{z^2 e^6}{96\pi^3 \epsilon_0^3 c^3 m_e^2} \frac{1}{p^4} \quad (\text{W})$$

Bremsstrahlung emission



- The electron begins to accelerate when it is about a distance p from the ion. It continues to accelerate until it travels a distance p away from the ion.

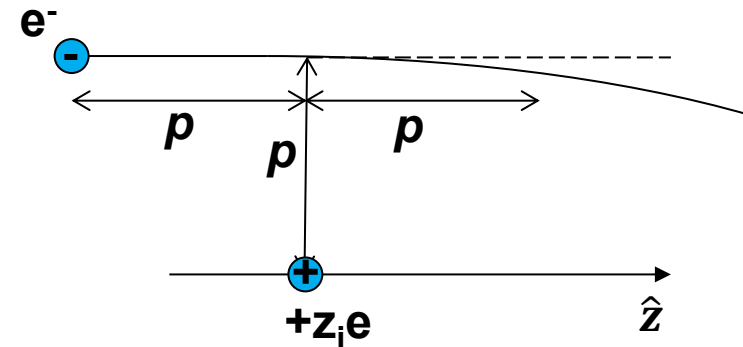
$$\Delta t = \frac{2p}{v}$$

- Therefore, the energy loss by one electron colliding one ion is:

$$E_{B,e1,i1} \approx P_{B,e1,i1} \Delta t = \frac{z^2 e^6}{48\pi^3 \epsilon_0^3 c^3 m_e^2} \frac{1}{vp^3} \quad (\text{J})$$

- With careful integration:

$$\begin{aligned} E_{B,e1,i1} &= \int_{-\infty}^{\infty} P_{B,e1,i1} dt = \frac{2z^2 e^6}{3(4\pi\epsilon_0)^3 m_e^2 c^3} \int_{-\infty}^{\infty} \frac{1}{[p^2 + (vt)^2]^2} dt \\ &= \frac{\pi z^2 e^6}{3(4\pi\epsilon_0)^3 m_e^2 c^3} \frac{1}{vp^3} \end{aligned}$$



Bremsstrahlung emission



- To consider the electron colliding with all ions with impact parameter p from 0 to ∞ and include the distribution function of ions $f_i(\vec{v}_i)$.

- Number of ions within the cylinder collided with the incident electron is:

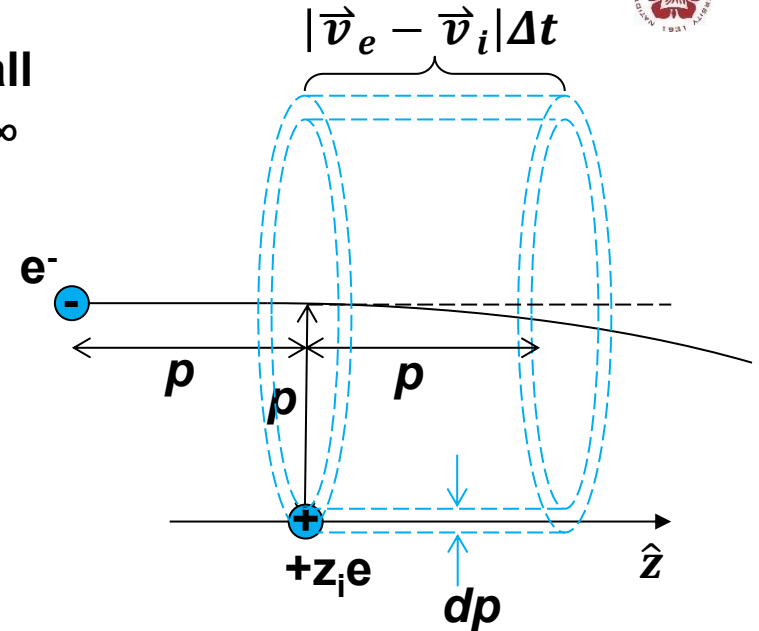
$$N_i = n_i 2\pi p dp |\vec{v}_e - \vec{v}_i| \Delta t$$

$$= \int \int_0^\infty f_i(\vec{v}_i) |\vec{v}_e - \vec{v}_i| \Delta t 2\pi p dp d\vec{v}_i$$

- The averaged radiation power is:

$$\bar{P}_{B,e1} = \frac{\bar{E}_{B,e1,i1} N_i}{\Delta t} = \int \int_0^\infty \frac{\bar{E}_{B,e1,i1} f_i(\vec{v}_i) |\vec{v}_e - \vec{v}_i| \Delta t 2\pi p dp d\vec{v}_i}{\Delta t}$$

$$= \int d\vec{v}_i \int_0^\infty \bar{E}_{B,e1,i1} |\vec{v}_e - \vec{v}_i| f_i(\vec{v}_i) 2\pi p dp$$



Bremsstrahlung emission



- Power of the electron colliding with all ions with impact parameter p from 0 to ∞ and include the distribution function of ions $f_i(\vec{v}_i)$:

$$\bar{P}_{B,e1} = \int d\vec{v}_i \int_0^\infty \bar{E}_{B,e1,i1} |\vec{v}_e - \vec{v}_i| f_i(\vec{v}_i) 2\pi p dp$$

- In addition, we need to consider the distribution function of electrons $f_e(\vec{v}_e)$.

The total power loss is:

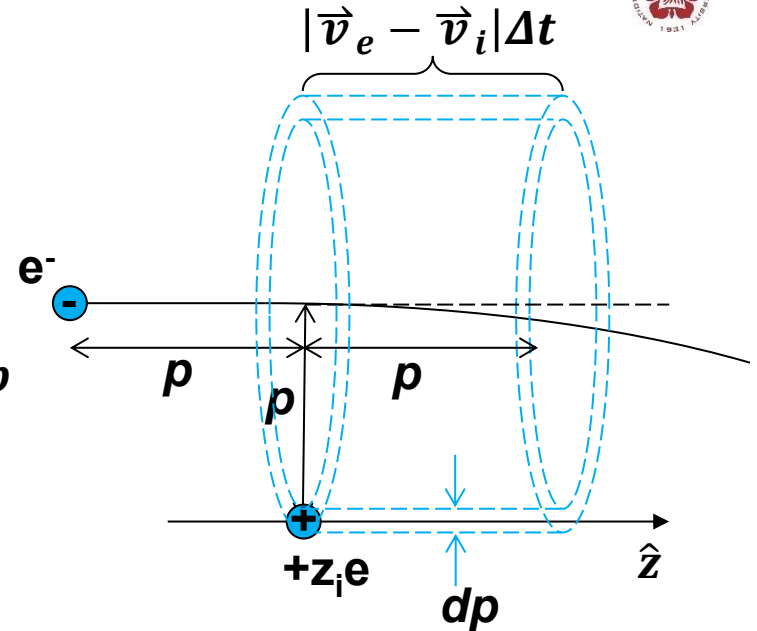
$$\bar{P}_B = \int d\vec{v}_i \int d\vec{v}_e \int_0^\infty \bar{E}_{B,e1,i1} |\vec{v}_e - \vec{v}_i| f_i(\vec{v}_i) f_e(\vec{v}_e) 2\pi p dp$$

- Since $|\vec{v}_e| \gg |\vec{v}_i|$, $|\vec{v}_e - \vec{v}_i| \approx v_e$.

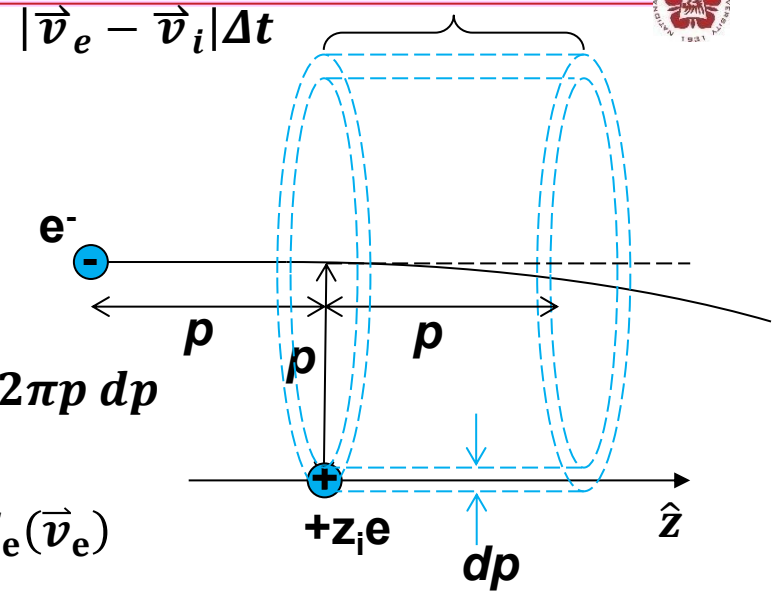
- In addition: $\int f_i(\vec{v}_i) d\vec{v}_i \equiv n_i$

$$d\vec{v}_e = dv_x dv_y dv_z = v_e^2 \sin\theta dv d\theta d\phi \rightarrow 4\pi v_e^2 dv_e$$

$$f_e = n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{m_e v_e^2}{2T_e} \right)$$



Bremsstrahlung emission



$$\begin{aligned}
 \bar{P}_B &= \int d\vec{v}_i \int d\vec{v}_e \int_0^\infty \bar{E}_{B,e1,i1} |\vec{v}_e - \vec{v}_i| f_i(\vec{v}_i) f_e(\vec{v}_e) 2\pi p dp \\
 &= 2\pi \int f_i(\vec{v}_i) d\vec{v}_i \int 4\pi v_e^2 dv_e \int_0^\infty p dp E_{B,e1,i1} v_e f_e(\vec{v}_e) \\
 &= 8\pi^2 n_i \int v_e^3 dv_e \int_0^\infty p dp E_{B,e1,i1} n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{m_e v_e^2}{2T_e} \right) \\
 &= 8\pi^2 n_i n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \int_0^\infty v_e^3 dv_e \int_0^\infty p dp \left(\frac{z^2 e^6}{48\pi^3 \epsilon_0^3 c^3 m_e^2 v_e p^3} \right) \exp\left(-\frac{m_e v_e^2}{2T_e} \right) \\
 &= 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_0^3 c^3 m_e^2} \right) \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \int_0^\infty v_e^2 \exp\left(-\frac{m_e v_e^2}{2T_e} \right) dv_e \int_0^\infty \frac{dp}{p^2}
 \end{aligned}$$

Bremsstrahlung emission



- Notice that we are using classical physics. We are not taking account of quantum effects which happen on a length scale of deBroglie wavelength $\Delta x = \hbar/(m_e v)$. Therefore, we have $p_{\min} = \hbar/(m_e v)$.

$$\int_0^{\infty} \frac{dp}{p^2} \rightarrow \int_{p_{\min}}^{\infty} \frac{dp}{p^2} = \frac{1}{p_{\min}} = \frac{m_e v_e}{\hbar} = \frac{2\pi m_e v_e}{h}$$

$$\begin{aligned} \bar{P}_B &= 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_0^3 c^3 m_e^2} \right) \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \int_0^{\infty} v_e^2 \exp\left(-\frac{m_e v_e^2}{2T_e}\right) dv_e \int_0^{\infty} \frac{dp}{p^2} \\ &= 8\pi^2 n_i n_e \left(\frac{z^2 e^6}{48\pi^3 \epsilon_0^3 c^3 m_e^2} \right) \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \frac{2\pi m_e}{h} \int_0^{\infty} v_e^3 \exp\left(-\frac{m_e v_e^2}{2T_e}\right) dv_e \end{aligned}$$

