Introduction to Nuclear Fusion as An Energy Source

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Institute of Space and Plasma Sciences, National Cheng Kung University

Lecture 13

2024 spring semester

Wednesday 9:10-12:00

Materials:

https://capst.ncku.edu.tw/PGS/index.php/teaching/

Online courses:

https://nckucc.webex.com/nckucc/j.php?MTID=ma76b50f97b1c6d72db61de9eaa9f0b27

2024/6/5 updated

Note!

- **Final exam 6/12 (One double-sided A4 cheating sheet is allowed.)**
- **Last class 6/19**

Course Outline

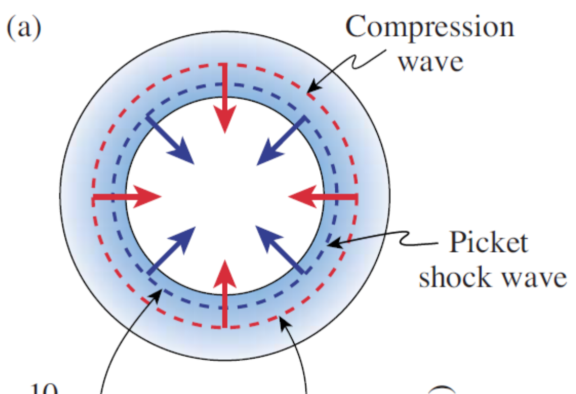
- **Inertial confinement fusion (ICF)**
  - **Plasma frequency and critical density**
  - **Direct- and indirect- drive**
  - **Laser generated pressure (Inverse bremsstrahlung and Ablation pressure)**
  - **Burning fraction, why compressing a capsule?**
  - **Implosion dynamics**
  - **Shock (Compression with different adiabat)**
  - **Laser pulse shape**
  - **Rocket model, shell velocity**
  - **Laser-plasma interaction (Stimulated Raman Scattering, SRS; Stimulated Brillouin Scattering, SBS; Two-plasmon decay)**
  - **Instabilities (Rayleigh-taylor instability, Kelvin-Helmholtz instability, Richtmeyer-Meshkov instability)**

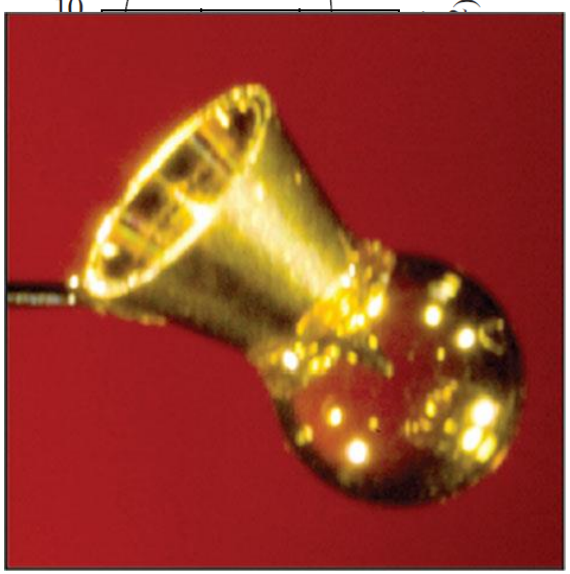
Reference for ICF

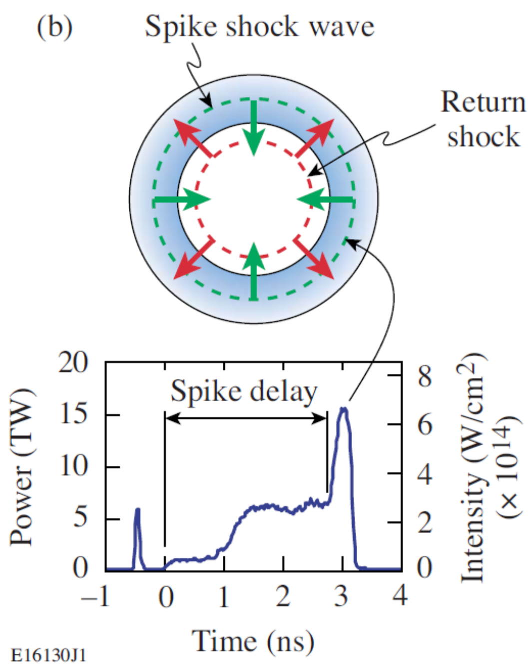
- Riccardo Betti, University of Rochester, HEDSA HEDP summer school, San Diego, CA, August 16-21, 2015
- ICF lectures for course PHY558/ME533
- The physics of inertial fusion, by S. Atzeni, J. Meyer-Ter-Vehn

External “spark” can be used for ignition

- **Shock ignition**

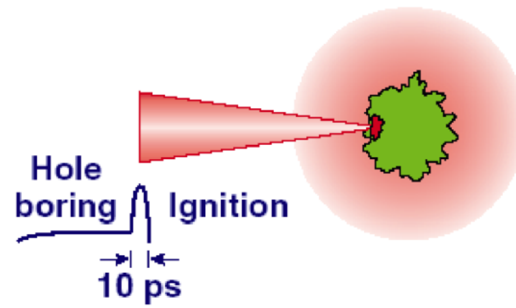




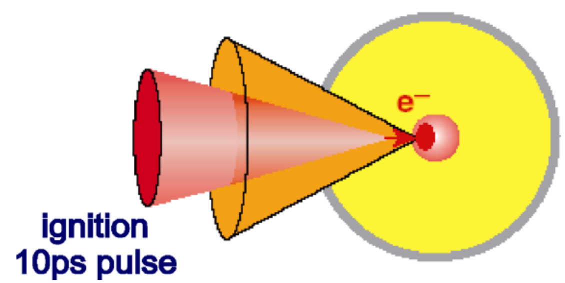


- **Fast ignition**

a) channeling FI concept

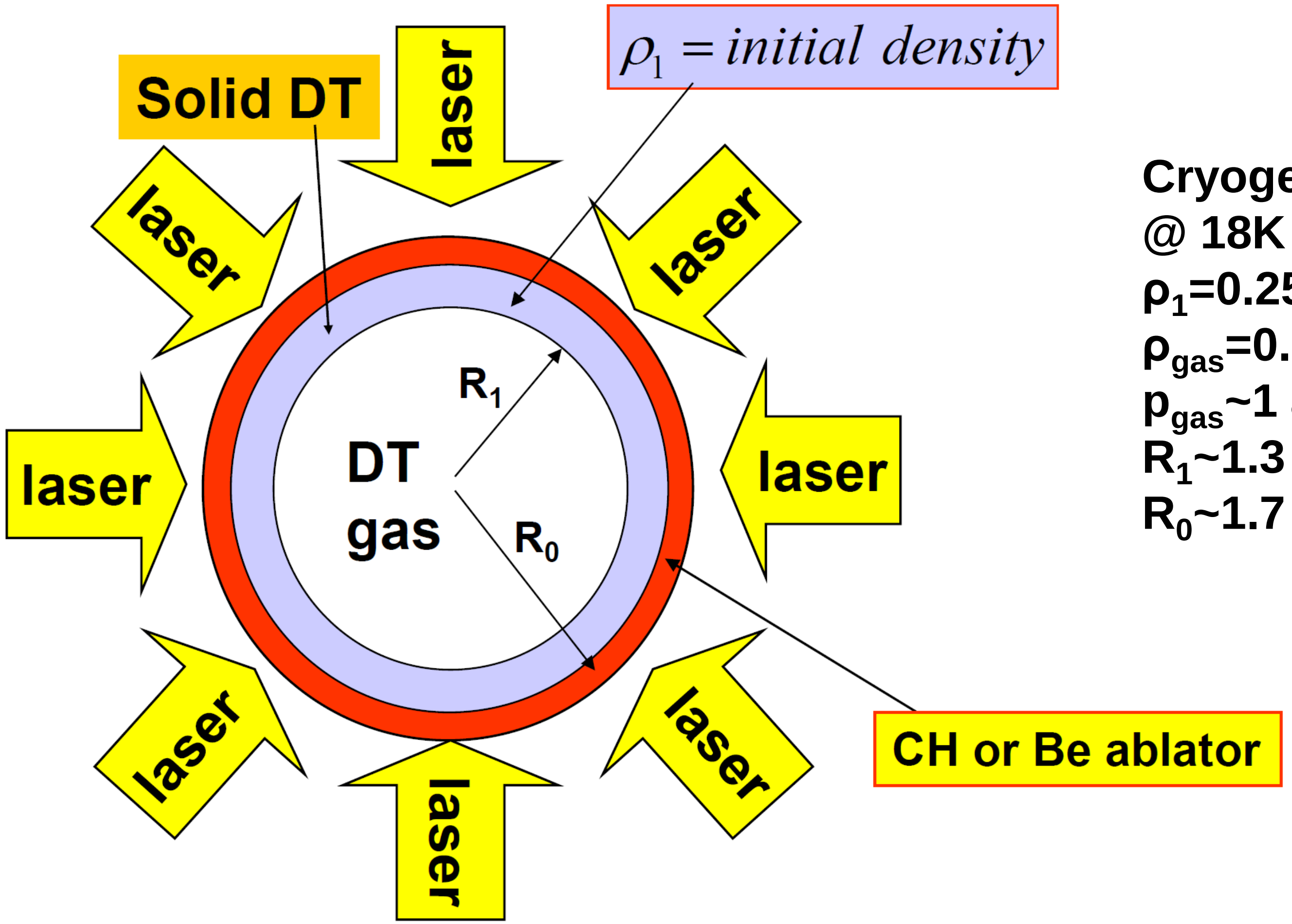


b) cone-in-shell FI concept



J. Badziak, Bull.Polish Acad. Sci. Tech. Sci.Phys. Plasmas 15, 056306 (2008)
T. Ditmire, etc., J. Fusion Energy **42**, 27 (2023)

Laser-driven imploding capsules are mm-size shells with hundreds of μm thick layers of cryogenic solid DT



Cryogenic solid DT ice @ 18K
ρ_1=0.25 g/cc
ρ_{gas}=0.3-0.6 mg/cc
p_{gas}~1 atm
R_1~1.3 mm
R_0~1.7 mm