- With $\int_0^{\infty} x^3 e^{-x^2} dx = \frac{1}{2}$, a better value: $\left(\frac{2^{1/2}}{3\pi^{5/2}} \right)$

$$\bar{P}_B = \left(\frac{2^{1/2}}{6\pi^{3/2}} \right) \left(\frac{e^6}{\epsilon_0^3 c^3 h m_e^{3/2}} \right) z^2 n_i n_e T_e^{1/2} \left(\frac{W}{m^3} \right)$$

Bremsstrahlung emission



- For multiple ion species: n_j, z_j

$$\begin{aligned}\bar{P}_B &= \left(\frac{2^{1/2}}{3\pi^{5/2}} \right) \left(\frac{e^6}{\epsilon_0^3 c^3 h m_e^{3/2}} \right) n_e T_e^{1/2} \sum_j z_j^2 n_{i,j} \left(\frac{W}{m^3} \right) \\ &= \left(\frac{2^{1/2}}{3\pi^{5/2}} \right) \left(\frac{e^6}{\epsilon_0^3 c^3 h m_e^{3/2}} \right) Z_{\text{eff}} n_e^2 T_e^{1/2} \left(\frac{W}{m^3} \right)\end{aligned}$$

where

$$Z_{\text{eff}} \equiv \frac{\sum_j z_j^2 n_j}{n_e} = \frac{\sum_j z_j^2 n_j}{\sum_j z_j n_j} \quad n_e = \sum_j z_j n_j$$

$$\bar{P}_B = 5.35 \times 10^{-37} Z_{\text{eff}} n_{e(m^{-3})}^2 T_{e(\text{keV})}^{1/2} \left(\frac{W}{m^3} \right)$$

$$\bar{P}_B \equiv C_B Z_{\text{eff}} n_{e(m^{-3})}^2 T_{e(\text{keV})}^{1/2} \left(\frac{W}{m^3} \right)$$

Ignition condition (Lawson criterion) revision



- Steady state 0-D power balance:

$$S_{\alpha} + S_h = S_B + S_k$$

S_h : external heating

S_{α} : α particle heating



$$S_f = E_f n_1 n_2 \langle \sigma v \rangle (\text{W/m}^3) \quad n_D = n_T = \frac{1}{2} n$$

$$S_{\alpha} = \frac{1}{4} E_{\alpha} n^2 \langle \sigma v \rangle = \frac{1}{16} E_{\alpha} \frac{p^2 \langle \sigma v \rangle}{T^2}$$

$$E_{\alpha} = 3.5 \text{ MeV}$$

$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

S_B : Bremsstrahlung radiation

$$S_B = C_B Z_{\text{eff}} n_e^2 (\text{m}^{-3}) T_e^{1/2} (\text{keV}) \left(\frac{\text{W}}{\text{m}^3} \right)$$

$$= \frac{1}{4} C_B \frac{p^2}{T^{3/2}}$$

S_k : heat conduction lost

$$S_k = \frac{3}{2} \frac{p}{\tau}$$

$$\frac{1}{16} E_{\alpha} \frac{p^2 \langle \sigma v \rangle}{T^2} \geq \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

Ignition condition (Lawson criterion) revision



- Steady state 0-D power balance:

$$S_{\alpha} + S_h = S_B + S_k$$

$$\frac{1}{16} E_{\alpha} \frac{p^2 \langle \sigma v \rangle}{T^2} \geq \frac{1}{4} C_B \frac{p^2}{T^{3/2}} + \frac{3}{2} \frac{p}{\tau}$$

$$p\tau \geq \frac{6}{\frac{1}{4} E_{\alpha} \frac{\langle \sigma v \rangle}{T^2} - C_B \frac{1}{T^{3/2}}}$$

$$p = p_e + p_i = 2p_e = 2n_e T \equiv 2nT$$

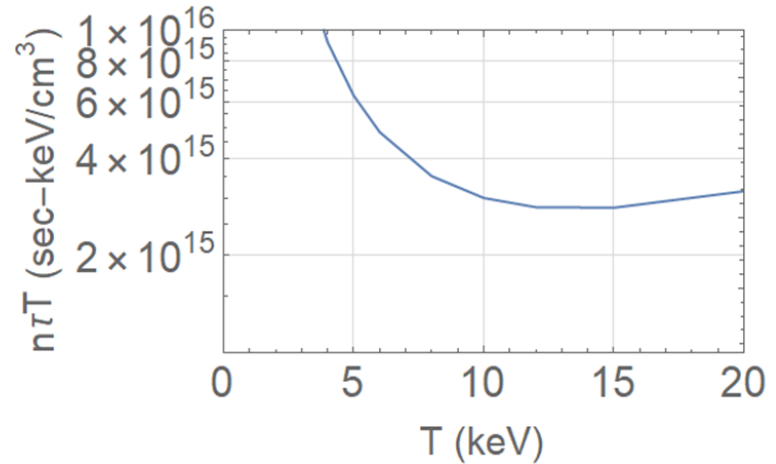
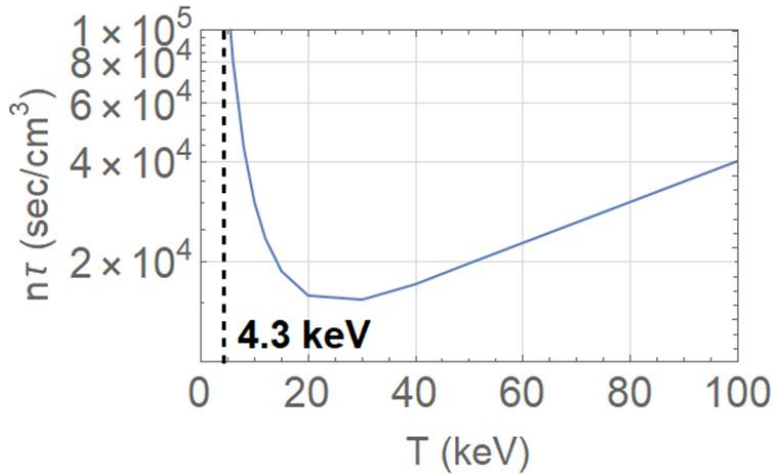
$$nT\tau > \frac{3T^2}{\frac{1}{4} \langle \sigma v \rangle E_{\alpha} - C_B \sqrt{T}}$$

$$n\tau > \frac{3T}{\frac{1}{4} \langle \sigma v \rangle E_{\alpha} - C_B \sqrt{T}}$$

Temperature needs to be greater than ~5 keV to ignite



$$n\tau > \frac{3T}{\frac{1}{4}\langle\sigma v\rangle E_\alpha - C_B\sqrt{T}}$$

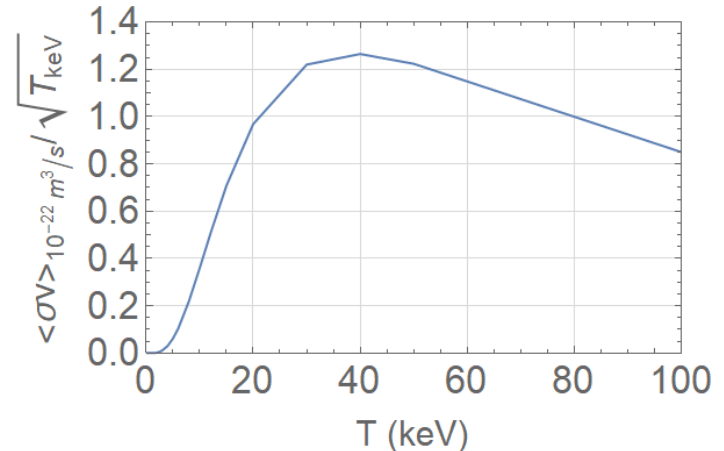


$$n\tau > 2 \times 10^4 \text{ sec/cm}^3$$

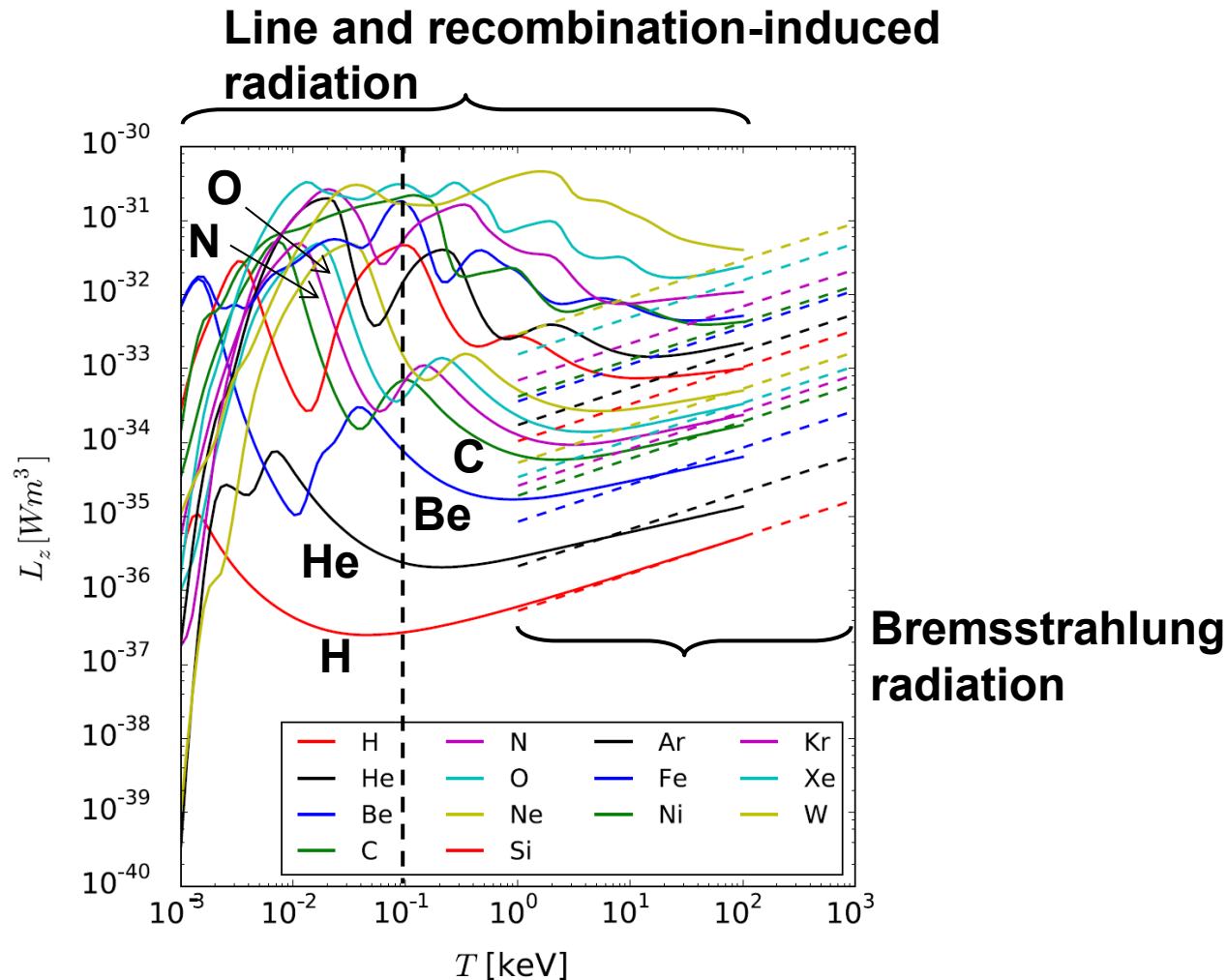
$$nT\tau > 3.5 \times 10^{15} \text{ keV - sec/cm}^3$$

$$S_\alpha > S_B \quad \frac{1}{4} E_\alpha n^2 \langle\sigma v\rangle > C_B n^2 T^{1/2}$$

$$\frac{\langle\sigma v\rangle}{T^{1/2}} > \frac{4C_B}{E_\alpha} \quad T > 4.3 \text{ keV}$$



Temperature of 100 eV is the threshold of radiation barrier by impurities



Under what conditions the plasma keeps itself hot?