Conservation equations of gas-dynamics and ideal gas EOS are used for DT plasma

Mass conservation: $\partial_t \rho + \partial_x(\rho \vec{v}) = 0$

Momentum conservation: $\partial_t(\rho \vec{v}) + \partial_x\left(p + \rho v^2\right) = \vec{F}$

Energy conservation: $\partial_t \epsilon + \partial_x(\vec{v}\,(\epsilon + p) - \kappa \partial_x T) = \text{source} + \text{sinks}$

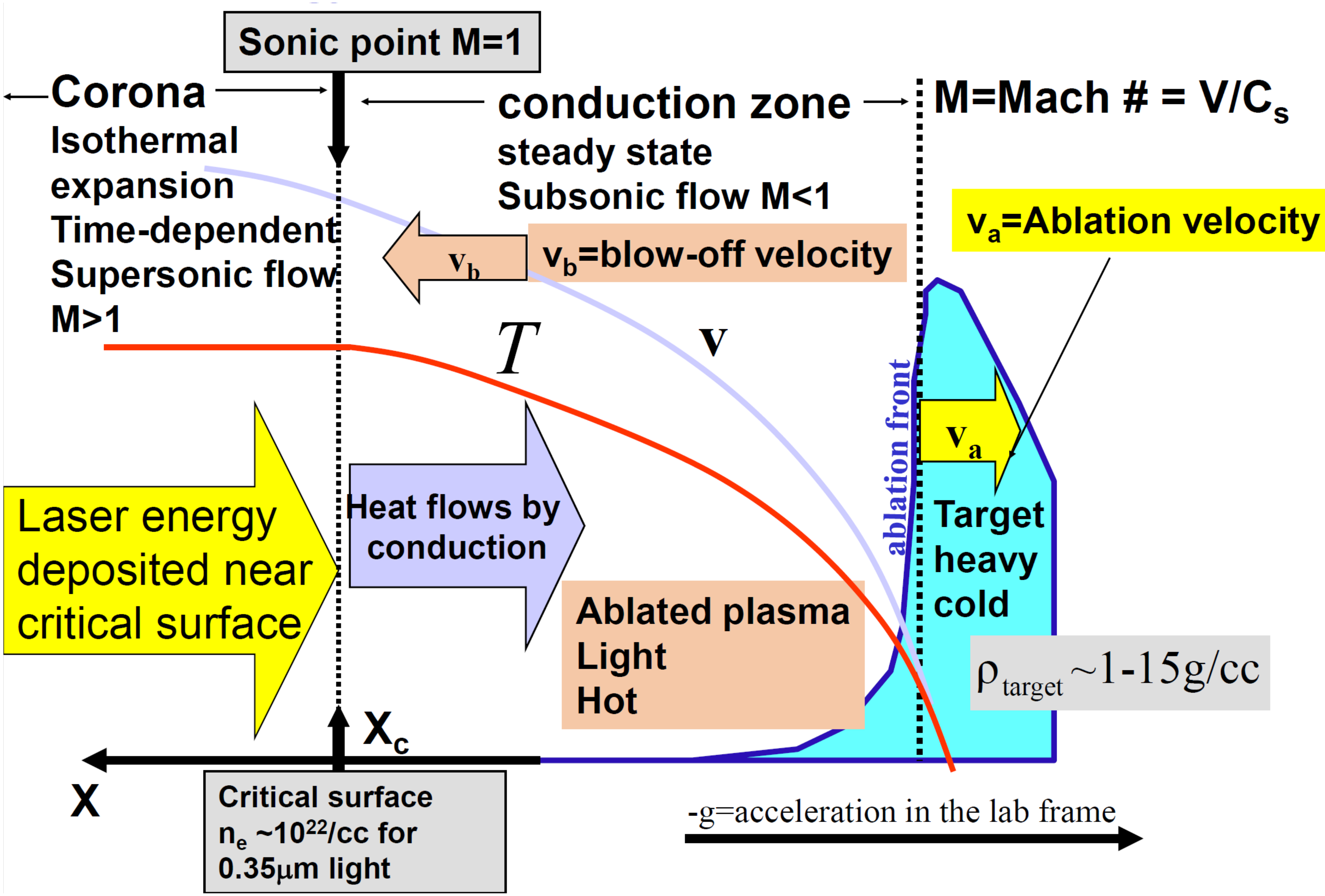
Ideal gas EOS: $p = (n_e T_e + n_i T_i) = 2nT = \frac{2}{m_i}\rho_i T = \frac{\rho T}{A}$

Total energy per unit volume: $\epsilon = \frac{3}{2}p + \rho\frac{v^2}{2}$

Mass density: $\rho = n_i m_i$

Plasma thermal conductivity: κ

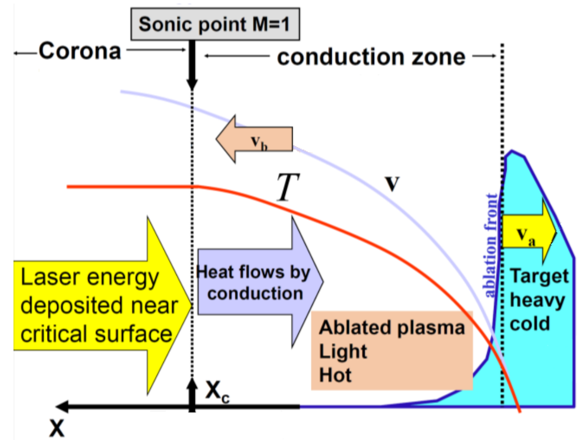
The laser generates a pressure by depositing energy at the critical surface



Laser produced ablation pressure

- **Total ablation pressure (static + dynamic):**

$$P_A = \frac{\rho_c T_c}{A} + \rho_c {v_c}^2 = 2\frac{\rho_c T_c}{A} \sim \rho_c \frac{I^{2/3}}{{\rho_c}^{2/3}} \sim {\rho_c}^{1/3} I^{2/3}$$



- **Temperature at critical surface:** $T_c \sim \left(\frac{I}{\rho_c}\right)^{2/3}$

- **Velocity at critical surface:** $v_c \sim \left(\frac{I}{\rho_c}\right)^{1/3}$

$$n_e = \frac{\pi c^2 m_e}{e^2 {\lambda_L}^2}$$

- **Ablation rate:** $\rho_c v_c \sim {\rho_c}^{2/3} I^{1/3}$

$$\rho_\mathrm{c} = m_\mathrm{i} n_\mathrm{cr,i} = m_\mathrm{i} \frac{n_\mathrm{cr,e}}{z} = \frac{\pi c^2 m_\mathrm{e} m_\mathrm{i}}{z e^2 {\lambda_\mathrm{L}}^2}$$

$$p_\mathrm{A}(\mathrm{Mbar}) \approx 83 \left(\frac{I_{15}}{\lambda_{\mathrm{L},\mu\mathrm{m}}/0.35}\right)^{2/3}$$

$$\dot{m}_\mathrm{a} = 3.3 \times 10^5 \frac{{I_{15}}^{1/3}}{{\lambda_\mathrm{L}}^{4/3}}\ \mathrm{g/cm^2\,s}$$

I_{15}: laser intensity in 10^{15}w/cm^2
$\lambda_{L,\mu m}$: laser wavelength in μm

Entropy of an ideal gas/plasma

- **The entropy S is a property of a gas just like p, T, and ρ**

$$S = c_v \ln\left[\frac{p}{\rho^{5/3}}\text{const}\right] = c_v \ln\alpha \qquad \alpha = \text{const}\frac{p}{\rho^{5/3}}$$

- **α is called the “adiabat”**
- **The entropy/adiabat S/α changes through dissipation or heat sources/sinks**

$$\rho\left(\frac{\partial S}{\partial t} + \vec{u}\cdot\nabla S\right) = \frac{DS}{Dt} = \mu\frac{|\nabla\,\vec{u}|^2}{T} + \frac{\nabla\cdot\kappa\nabla T}{T} + \text{sources/sinks}$$

- **In an ideal gas (no dissipation) and without sources and sinks, the entropy/adiabat is a constant of motion of each fluid element**

$$\frac{DS}{Dt} = 0 \quad \Rightarrow \quad S\,,\alpha = \text{const} \Rightarrow \quad p \sim \alpha\rho^{5/3}$$

It is easier to compress a low adiabat (entropy) gas

- **α is called the "adiabat"**
- **Smaller α -> less work to compress from low to high density**

$$W_{1\to 2} = -\int p dV \sim -\int_{\rho_1}^{\rho_2} \alpha \rho^{5/3} d\left(\frac{M}{\rho}\right) \sim \alpha M\left({\rho_2}^{2/3} - {\rho_1}^{2/3}\right)$$

- **Smaller α -> higher density for the same pressure**

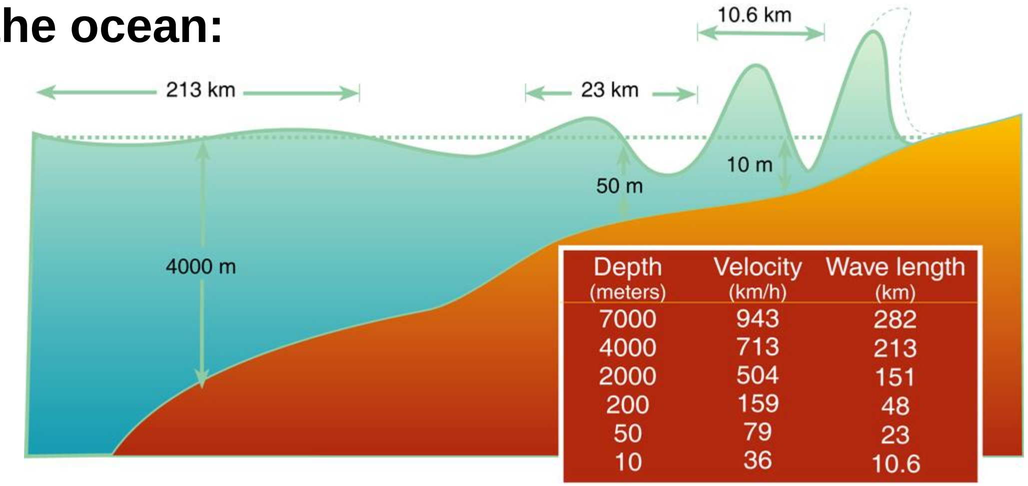
$$\alpha \sim \frac{p}{\rho^{5/3}} \Rightarrow \rho \sim \left(\frac{p}{\alpha}\right)^{3/5}$$

- **In HEDP, the constant in adiabat definition comes from the normalization of the pressure against the Fermi pressure.**
- **When thermal effects are negligible at very high densities, the pressure is proportional to $\rho^{5/3}$ due to the quantum mechanical effects (degenerate electron gas) just like isentropic flow**

$$\alpha \equiv \frac{p}{p_{\mathrm{F}}} \Rightarrow \alpha_{\mathrm{DT}} = \frac{p_{\mathrm{Mbar}}}{2.2{\rho_{\mathrm{g/cc}}}^{5/3}}$$

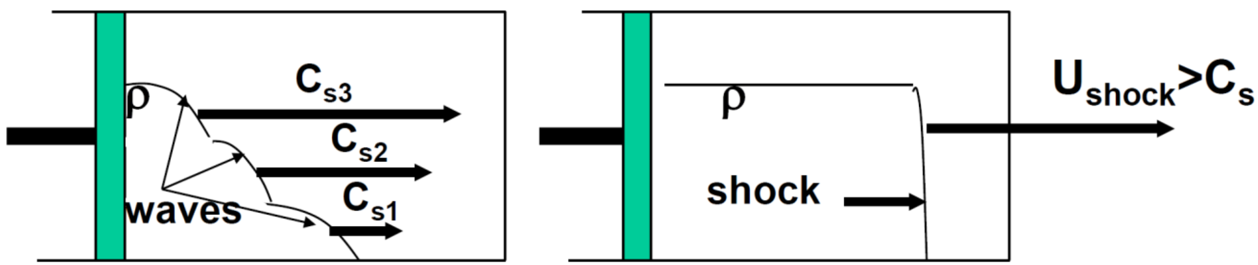
A shock is formed due to the increasing sound speed of a compressed gas/plasma

- **Wave in the ocean:**



Depth (meters)	Velocity (km/h)	Wave length (km)
7000	943	282
4000	713	213
2000	504	151
200	159	48
50	79	23
10	36	10.6

- **Acoustic/compression wave driven by a piston:**



$$C_s \sim \sqrt{\frac{p}{\rho}} \sim \sqrt{\frac{\alpha\rho^{5/3}}{\rho}} \sim \sqrt{\alpha}\rho^{1/3}$$

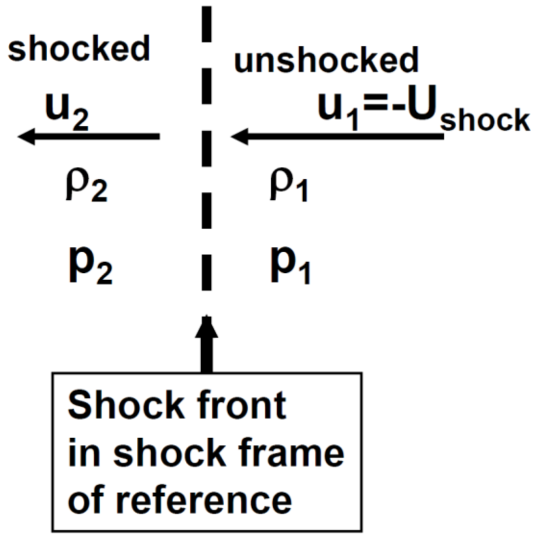
http://neamtic.ioc-unesco.org/tsunami-info/the-cause-of-tsunamis

Rankine-Hugoniot conditions are obtained using conservation of mass, momentum and energy across the shock front

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$u_1 \left(\varepsilon_1 + p_1\right) = u_2 \left(\varepsilon_2 + p_2\right)$$



- **Ideal gas/plasma:**

$$\varepsilon = \frac{3}{2}p + \rho\frac{u^2}{2}$$

- **With assigned ρ_1, p_1, and p_2, ρ_2, u_2, and u_1=-U_{shock} can be obtained using Rankine-Hugoniont conditions**

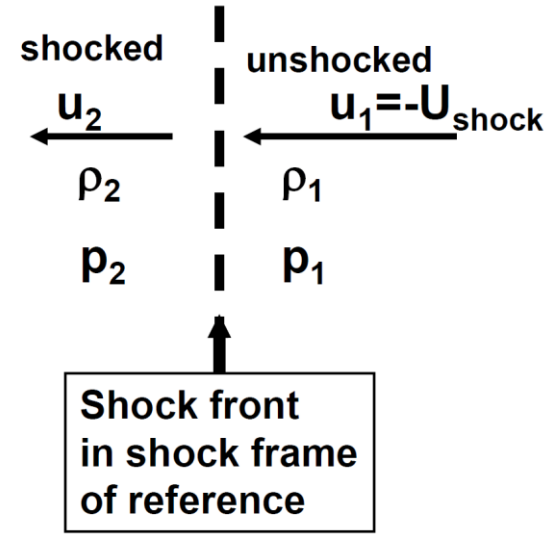
For a strong shock where $p_2 >> p_1$, the R-H conditions are simplified

$$\frac{\rho_2}{\rho_1} \approx 4$$

$$U_{\rm shock} = -u_1 \approx \sqrt{\frac{4p_2}{3\rho_1}}$$

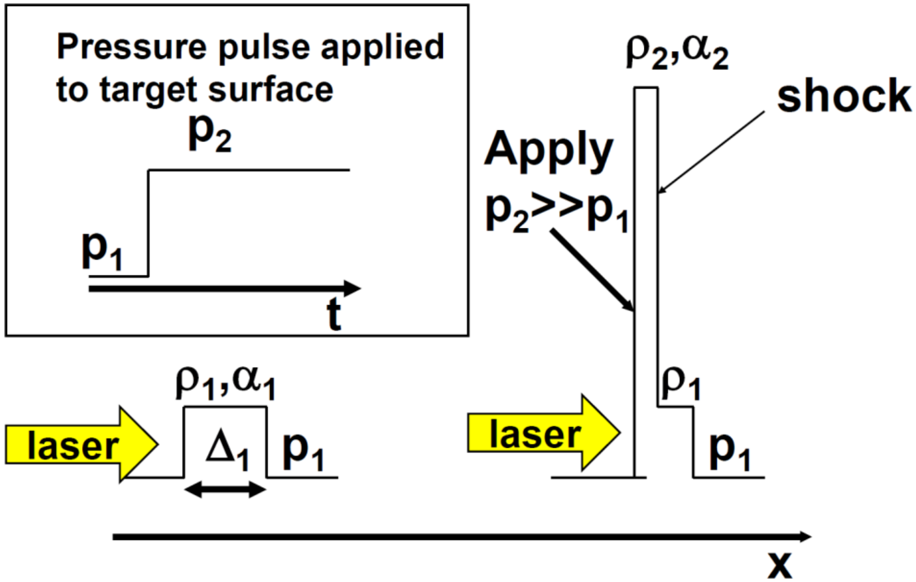
$$u_2 \approx \sqrt{\frac{p_2}{12\rho_1}}$$

$$\frac{\alpha_2}{\alpha_1} = \frac{p_2/\rho_2^{5/3}}{p_1/\rho_1^{5/3}} \approx \frac{1}{4^{5/4}}\frac{p_2}{p_1} >> 1$$



- **The adiabat increases through the shock.**

In an ideal gas/plasma, the adiabat α only raises when a shock is present



- **Post-shock density**

$$\rho_2 \approx 4\rho_1$$

- **Adiabat set by the shock for DT:**

$$\alpha_2 \approx \frac{p_{2,\mathrm{Mbar}}}{2.2\left(4\rho_{1,\mathrm{g/cc}}\right)^{5/3}}$$

- **Time required for the shock to reach the rear target surface (shock break-out time, t_{sb})**

$$t_{\mathrm{sb}} = \frac{\Delta_1}{u_{\mathrm{shock}}} = \Delta_1\sqrt{\frac{3\rho_1}{4p_2}} \propto \sqrt{\frac{1}{\alpha_2\rho_1^{2/3}}}$$

Higher laser intensity leads to higher adiabat

- **For a cryogenic solid DT target at 18 k:**

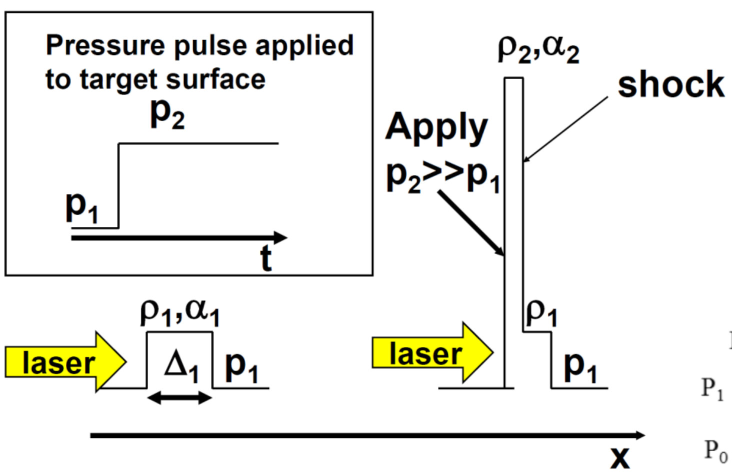
$$\rho_1 = 0.25\ \mathrm{g/cc} \qquad \alpha = \frac{p_{\mathrm{Mbar}}}{2.2} \qquad p \approx 83\left(\frac{I_{15}}{\lambda_{\mu\mathrm{m}}/0.35}\right)^{2/3}$$