- **Steady state 0-D power balance:**

$$S_{\alpha} + S_h = S_B + S_k$$

S_{α} : α particle heating

S_h : external heating

S_B : Bremsstrahlung radiation

S_k : heat conduction lost

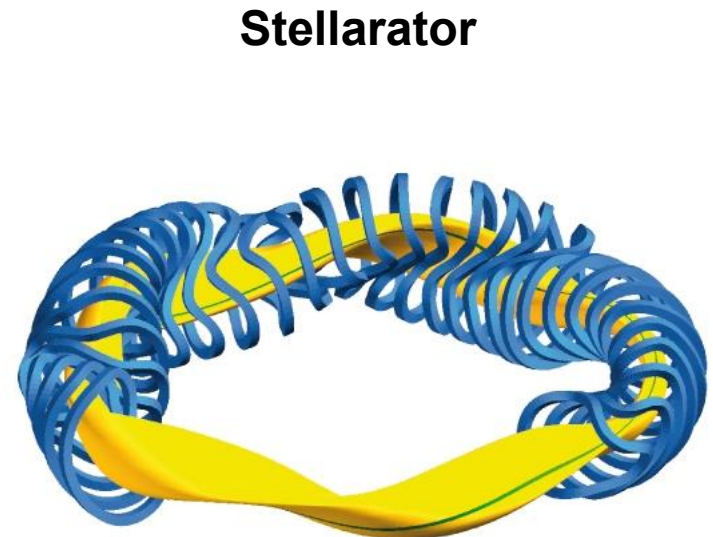
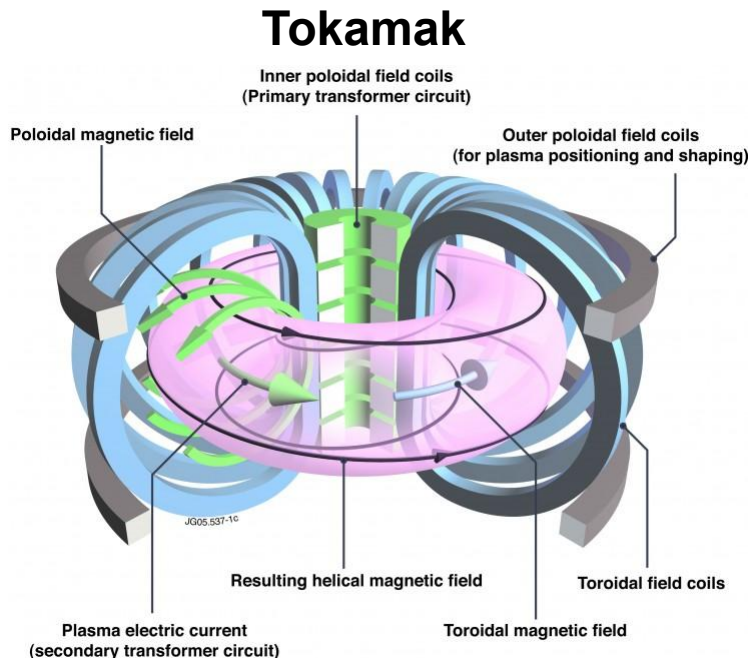
Ignition condition: $P\tau > 10 \text{ atm-s} = 10 \text{ Gbar - ns}$

- **P: pressure, or called energy density**
- **τ is confinement time**

The plasma is too hot to be contained



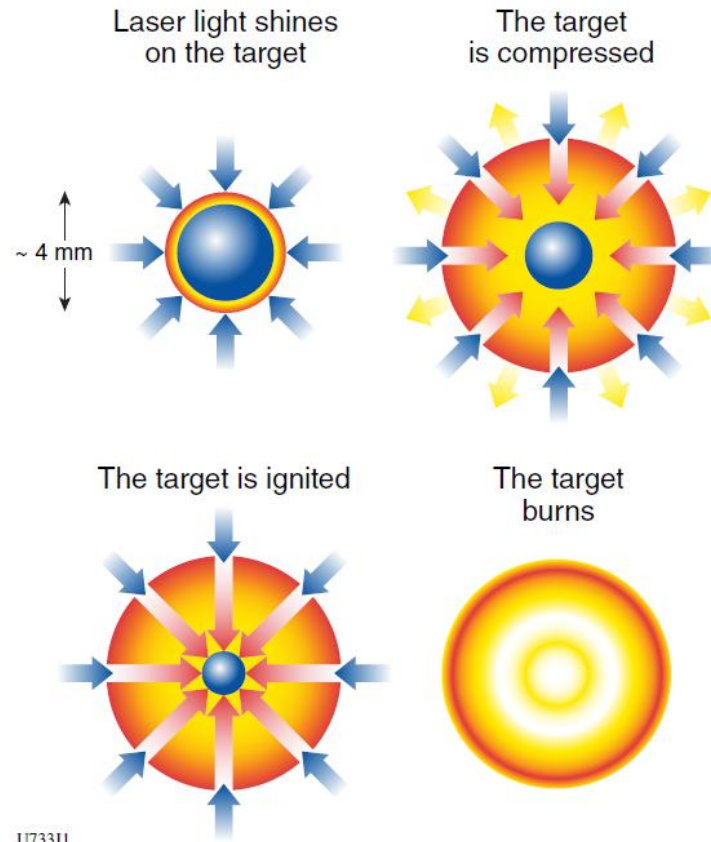
- **Solution 1: Magnetic confinement fusion (MCF), use a magnetic field to contain it. $P \sim \text{atm}$, $\tau \sim \text{sec}$, $T \sim 10 \text{ keV}$ ($10^8 \text{ }^\circ\text{C}$)**



Don't confine it!



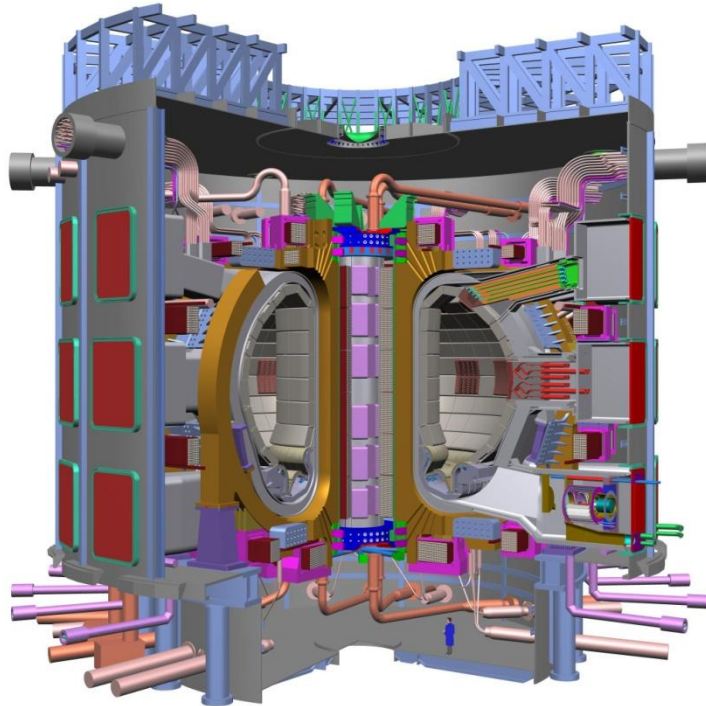
- **Solution 2: Inertial confinement fusion (ICF). Or you can say it is confined by its own inertia: P~Gigabar, τ ~nsec, T~10 keV (10^8 °C)**



To control? Or not to control?

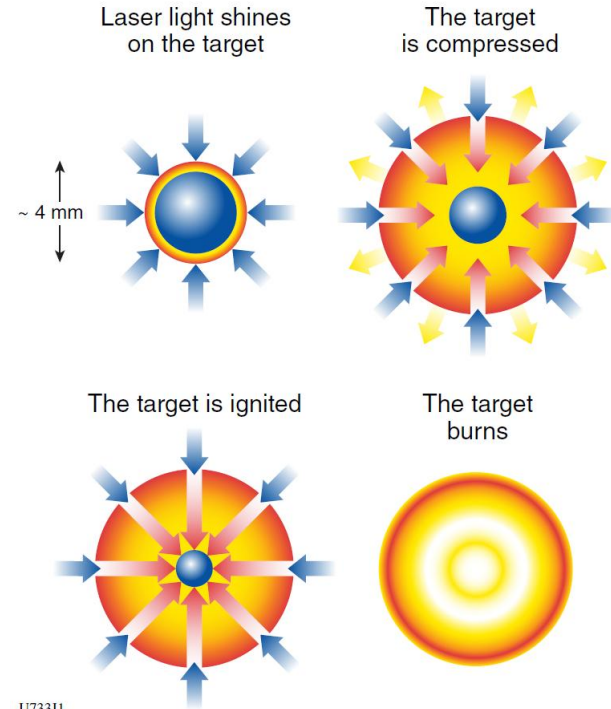


- **Magnetic confinement fusion (MCF)**



- Plasma is confined by toroidal magnetic field.

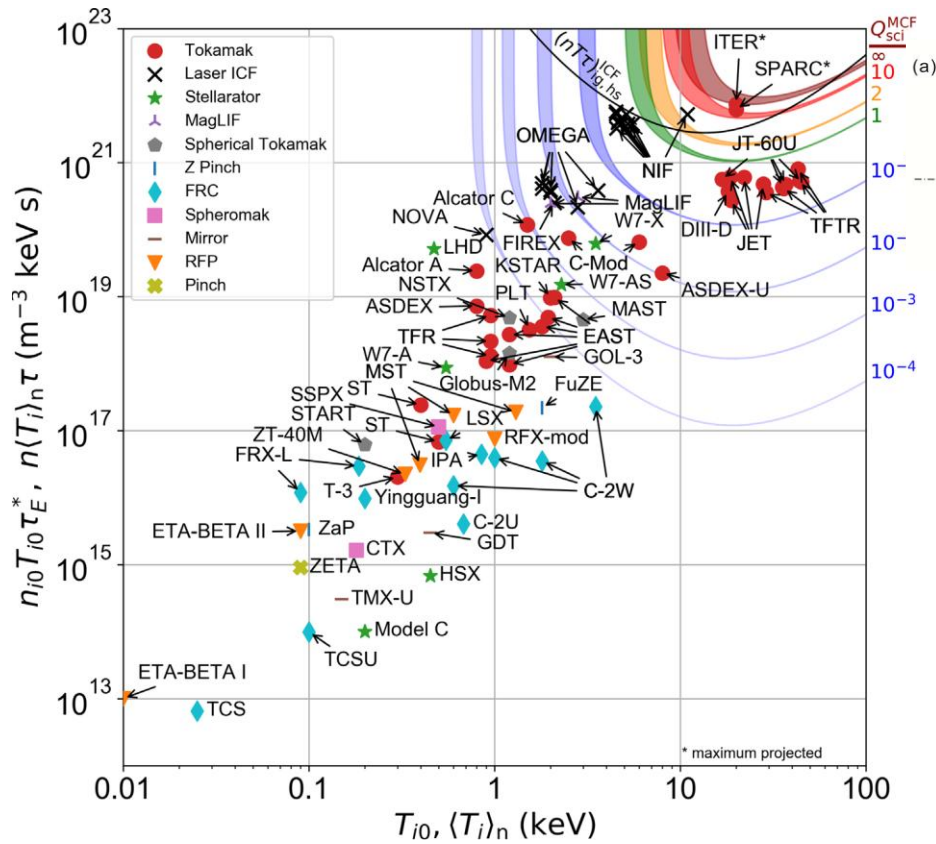
- **Inertial confinement fusion (ICF)**



- A DT ice capsule filled with DT gas is imploded by laser.

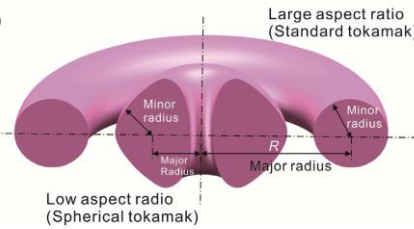
Laboratory for Laser Energetics, University of Rochester is a pioneer in laser fusion

We are closed to ignition!

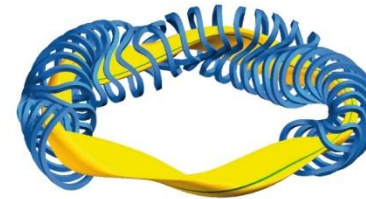


MCF

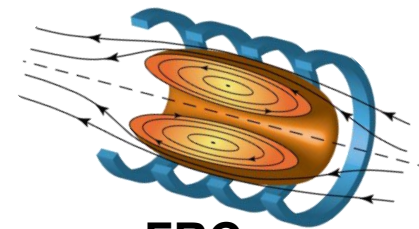
ICF



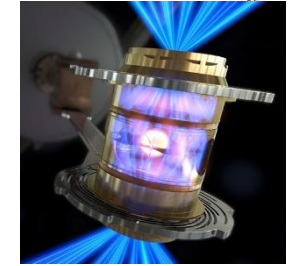
(Spherical) tokamak



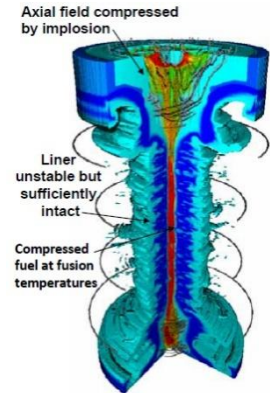
Stellarator



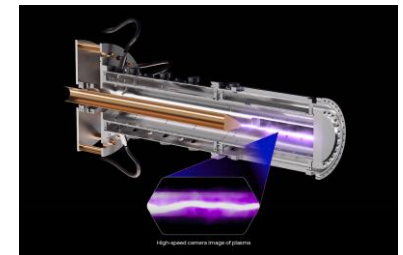
FRC



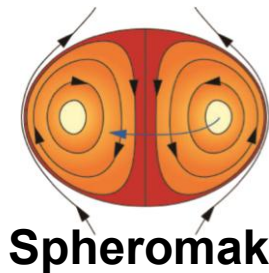
Laser ICF



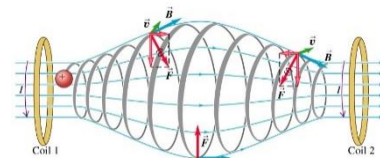
MagLIF



Sheared-flow Z pinch

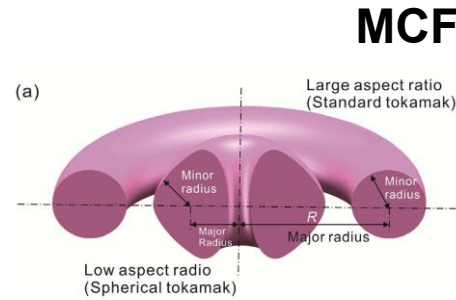
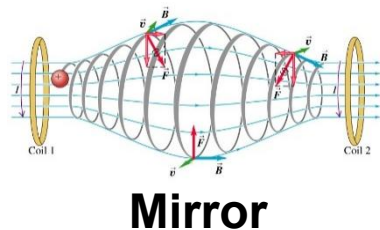
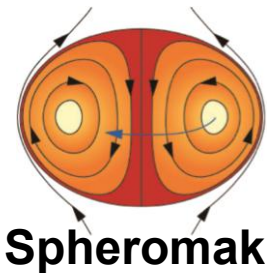
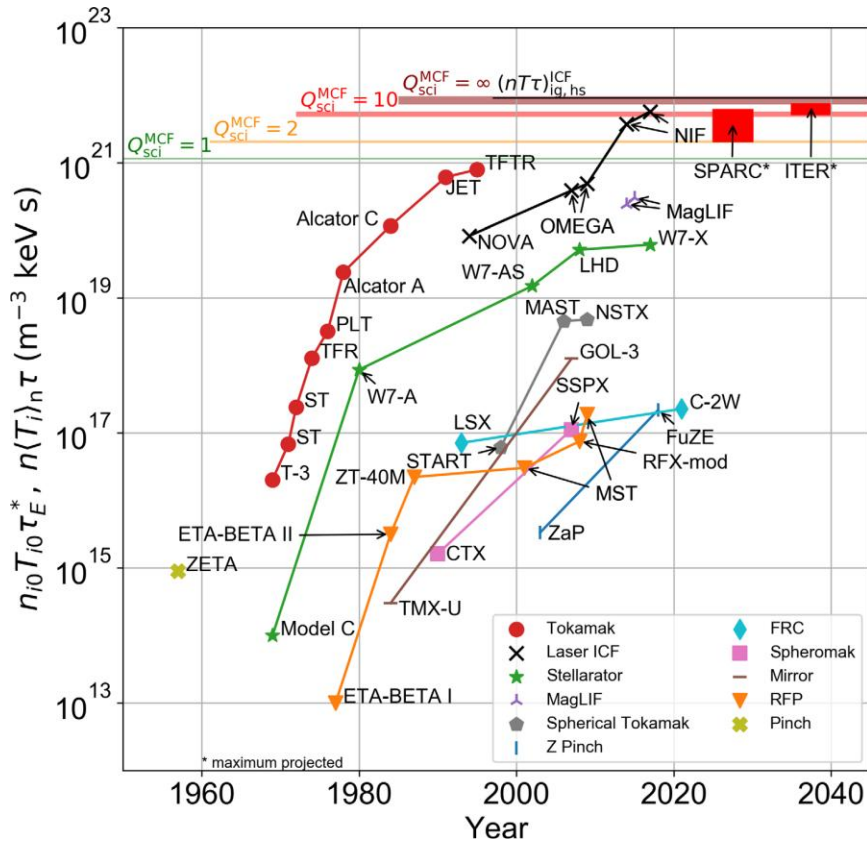


Spheromak



Mirror

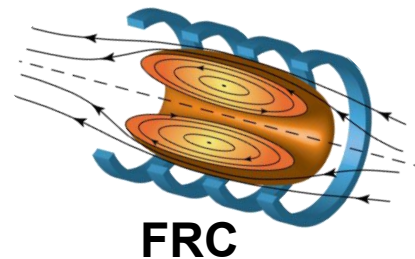
We are closed to ignition!



(Spherical) tokamak



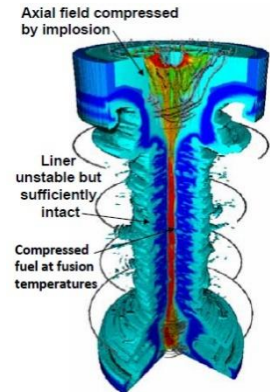
Stellerator



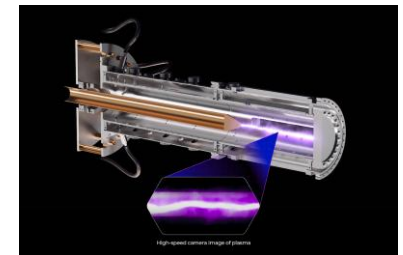
ICF



Laser ICF

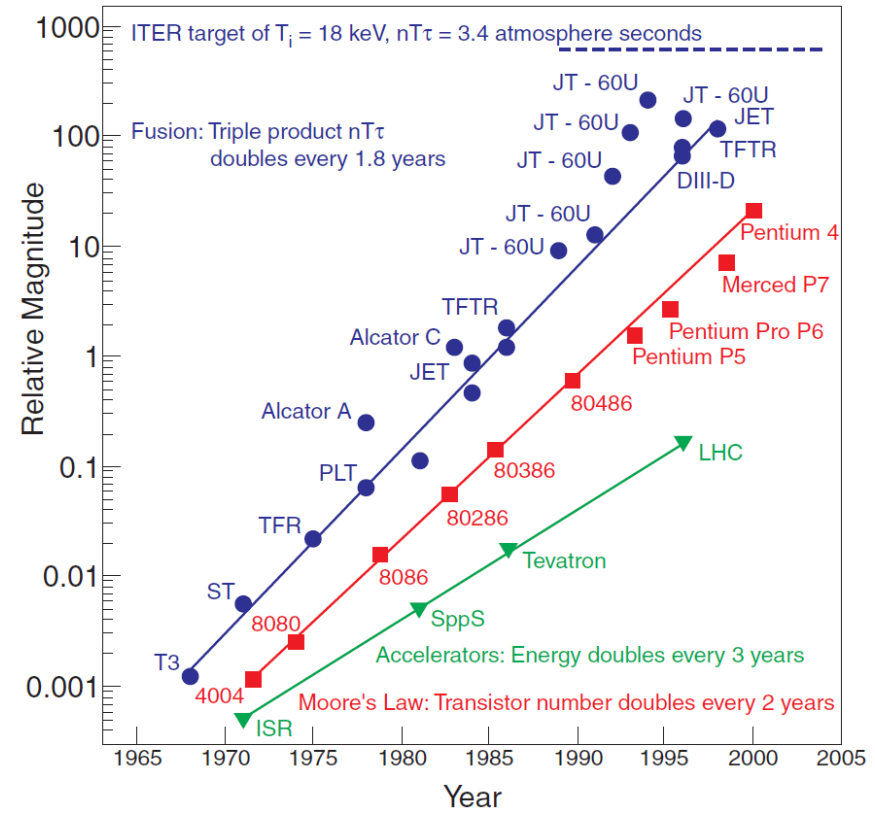
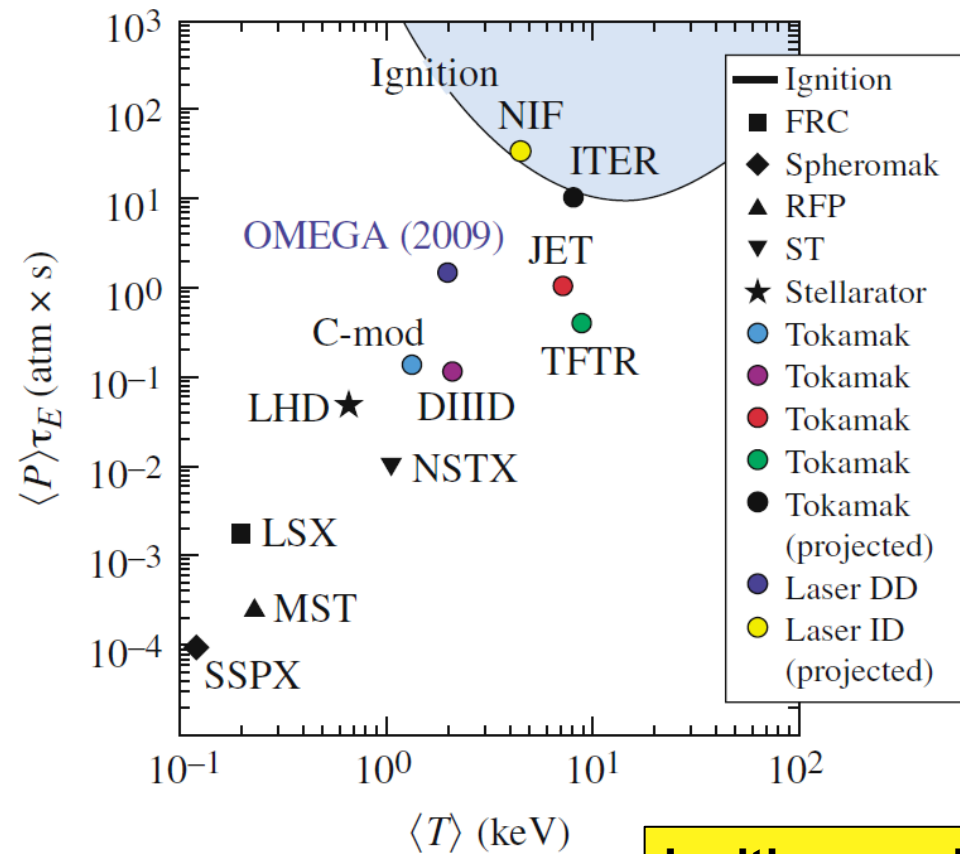