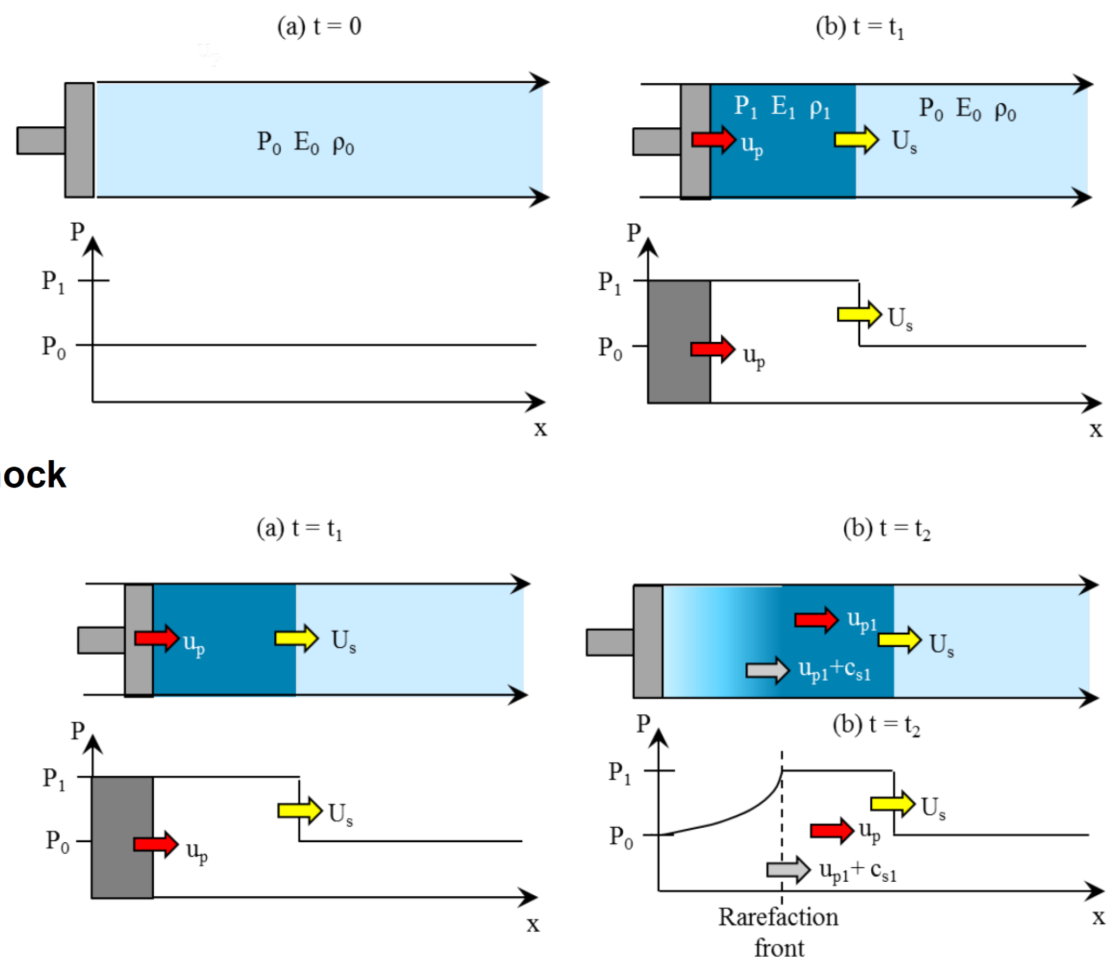
$$I \approx 4.3\times 10^{12}\ \mathrm{w/cm^2} \quad \Rightarrow \quad p = 2.2\ \mathrm{Mbar} \quad \Rightarrow \quad \alpha = 1$$

$$I \approx 1.2\times 10^{13}\ \mathrm{w/cm^2} \quad \Rightarrow \quad p = 4.4\ \mathrm{Mbar} \quad \Rightarrow \quad \alpha = 2$$

$$I \approx 2.2\times 10^{13}\ \mathrm{w/cm^2} \quad \Rightarrow \quad p = 6.6\ \mathrm{Mbar} \quad \Rightarrow \quad \alpha = 3$$

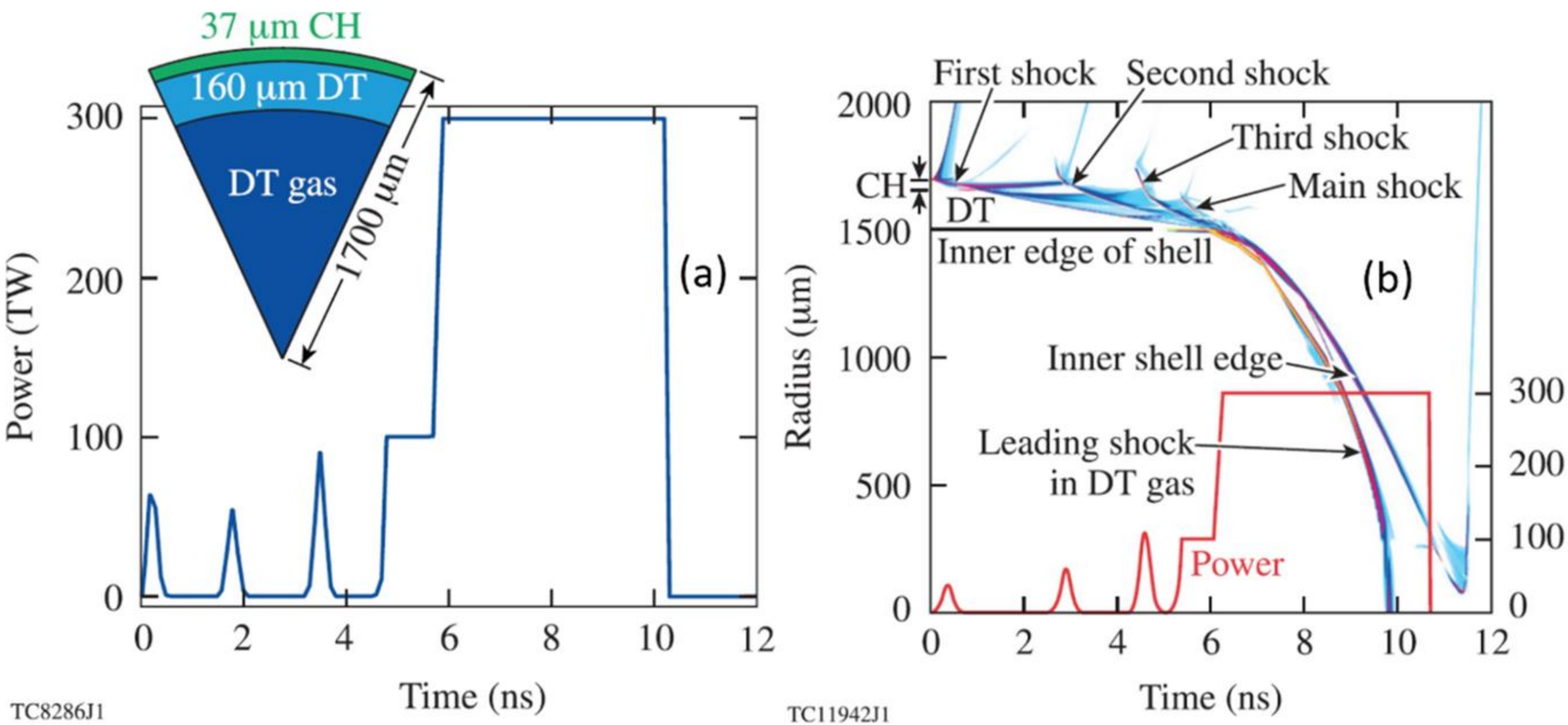
A shock or a rarefaction wave may be formed depending on the driving force from the piston





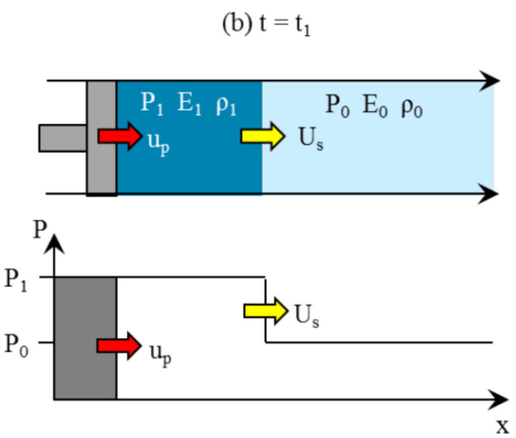
Michelle Colleen Gregor, PhD Thesis, U of Rochester, 2017

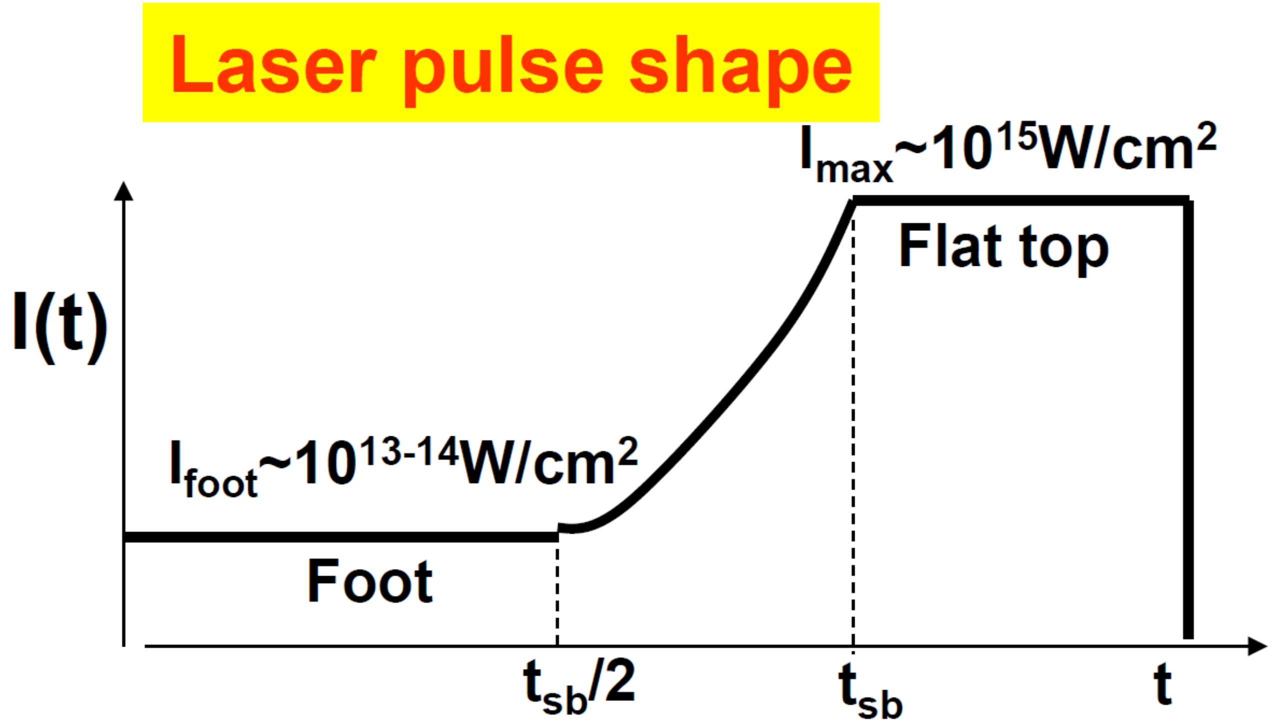
Multiple shock can be generated with multiple pickets