MagLIF



Sheared-flow Z pinch

We are closed to ignition!



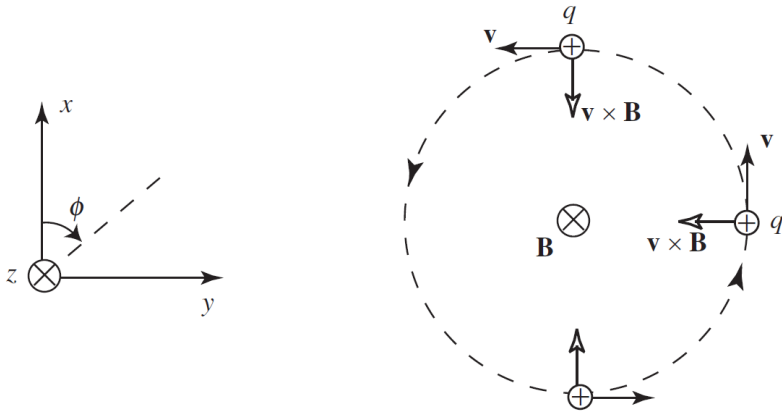
Ignition condition: $P\tau > 10$ atm-s = 10 Gbar - ns

Course Outline



- **Magnetic confinement fusion (MCF)**
 - Gyro motion, MHD
 - 1D equilibrium (z pinch, theta pinch)
 - Drift: ExB drift, grad B drift, and curvature B drift
 - Tokamak, Stellarator (toroidal field, poloidal field)
 - Magnetic flux surface
 - 2D axisymmetric equilibrium of a torus plasma: Grad-Shafranov equation.
 - Stability (Kink instability, sausage instability, Safety factor Q)
 - Central-solenoid (CS) start-up (discharge) and current drive
 - CS-free current drive: electron cyclotron current drive, bootstrap current.
 - Auxiliary Heating: ECRH, Ohmic heating, Neutral beam injection.

Charged particles gyro around the magnetic field line



$$m \frac{d\vec{v}}{dt} = q \vec{v} \times \vec{B}$$

- Assuming $\vec{B} = B\hat{z}$ and the electron oscillates in x-y plane

$$m\dot{v}_x = qBv_y$$

$$m\dot{v}_y = -qBv_x$$

$$m\dot{v}_z = 0 \quad v_z = v_{||} = \text{constant}$$

$$\ddot{v}_x = -\frac{qB}{m} \dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\frac{qB}{m} \dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

$$\omega_c \equiv \frac{|q|B}{m} \quad \text{Cyclotron frequency or gyrofrequency}$$

$$\ddot{v}_x + \omega_c^2 v_x = 0$$

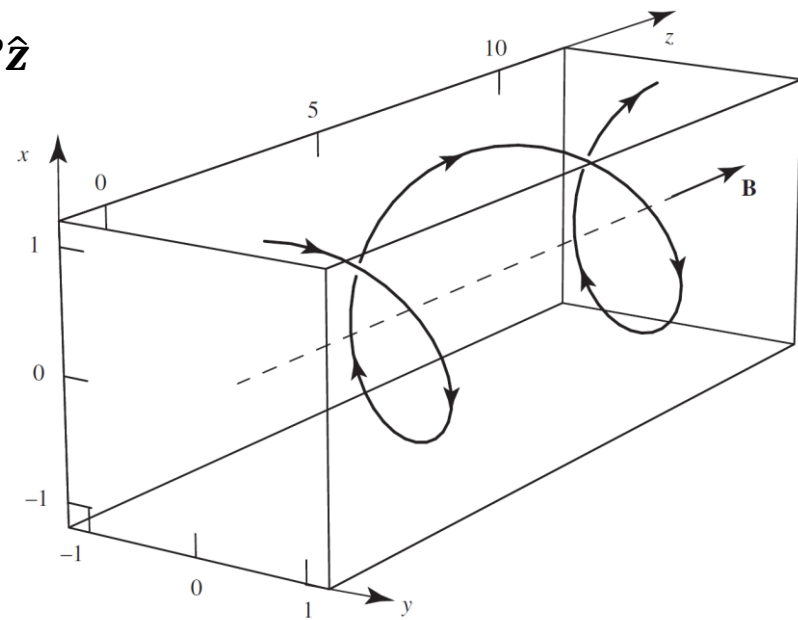
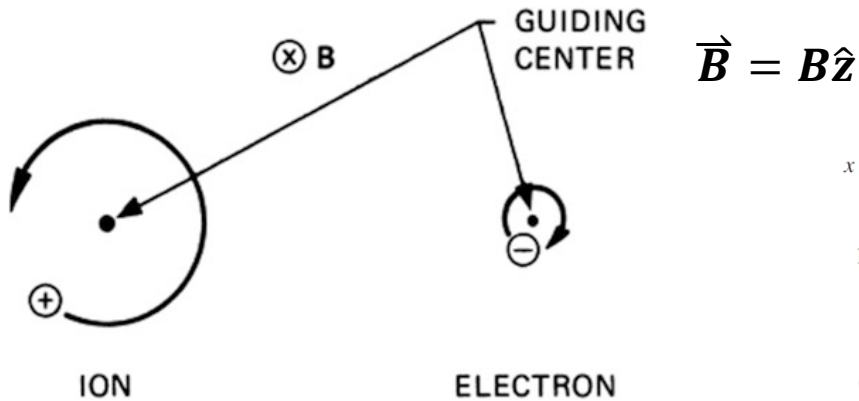
$$\ddot{v}_y + \omega_c^2 v_y = 0$$

$$v_x = v_{\perp} \cos(\pm\omega_c t + \psi)$$

$$v_y = -v_{\perp} \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{||}$$

Charged particles spiral around the magnetic field line



$$v_x = v_{\perp} \cos(\pm\omega_c t + \psi)$$

$$v_y = v_{\perp} \sin(\pm\omega_c t + \psi)$$

$$v_z = v_{\parallel}$$

$$\omega_c \equiv \frac{|q|B}{m}$$

$$\left| \frac{mv_{\perp}^2}{r} \right| = |q \vec{v} \times \vec{B}| = |qv_{\perp}B|$$

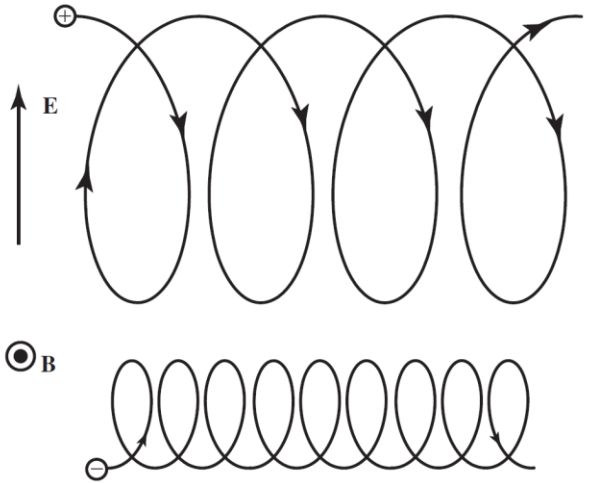
$$r_c = \frac{v_{\perp}}{\omega_c} = \frac{mv_{\perp}}{|q|B} \quad \text{Larmor radius or gyroradius}$$

$$x = \mp r_c \sin(\pm\omega_c t + \psi) + (x_0 - r_c \sin\psi)$$

$$y = \pm r_c \cos(\pm\omega_c t + \psi) + (y_0 + r_c \cos\psi)$$

$$z = z_0 + v_{\parallel} t$$

Charge particles drift across magnetic field lines when an electric field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_\perp + \hat{z}E_\parallel = \hat{x}E_\perp + \hat{z}E_\parallel$$

$$m \frac{dv_\parallel}{dt} = qE_\parallel$$

$$m \frac{d\vec{v}_\perp}{dt} = q(\hat{x}E_\perp + \vec{v}_\perp \times \hat{z}B)$$

$$v_\parallel(t) = \frac{qE_\parallel}{m}t + v_{\parallel,0}$$

$$\vec{v}_\perp(t) = \vec{v}_E + \vec{v}_{ac}(t)$$

$$m \frac{d}{dt} (\vec{v}_E + \vec{v}_{ac}(t)) = q[\hat{x}E_\perp + (\vec{v}_E + \vec{v}_{ac}(t)) \times \hat{z}B]$$

$$m \frac{d\vec{v}_{ac}(t)}{dt} = q[\hat{x}E_\perp + \vec{v}_E \times \hat{z}B + \vec{v}_{ac}(t) \times \hat{z}B]$$

No E field case: $m \frac{d\vec{v}}{dt} = q \vec{v} \times \vec{B}$



$$\hat{x}E_\perp + \vec{v}_E \times \hat{z}B = 0$$

$$\times \hat{z}B \quad (\vec{C} \times \vec{B}) \times \vec{A} = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$\vec{v}_E = \frac{\hat{x}E_\perp \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