T. Ditmire, etc., J. Fusion Energy **42**, 27 (2023)
R. S. Craxton, etc., Phys. Plasmas 22, 110501 (2015)

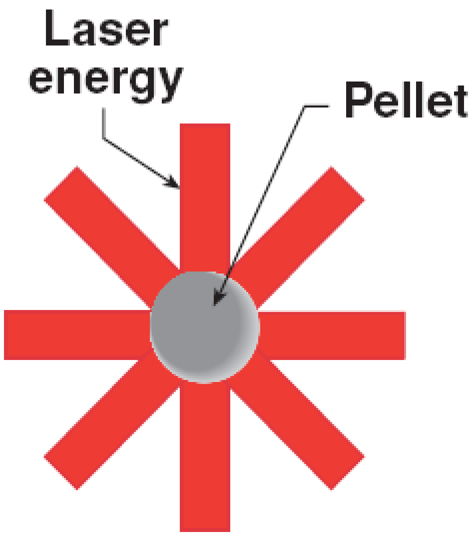
The pressure must be "slowly" increased after the first shock to avoid raising the adiabat

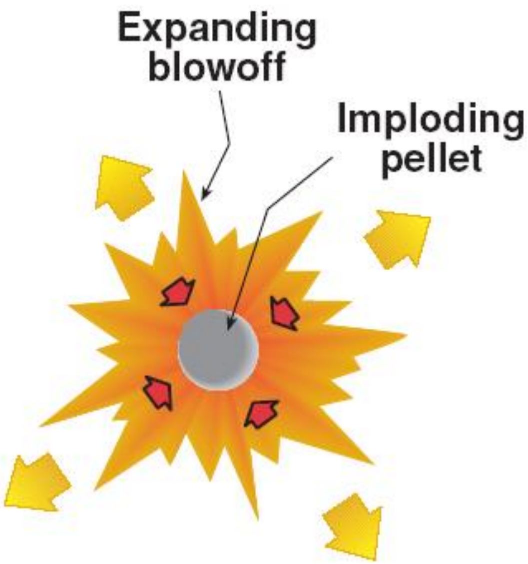




- After the foot of the laser pulse, the laser intensity must be raised starting at about $0.5t_{sb}$ and reach its peak at about t_{sb}
- Reaching I_{max} at t_{sb} prevents a rarefaction/decompression wave to propagate back from the rear target surface and decompress the target.

Most of the absorbed laser energy goes into the kinetic and thermal energy of the expanding blow-off plasma





- **The rocket model:**

Shell Newton's law

$$M\frac{du}{dt} = -4\pi R^2 p_{\mathrm{a}}$$

Shell mass decreases due to ablation

$$\frac{dM}{dt} = -4\pi R^2 \dot{m}_{\mathrm{a}}$$

p_a =ablation rate x exhaust velocity

$$p_{\mathrm{a}} = \dot{m}_{\mathrm{a}} u_{\mathrm{exhaust}}$$

Shell velocity can be obtained by integrating the rocket equations

$$M\frac{du}{dt} = -4\pi R^2 p_\mathrm{a} \qquad \frac{dM}{dt} = -4\pi R^2 \dot{m}_\mathrm{a} \qquad p_\mathrm{a} = \dot{m}_\mathrm{a} u_\mathrm{exhaust}$$

$$\begin{aligned}
M\frac{du}{dt} &= -4\pi R^2 p_\mathrm{a} = -4\pi R^2 \dot{m}_\mathrm{a} u_\mathrm{exhaust} \\
&= -4\pi R^2 u_\mathrm{exhaust} \frac{1}{-4\pi R^2}\frac{dM}{dt} \\
&= u_\mathrm{exhaust}\frac{dM}{dt} \\
\int du &= u_\mathrm{exhaust}\int \frac{dM}{M} \\
u_\mathrm{shell} &= u_\mathrm{exhaust} \ln\left(\frac{M_\mathrm{initial}}{M_\mathrm{final}}\right)
\end{aligned}$$

$$E_\mathrm{kin}^\mathrm{shell} = \frac{M_\mathrm{final}}{2} u_\mathrm{shell}^2 = \frac{M_\mathrm{final}}{2}\left[u_\mathrm{exhaust}\ln\left(\frac{M_\mathrm{initial}}{M_\mathrm{final}}\right)\right]^2$$

$$E_\mathrm{exhaust} = (M_\mathrm{initial} - M_\mathrm{final})\left(\frac{u_\mathrm{exhaust}^2}{2} + \frac{3}{2}\frac{p_\mathrm{ex}}{\rho_\mathrm{ex}}\right)$$

(dynamic + static)

$M_\mathrm{exhaust} = M_\mathrm{initial} - M_\mathrm{final}$

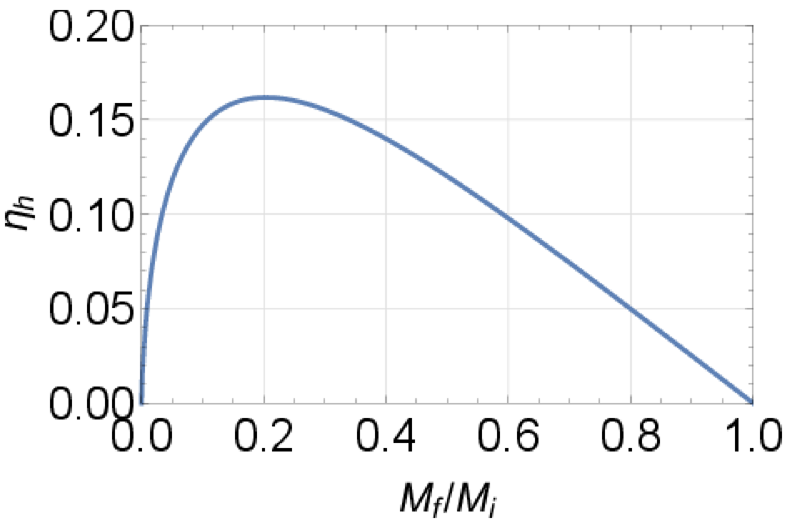
Maximum hydro efficiency is about 15%

$$E_{\rm kin}^{\rm shell} = \frac{M_{\rm final}}{2} u_{\rm shell}^2 = \frac{M_{\rm final}}{2} \left[u_{\rm exhaust} \ln\left(\frac{M_{\rm initial}}{M_{\rm final}} \right) \right]^2$$

$$E_{\rm exhaust} = (M_{\rm initial} - M_{\rm final}) \left(\frac{u_{\rm exhaust}^2}{2} + \frac{3}{2} \frac{p_{\rm ex}}{\rho_{\rm ex}} \right)$$

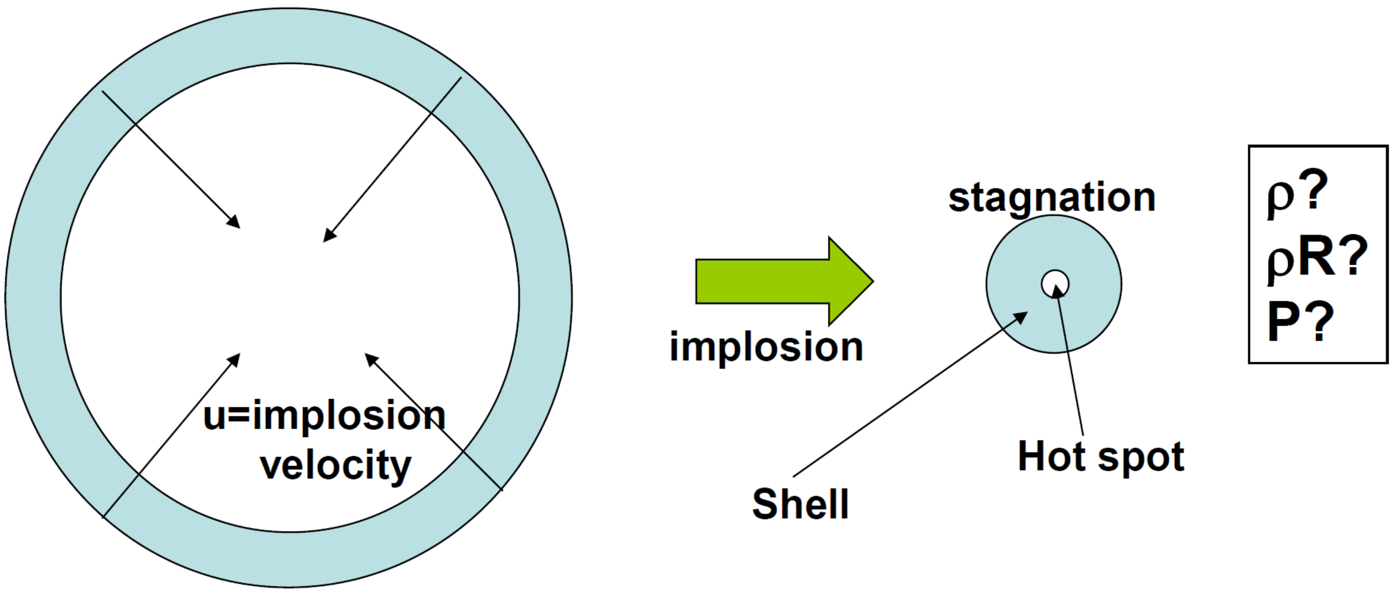
Take $u_{\rm exhaust}^2 \approx C_{\rm s}^2 \approx \dfrac{p_{\rm ex}}{\rho_{\rm ex}}$

$$\eta_{\rm h} = \frac{E_{\rm kin}^{\rm shell}}{E_{\rm exhaust}} = \frac{M_{\rm f}/M_{\rm i} \left[\ln\left(M_{\rm f}/M_{\rm i} \right) \right]^2}{4\left(1 - M_{\rm f}/M_{\rm i}\right)}$$



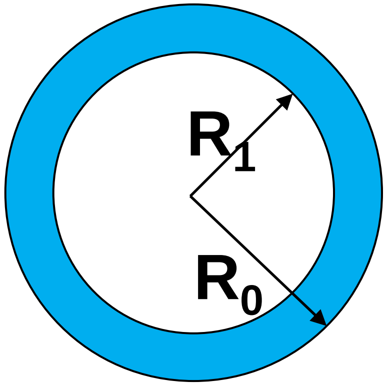
One dimensional implosion hydrodynamics

- What are the stagnation values of the relevant hydrodynamic properties?
- What's the requirement of the final density *ρ*, *ρR*, *P*, *T*?

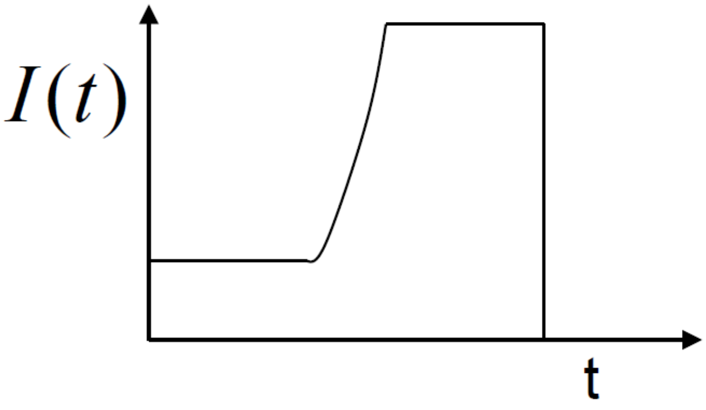


What variables can be controlled?

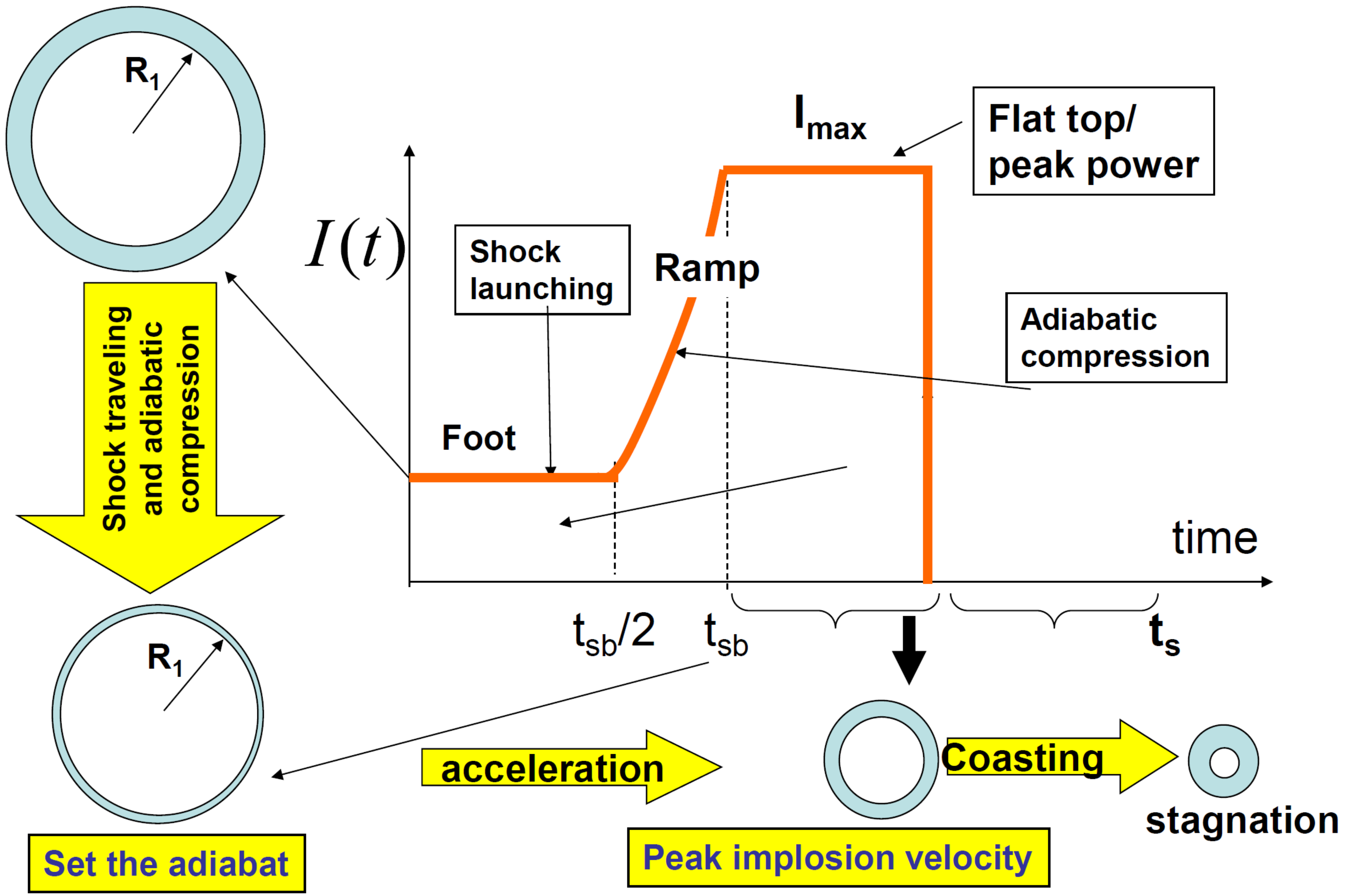
- Shell outer radius R_0 at time t=0
- Shell inner radius R_1 at time t=0
- The total laser energy on target
- Adiabat α through shocks
- Applied pressure p(t) through the pulse shape I(t)



$$\alpha \sim \frac{p}{\rho^{5/3}} \qquad p \sim I^{2/3}$$

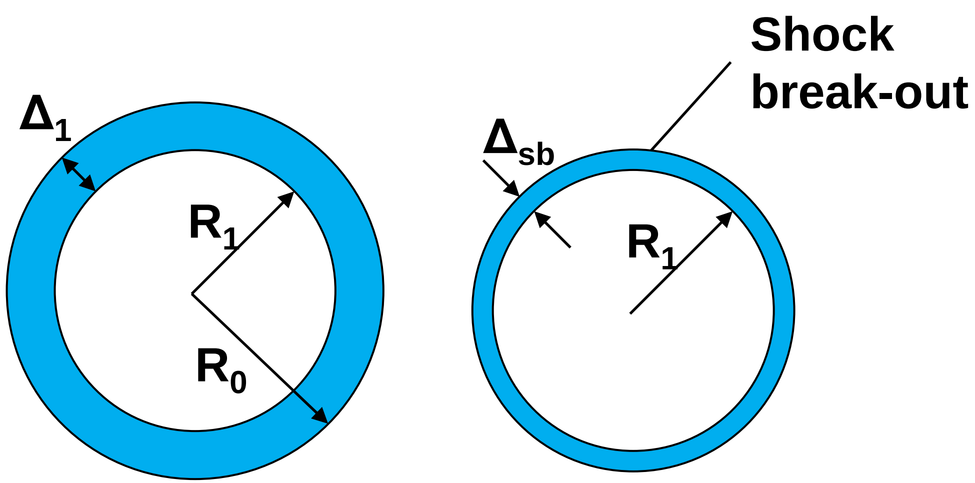


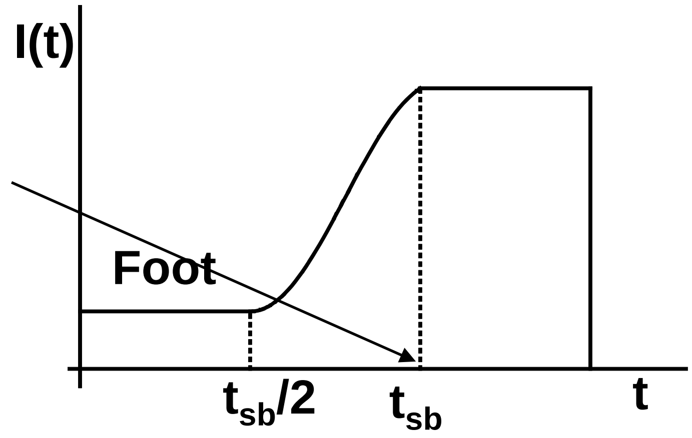
There are three stages in the laser pulse: foot, ramp, and flat top



The adiabat is set by the shock launched by the foot of the laser pulse

$\rho_1 = \text{initial density}$





$$\alpha \sim \frac{p}{\rho^{5/3}} \sim \frac{p_{\text{foot}}}{(4\rho_1)^{5/3}}$$

$$\rho_{sb} \sim \left(\frac{p_{\max}}{\alpha}\right)^{5/3} = 4\rho_1 \left(\frac{p_{\max}}{p_{\text{foot}}}\right)^{5/3}$$

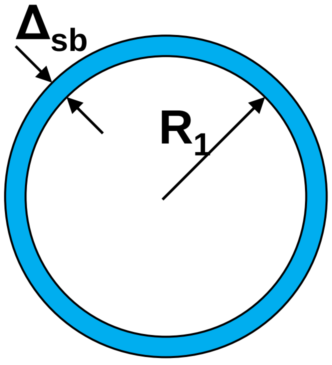
$$m_{sb} \sim 4\pi {R_1}^2 \Delta_1 \rho_1 = 4\pi {R_1}^2 \Delta_{sb} \rho_{sb}$$

$$\Delta_1 \rho_1 = \Delta_{sb} \rho_{sb}$$

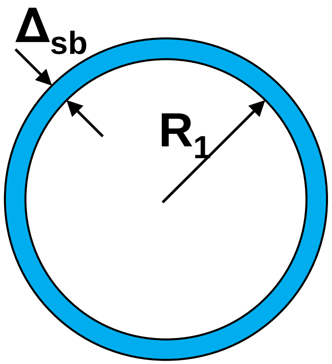
$$\Delta_{sb} = \Delta_1 \frac{\rho_1}{\rho_{sb}} \sim \Delta_1 \frac{\rho_1}{4\rho_1 \left(\frac{p_{\max}}{p_{\text{foot}}}\right)^{3/5}} = \frac{\Delta_1}{4} \left(\frac{p_{\text{foot}}}{p_{\max}}\right)^{3/5}$$