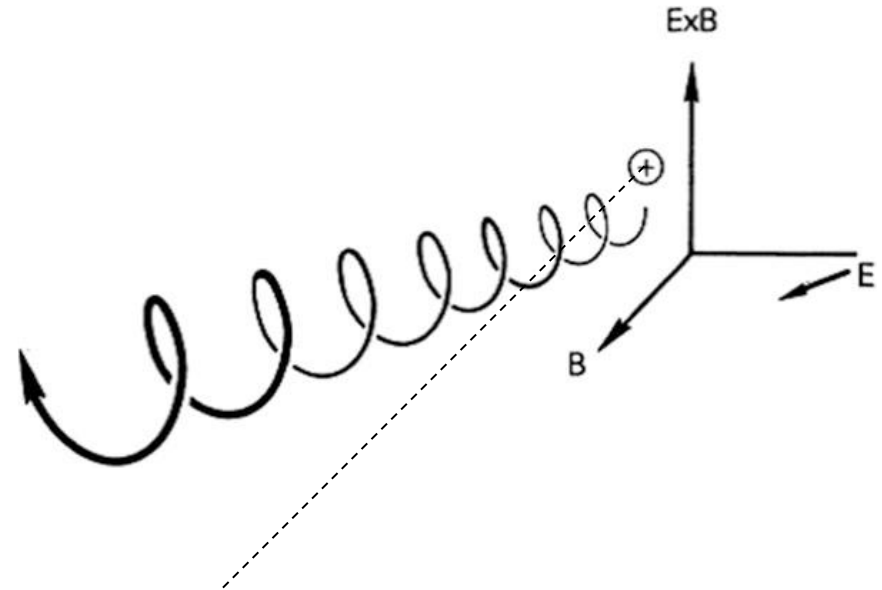
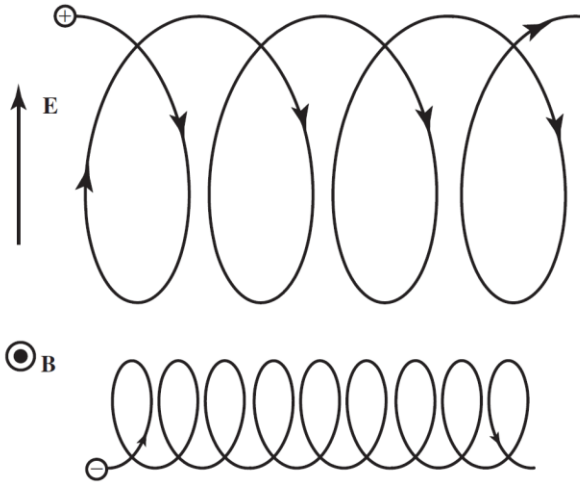
$$m \frac{d\vec{v}_{ac}(t)}{dt} = q \vec{v}_{ac}(t) \times \hat{z}B \quad \text{Gyro motion}$$

$$\vec{v}(t) = \hat{z}v_\parallel(t) + \vec{v}_E + \vec{v}_{ac}(t)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_\parallel(t) + \vec{v}_E$$

• Electrons and ions drift in the same direction.

No current is generated in ExB drift



$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2} \quad \text{ExB drift velocity}$$

- Electrons and ions drift in the same direction.

Charge particles drift across magnetic field lines when an external field not parallel to the magnetic field occurs



$$\vec{E} = \vec{E}_{\perp} + \hat{z}E_{\parallel} = \hat{x}E_{\perp} + \hat{z}E_{\parallel}$$

$$m \frac{dv_{\parallel}}{dt} = qE_{\parallel}$$

$$m \frac{d\vec{v}_{\perp}}{dt} = q(\hat{x}E_{\perp} + \vec{v}_{\perp} \times \hat{z}B)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_E$$

$$\vec{v}_E = \frac{\hat{x}E_{\perp} \times \hat{z}B}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift velocity

$$\vec{F} = \vec{F}_{\perp} + \hat{z}F_{\parallel} = \hat{x}F_{\perp} + \hat{z}F_{\parallel}$$

$$m \frac{dv_{\parallel}}{dt} = F_{\parallel}$$

$$m \frac{d\vec{v}_{\perp}}{dt} = q \left(\hat{x} \frac{F_{\perp}}{q} + \vec{v}_{\perp} \times \hat{z}B \right)$$

$$\langle \vec{v}(t) \rangle = \frac{1}{T_c} \int_0^{T_c} \vec{v}(t) dt = \hat{z}v_{\parallel}(t) + \vec{v}_F$$

$$\vec{v}_F = \frac{\hat{x}(F_{\perp}/q) \times \hat{z}B}{B^2} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

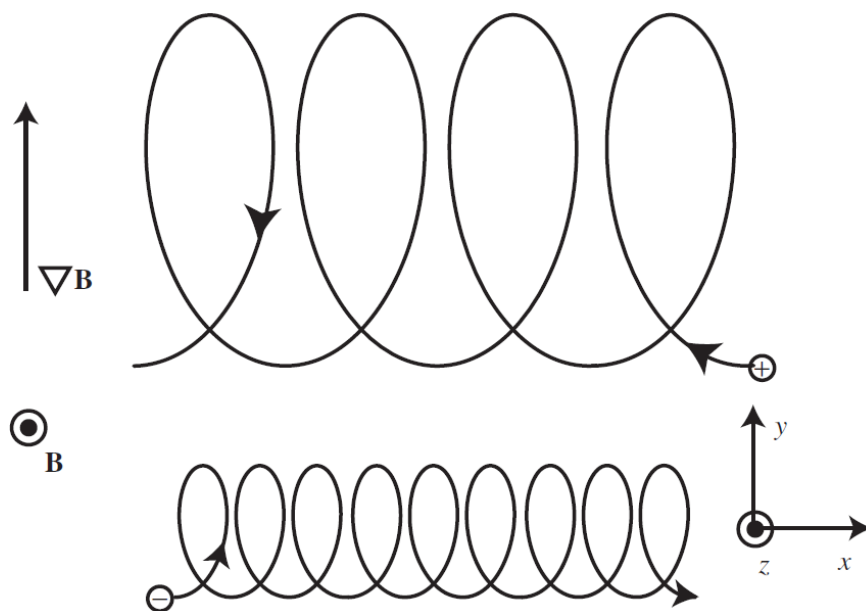
Gravitational drift velocity

- Electrons and ions drift in the opposite directions in the gravitational drift. Therefore, currents are generated.

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved

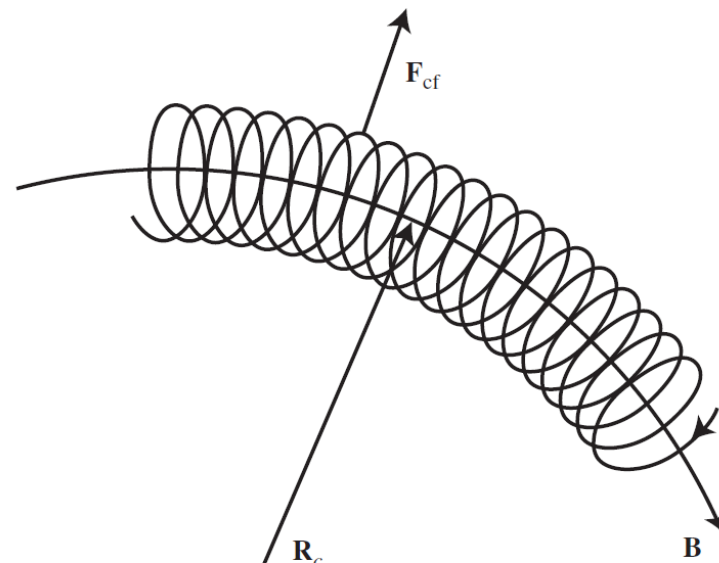


- Gradient-B drift



$$\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$$

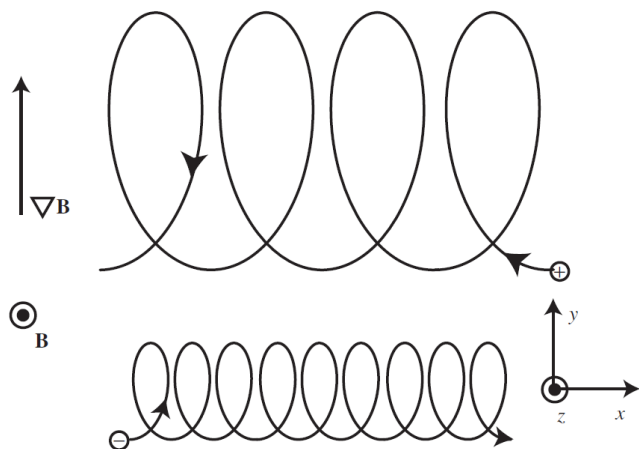
- Curvature drift



$$\vec{v}_R = \frac{mv_\parallel^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



- In the case with no gradient B

$$x_c = \mp r_c \sin(\pm \omega_c t + \psi)$$

$$y_c = \pm r_c \cos(\pm \omega_c t + \psi)$$

$$v_x = v_{\perp} \cos(\pm \omega_c t + \psi)$$

$$v_y = -v_{\perp} \sin(\pm \omega_c t + \psi)$$

$$\begin{aligned} \vec{F} &= q(\vec{v} \times \vec{B}) = \hat{x}qv_y B_z - \hat{y}qv_x B_z \\ &\simeq \hat{x}qv_y \left(B_0 + y \frac{\partial B_z}{\partial y} \right) - \hat{y}qv_x \left(B_0 + y \frac{\partial B_z}{\partial y} \right) \end{aligned}$$

$$B_z(y) = B_0 + y \frac{\partial B_z}{\partial y} + y^2 \frac{1}{2} \frac{\partial^2 B_z}{\partial y^2} + \dots$$

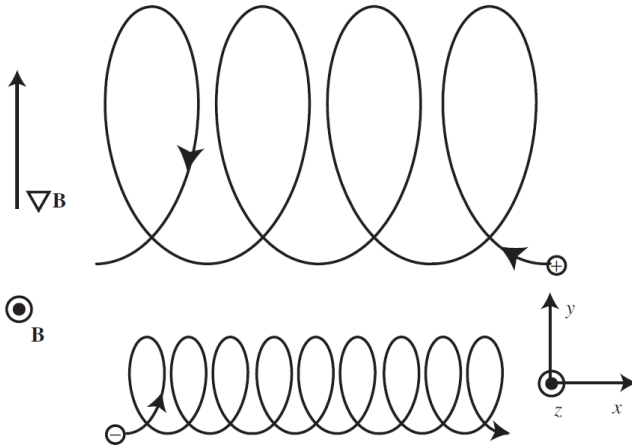
$$F_x = qv_y \left(B_0 + y \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -qv_x \left(B_0 + y \frac{\partial B_z}{\partial y} \right)$$

$$F_x \simeq -qv_{\perp} \sin(\pm \omega_c t + \psi) \times \left(B_0 \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

$$F_y = -qv_{\perp} \cos(\pm \omega_c t + \psi) \times \left(B_0 \pm r_c \cos(\pm \omega_c t + \psi) \frac{\partial B_z}{\partial y} \right)$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform



$$F_x \simeq -qv_{\perp} \sin(\pm\omega_c t + \psi) \left[B_0 \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y} \right]$$

$$F_y \simeq -qv_{\perp} \cos(\pm\omega_c t + \psi) \left[B_0 \pm r_c \cos(\pm\omega_c t + \psi) \frac{\partial B_z}{\partial y} \right]$$