Density and thickness at shock break out time are expressed in laser intensity

- Use $p \sim I^{2/3}$
- Shell density $\rho_{sb} \sim \rho_1 \left(\frac{p_{max}}{p_{foot}}\right)^{5/3} = 4\rho_1 \left(\frac{I_{max}}{I_{foot}}\right)^{2/5}$
- Shell thickness $\Delta_{sb} \sim \frac{\Delta_1}{4} \left(\frac{p_{foot}}{p_{max}}\right)^{3/5} = \frac{\Delta_1}{4} \left(\frac{I_{foot}}{I_{max}}\right)^{2/5}$
- Shell radius $R \approx R_1$



The aspect ratio is maximum at shock break out



$$\text{Aspect ratio} \equiv \frac{R}{\Delta}$$

$$A_1 = \frac{R_1}{\Delta_1} = \text{initial aspect ratio}$$

$$A_{\text{sb}} = IFAR = \frac{R_1}{\Delta_{\text{sb}}} = 4A_1\left(\frac{I_{\max}}{I_{\text{foot}}}\right)^{2/5}$$

$$A_{\text{sb}} = A_{\max}$$

IFAR ≡ Maximum In-Flight-Aspect-Ratio = aspect ratio at shock break-out

The IFAR scales with the Mach number

- The shell kinetic energy = the work done on the shell

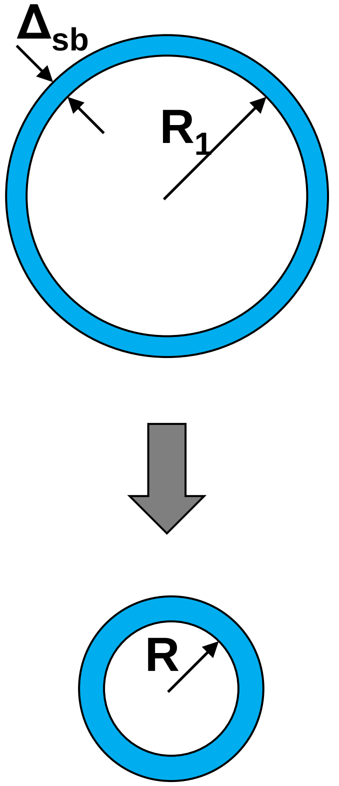
$$Mu_{max}^2 \sim -\int_{R_1}^{R} pr^2 dr \sim p\left(R_1^3 - R^3\right) \approx pR_1^3 \qquad R_1^3 = \frac{Mu_{max}^2}{p}$$

$$M \sim \rho_{sb} \Delta_{sb} R_1^2 \qquad \Delta_{sb} \sim \frac{M}{\rho_{sb} R_1^2} \qquad R_1 >> R$$

$$IFAR = \frac{R_1}{\Delta_{sb}} = \frac{R_1}{\frac{M}{\rho_{sb} R_1^2}} = \frac{\rho_{sb} R_1^3}{M} = \frac{\rho_{sb}}{M} \frac{Mu_{max}^2}{p}$$

$$= \frac{u_{max}^2}{p/\rho_{sb}} \sim Mach_{max}^2$$

$$\alpha \sim \frac{p}{\rho^{5/3}} \qquad p \sim I^{2/3} \qquad IFAR \sim \frac{u_{max}^2}{\alpha^{3/5} I^{4/15}}$$



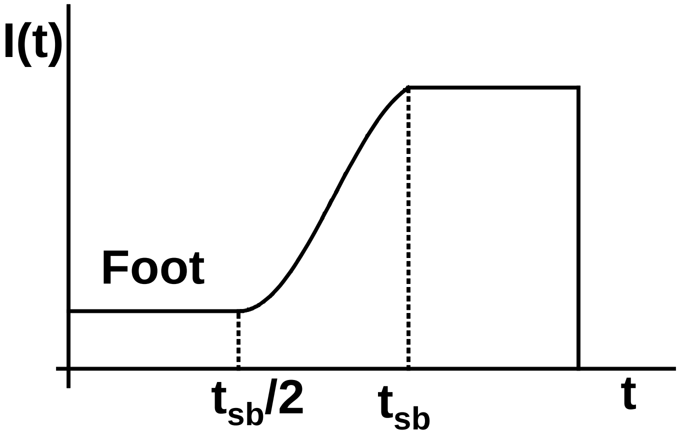
The final implosion velocity can be found using IFAR

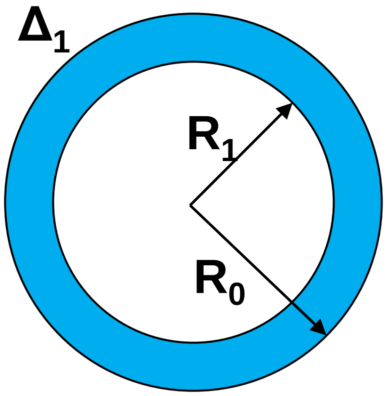
$$u_{\max}{}^{2} \sim IFAR \times \alpha^{3/5} I^{4/15}$$

$$IFAR = 4A_1 \left(\frac{I_{\max}}{I_{\text{foot}}}\right)^{2/5}$$

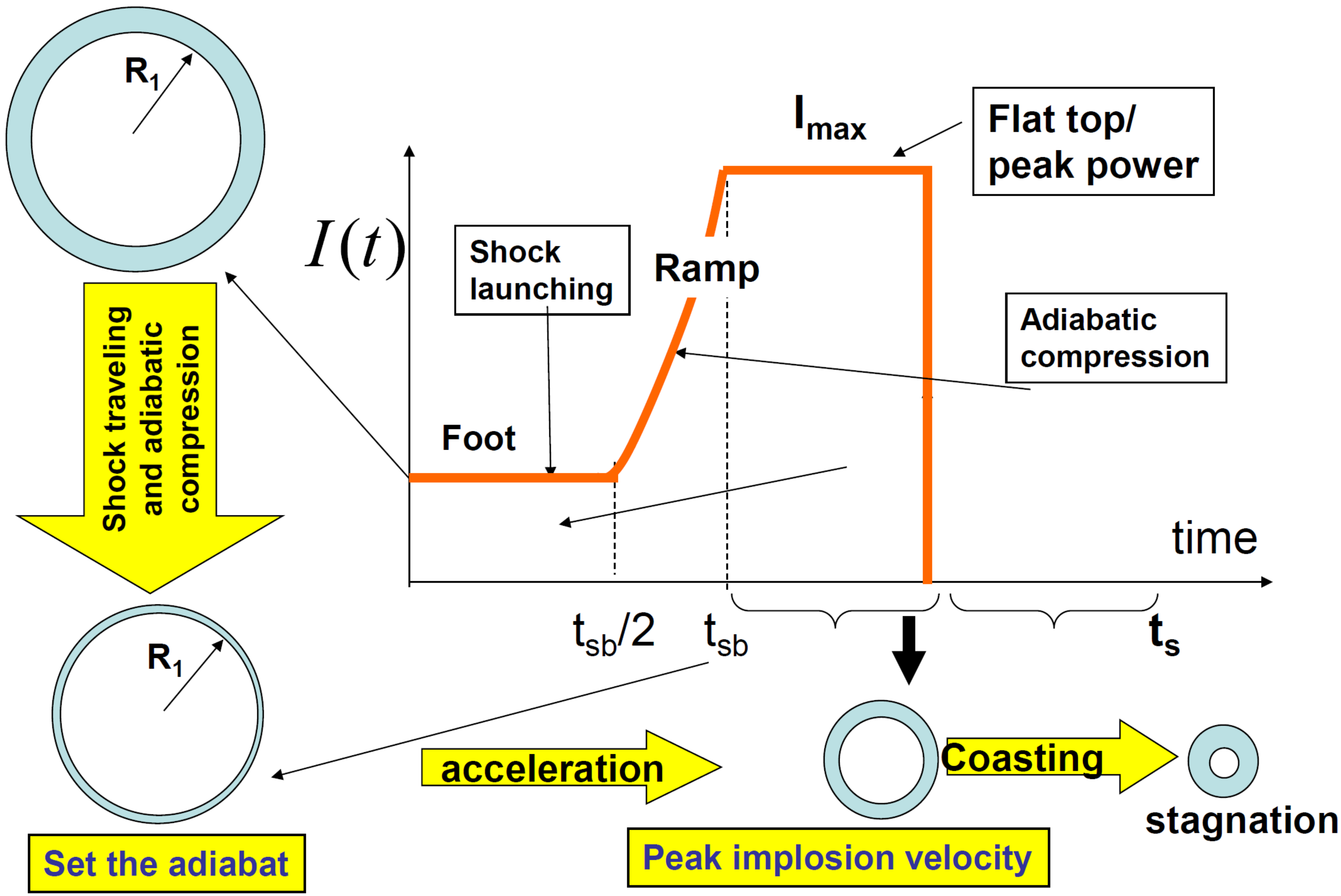
$$A_1 = \frac{R_1}{\Delta_1}$$

$$u_{\max,\text{cm}/s} \approx 10^7 \sqrt{0.7 A_1 \alpha^{3/5} I_{15,\max}{}^{4/15} \left(\frac{I_{\max}}{I_{\text{foot}}}\right)^{2/5}}$$





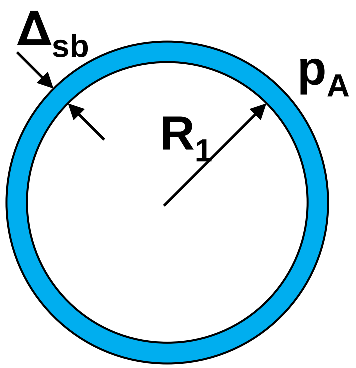
There are three stages in the laser pulse: foot, ramp, and flat top



A simple implosion theory can be derived in the limit of infinite initial aspect ratio