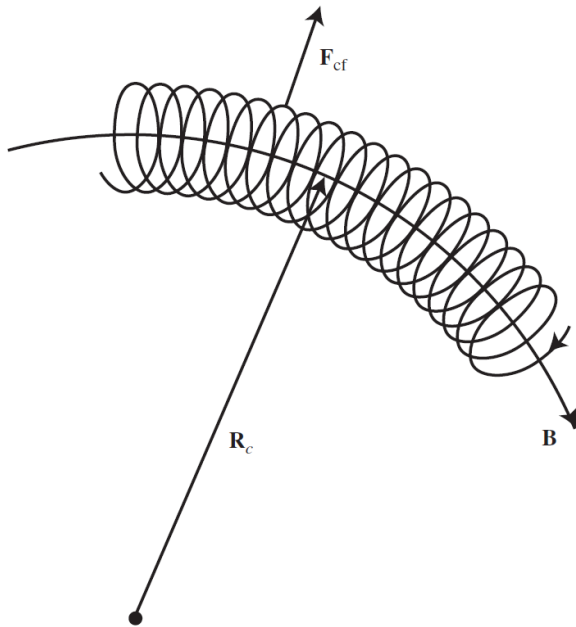
$$\langle F_x \rangle = 0$$

$$\langle F_y \rangle = \mp \frac{qv_{\perp} r_c}{2} \frac{\partial B_z}{\partial y} = -\frac{mv_{\perp}^2}{2B} \frac{\partial B_z}{\partial y} \quad r_c = \frac{v_{\perp}}{\omega_c} \quad \omega_c \equiv \frac{|q|B}{m}$$

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2} \quad \vec{v}_{\nabla} = \frac{1}{q} \frac{\langle F_y \rangle \hat{y} \times \hat{z} B_z}{B_z^2} = -\frac{mv_{\perp}^2}{2qB_z^2} \frac{\partial B_z}{\partial y} \hat{x}$$

- More general:
$$\vec{v}_{\nabla} = \frac{mv_{\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$$

Charge particles drift across magnetic field lines when the magnetic field line is curved

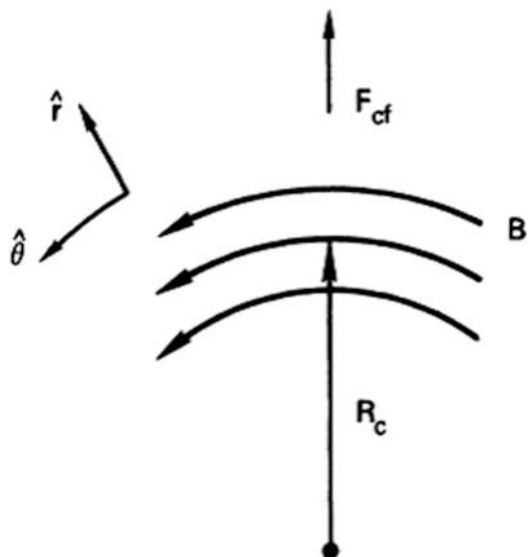


$$\vec{F}_{cf} = mv_{\parallel}^2 \frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_R = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{mv_{\parallel}^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

Charge particles drift across magnetic field lines when the magnetic field is not uniform or curved



$$\vec{B} = B\hat{\theta}$$

$$\nabla B = \nabla B\hat{r}$$

Cylindrical coordinate

$$\vec{v}_{\nabla} = \frac{mv_{\perp}^2}{2q} \frac{\vec{B} \times \nabla B}{B^3} \quad \vec{v}_R = \frac{mv_{\parallel}^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$$

$$\nabla \times \vec{B} = 0 \quad (\nabla \times \vec{B})_r = (\nabla \times \vec{B})_{\theta} = 0$$

$$(\nabla \times \vec{B})_r = \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_{\theta}}{\partial z} \quad (\nabla \times \vec{B})_{\theta} = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}$$

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (rB_{\theta}) - \frac{1}{r} \frac{\partial B_r}{\partial \theta}$$

$$\nabla \times \vec{B} = (\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (rB_{\theta}) = 0 \quad B_{\theta} \propto \frac{1}{r}$$

$$\frac{\nabla |B|}{|B|} = -\frac{\vec{R}_c}{R_c^2}$$

$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_{\nabla} = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)$$

- Electrons and ions drift in the opposite directions in the grad-B and curvature drifts. Therefore, currents are generated.

Quick summary of different drifts



- **ExB drift:** $\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$ Independent to charge
- **Gravitational drift:** $\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$ Depended on charge
- **Grad-B drift:** $\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$ Depended on charge
- **Curvature drift:** $\vec{v}_R = \frac{mv_{||}^2}{2q} \frac{\vec{R}_c \times \vec{B}}{R_c B^2}$ Depended on charge
- **Non-uniform B drift:**

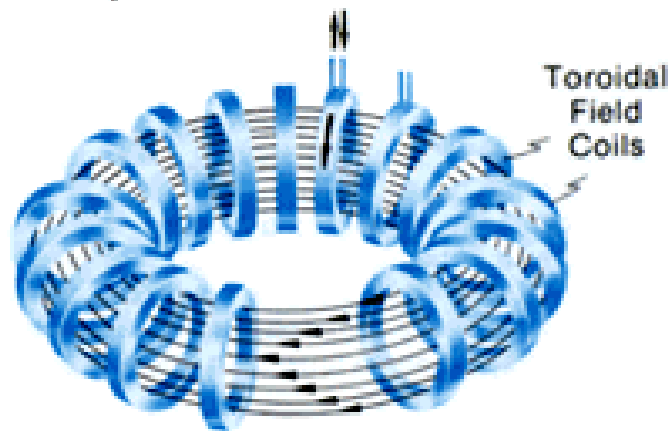
$$\vec{v}_{\text{total}} = \vec{v}_R + \vec{v}_\nabla = \frac{\vec{B} \times \nabla B}{\omega_c B^2} \left(v_{||}^2 + \frac{1}{2} v_\perp^2 \right) = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_{||}^2 + \frac{1}{2} v_\perp^2 \right)$$

Plasma can be confined in a doughnut-shaped chamber with toroidal magnetic field



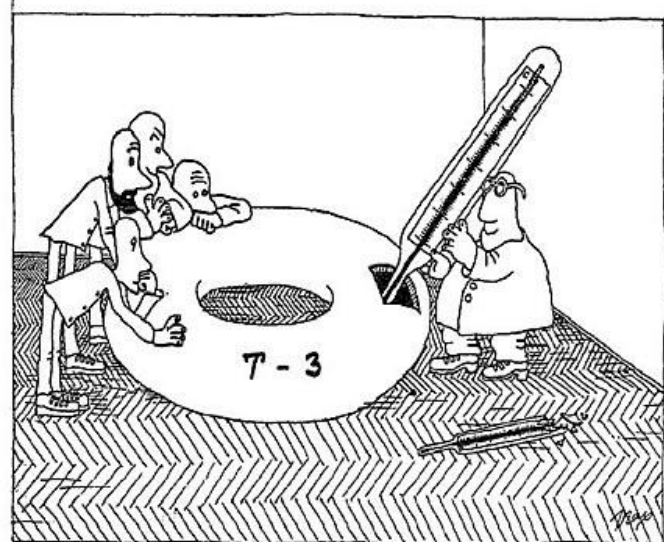
- Tokamak - "toroidal chamber with magnetic coils" (тороидальная камера с магнитными катушками)

Relatively Constant Electric Current



Nature

Constant Toroidal Field



Measurement of the Electron Temperature by Thomson Scattering in Tokamak T3

by

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D. C. ROBINSON
M. J. FORREST
P. D. WILCOCK
UKAEA Research Group,
Culham Laboratory,
Abingdon, Berkshire

V. V. SANNIKOV
I. V. Kurchatov Institute,
Moscow

$$T_e = 100 \sim 1 \text{ keV}$$

$$n_e = 1\text{-}3 \times 10^{13} \text{ cm}^{-3}$$

Electron temperatures of 100 eV up to 1 keV and densities in the range $1\text{-}3 \times 10^{13} \text{ cm}^{-3}$ have been measured by Thomson scattering on Tokamak T3. These results agree with those obtained by other techniques where direct comparison has been possible.

<https://www.iter.org/mach/tokamak>

https://en.wikipedia.org/wiki/Tokamak#cite_ref-4

Drawing from the talk "Evolution of the Tokamak" given in 1988 by B.B. Kadomtsev at Culham.

N. J. Peacock, et al., Nature **224**, 488 (1969)

Quick summary of different drifts



- ExB drift: $\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$

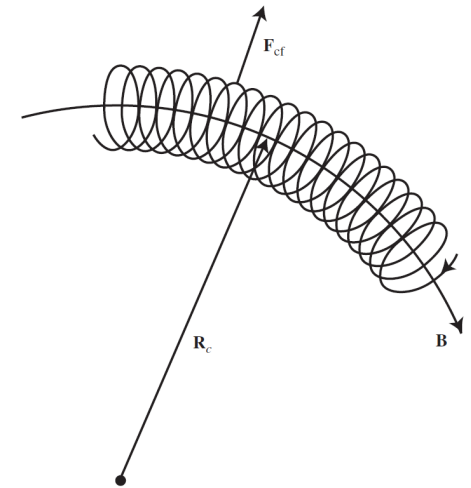
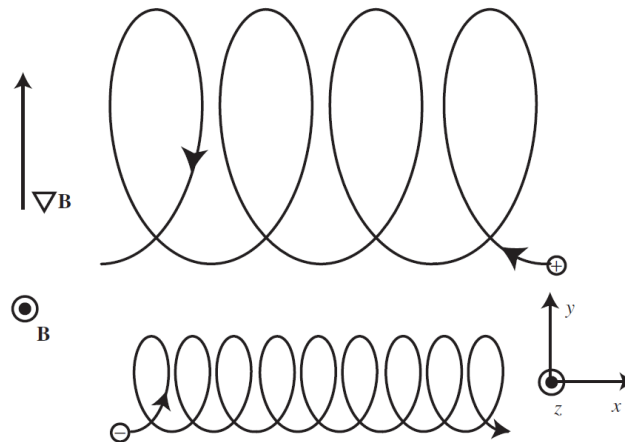
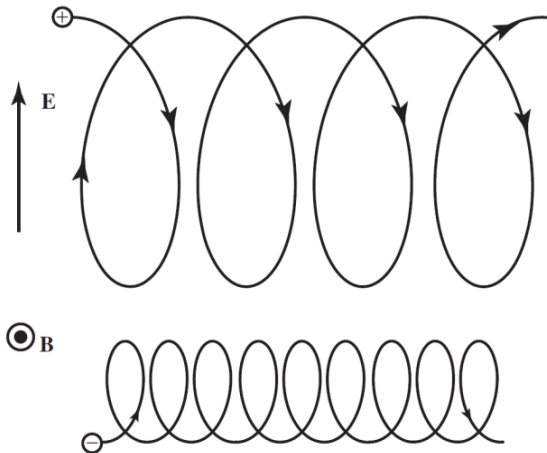
Independent to charge

- Grad-B drift: $\vec{v}_\nabla = \frac{mv_\perp^2}{2q} \frac{\vec{B} \times \nabla B}{B^3}$

Depended on charge

- Curvature drift: $\vec{v}_R = \frac{mv_{||}^2}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$

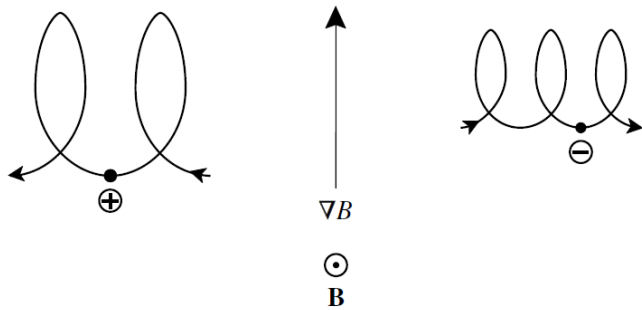
Depended on charge



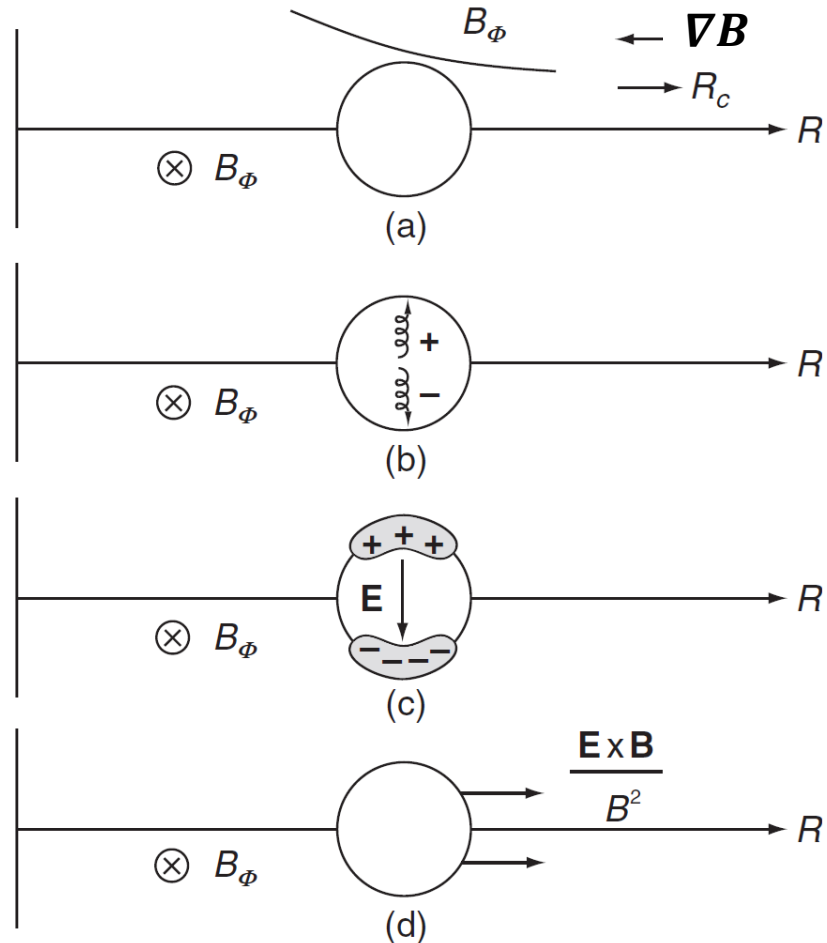
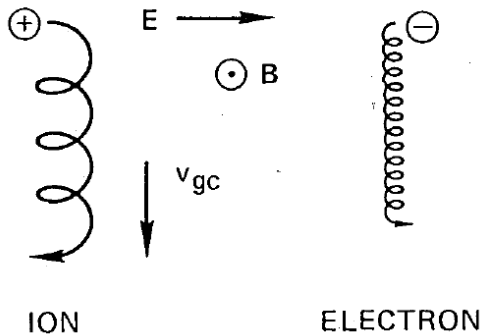
Charged particles drift across field lines



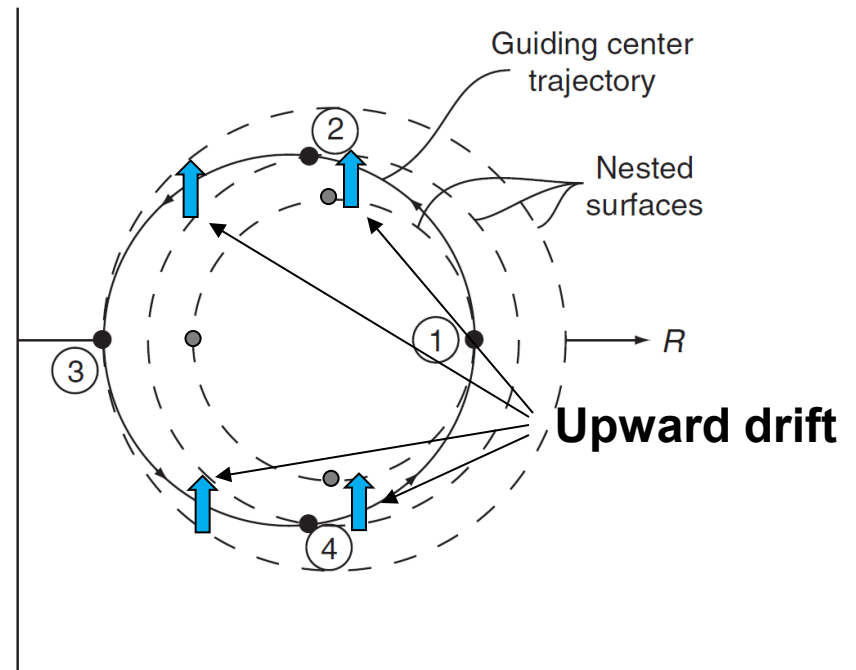
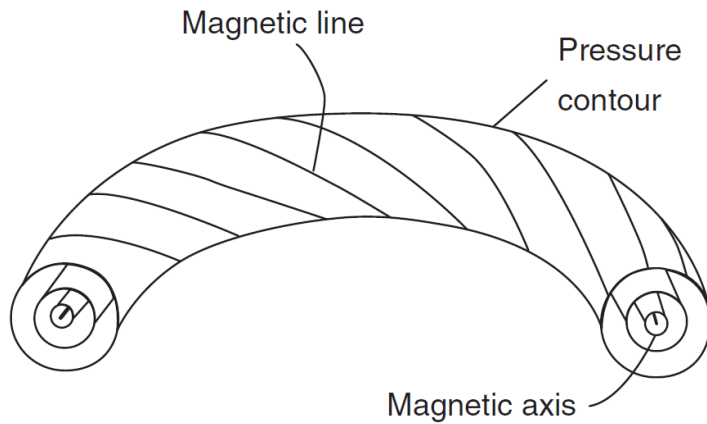
- **Grad-B drift**



- **ExB drift**

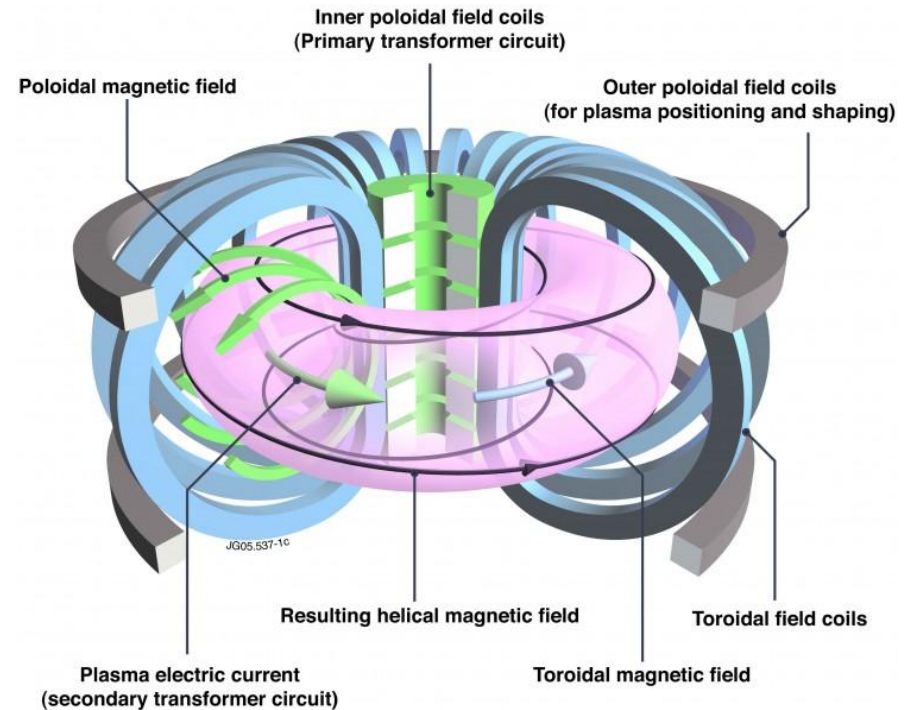
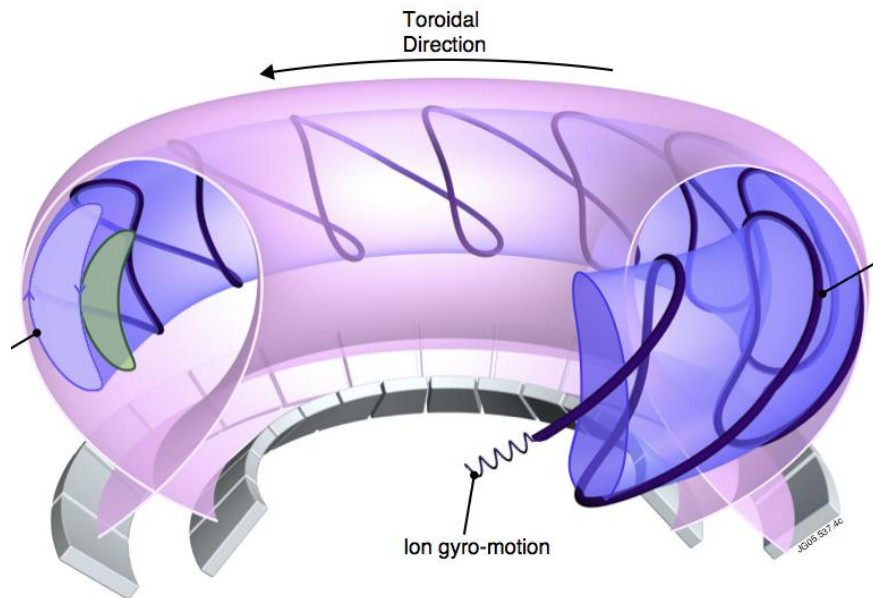


The particle drifts back to the original position if a small poloidal field is superimposed on the toroidal field

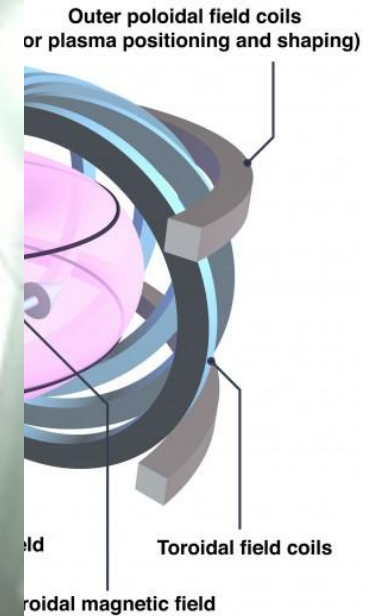
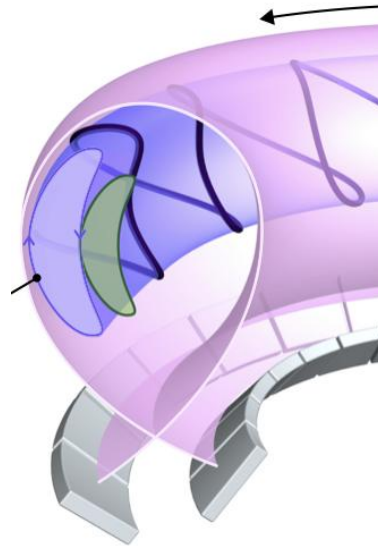


• Points with no drift

A poloidal magnetic field is required to reduce the drift across field lines



A poloidal magnetic field is required to reduce the drift across field lines



Stellarator uses twisted coil to generate poloidal magnetic field