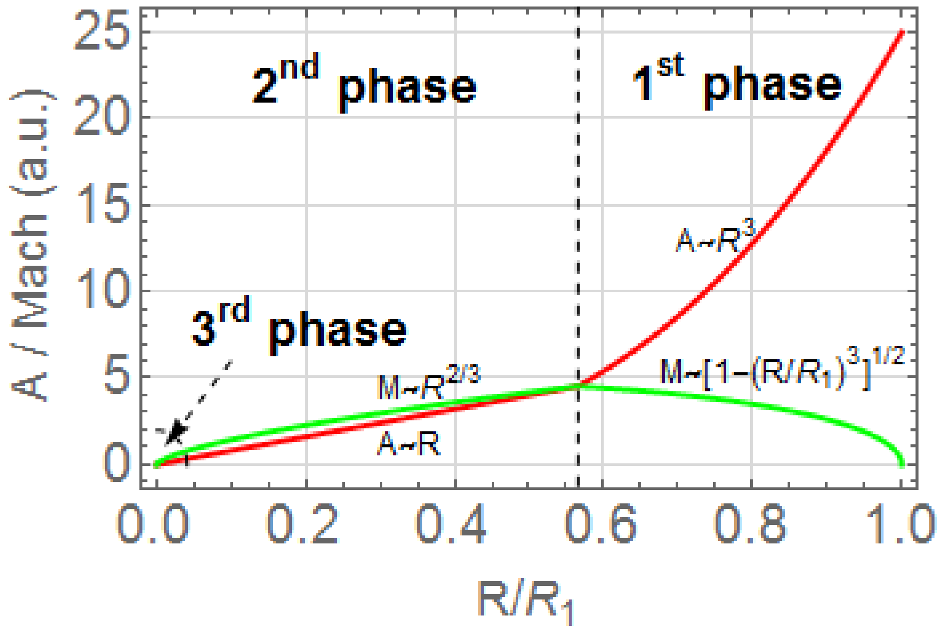
- Start from a high aspect ratio shell (thin shell) at the beginning of the acceleration phase
  - Constant ablated pressure
  - The adiabat is set and kept fixed by the first and the only shock

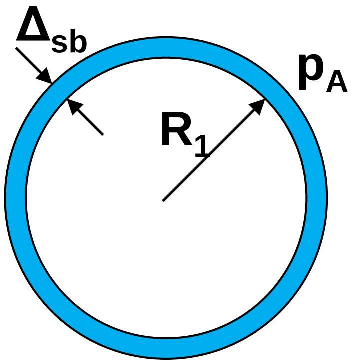
$$IFAR = A_{\mathrm{sb}} = \frac{R_1}{\Delta_{\mathrm{sb}}} \gg 1$$



Basko and Meyer-ter-vehn, Phys. Rev. Lett. *88*, 244502-1, 2002

The implosion are divided in 3 phases after the shock break out





- 1^{st} phase: acceleration
- 2^{nd} phase: coasting
- 3^{rd} phase: stagnation

The shell density is constant

- Shell expansion/contraction: $t_{\mathrm{ex}} \sim \frac{\Delta}{C_s}$
- Implosion time: $t_{\mathrm{i}} \sim \frac{R}{u_{\mathrm{i}}}$

$$\frac{t_{\mathrm{i}}}{t_{\mathrm{ex}}} \sim \frac{R}{\Delta}\frac{C_s}{u_{\mathrm{i}}} = \frac{A}{Mach} \qquad A = \frac{R}{\Delta} \qquad Mach = \frac{u}{C_s}$$

- In the acceleration phase $A \sim Mach^2$ $\quad IFAR \sim {Mach_{max}}^2$ (p29)

$$\frac{t_{\mathrm{i}}}{t_{\mathrm{ex}}} \sim \frac{A}{Mach} \sim Mach \sim \sqrt{A} \gg 1 \quad \Rightarrow \rho \approx \mathrm{const}$$

(Thin shell)

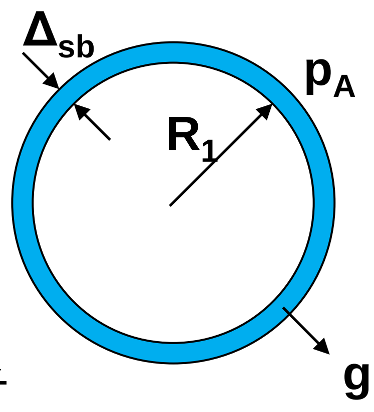
(implosion time >> expansion/contraction time)

- From mass conservation:

$$M \sim 4\pi R^2 \Delta\rho \quad \Rightarrow \quad \Delta \sim R^{-2} \qquad A = \frac{R}{\Delta} \sim R^3 \Rightarrow \quad A = A_{\mathrm{sb}}\left(\frac{R}{R_1}\right)^3$$

The shell density is constant

- **Shell expansion/contraction:** $t_{\mathrm{ex}} \sim \dfrac{\Delta_{\mathrm{sb}}}{C_s}$
- **Implosion time:** $t_{\mathrm{imp}} \sim \dfrac{R_1}{u_{\mathrm{i}}}$

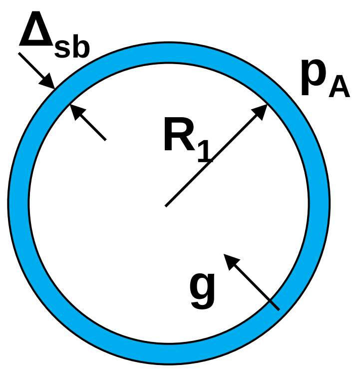


$$\frac{t_{imp}}{t_{ex}} \sim \frac{R_1}{\Delta_{sb}} \frac{C_s}{u_i} = \frac{A_{sb}}{Mach} \gg 1 \qquad A_{sb} = \frac{R_1}{\Delta_{sb}} \qquad Mach = \frac{u_{\mathrm{i}}}{C_s}$$

- **The pressure and the density are constant throughout the whole shell.**

The shell density is constant

- **In the shell frame of reference:**



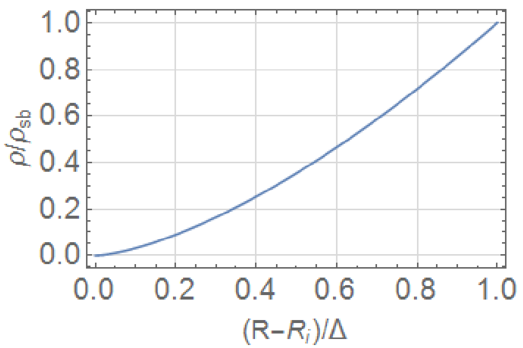
$$\rho(\partial_t \vec{w} + \vec{w} \cdot \nabla \vec{w}) = -\nabla p + \rho g \hat{r}$$

Neglect the first two term (check later) $\Rightarrow \dfrac{dp}{dr} = -\rho \ddot{R}$

Use $p = \alpha_0 \rho^{5/3}$ **and integrate along** ***r*****:**

$$\alpha_0 \frac{d\rho^{5/3}}{dr} = -\rho \ddot{R} \quad \Rightarrow \quad \alpha_0 \frac{d\rho^{5/3}}{\rho} = -\ddot{R} dr \quad \Rightarrow \quad \alpha_0 \frac{5}{3} \int_{@R_i}^{@R} \frac{\rho^{2/3}}{\rho} d\rho = -\ddot{R}(t) \int_{R_i}^{R} dr$$

$$\rho = \rho_{sb} \left(\frac{R - R_i}{\Delta} \right)^{3/2}$$



$$\text{where } \Delta = -\frac{5}{2} \frac{\alpha_0 \rho_{sb}^{2/3}}{\ddot{R}} = -\frac{3}{2} \frac{5}{3} \frac{p_A}{\rho_{sb}} \frac{1}{\ddot{R}}$$

$$\alpha \sim \frac{p}{\rho^{5/3}} \qquad C_s^2 \approx \frac{p_{ex}}{\rho_{ex}} \qquad = -\frac{3}{2} \frac{C_s^2}{\ddot{R}(t)}$$

The requirement of the 1st phase is obtained using mass conservation

- **Mass conservation:**

$$m=\int_{R_i}^{R_i+\Delta}\rho r^2 dr=\rho_{\rm sb}\int_{R_i}^{R_i+\Delta}\left(\frac{r-R_i}{\Delta}\right)^{3/2} r^2 dr \qquad \rho=\rho_{\rm sb}\left(\frac{R-R_i}{\Delta}\right)^{3/2}$$

$$\simeq \rho_{sb}{R_i}^2\Delta\int_{R_i}^{R_i+\Delta}\left(\frac{r-R_i}{\Delta}\right)^{3/2} d\left(\frac{r-R_i}{\Delta}\right)=\frac{2}{5}\rho_{sb}{R_i}^2\Delta\sim\frac{2}{5}\rho_{sb}R^2\Delta$$

$$\Delta=\frac{5}{2}\frac{m}{\rho_{\rm sb}R^2}\Rightarrow\dot{\Delta}=\frac{5}{2}\frac{m}{\rho_{\rm sb}}(-2)\frac{\dot{R}}{R^3}=-2\frac{\dot{R}}{R}\Delta=-2\frac{v}{A} \qquad \dot{\Delta}=-2\frac{v}{A} \qquad t_{\rm imp}\sim\frac{R_1}{u_{\rm i}}$$

$$\rho(\partial_{\rm t}\vec{w}+\vec{w}\cdot\nabla\vec{w})=-\nabla p+\rho g\hat{r} \qquad \vec{w}\sim\dot{\Delta} \qquad \partial_t\sim 1/t_{\rm imp} \qquad \nabla\sim 1/\Delta$$

$$\Rightarrow\ \rho\left(\frac{\dot{\Delta}}{t_{\rm imp}}+\frac{\dot{\Delta}^2}{\Delta}\right)\sim-\frac{p}{\Delta}+\rho\ddot{R} \qquad \rho\frac{\dot{\Delta}}{t_{\rm imp}}\sim\rho\frac{v}{At_{\rm imp}}\sim\rho\frac{v^2}{AR} \qquad \rho\frac{\dot{\Delta}^2}{\Delta}\sim\rho\frac{v^2}{A^2\Delta}\sim\rho\frac{v^2}{AR}$$

$$\rho\frac{\dot{\Delta}}{t_{\rm imp}}\Big/\frac{p}{\Delta}\sim\rho\frac{v^2}{AR}\frac{\Delta}{p}\sim\frac{v^2}{{C_s}^2}\frac{1}{A^2}=\frac{Mach^2}{A^2}$$

- $Mach << A$ **is the requirement for the 1st phase**

Aspect ratio and Mach number are functions of radius

$$A = \frac{R}{\Delta} = R^3 \left(\frac{2}{5}\frac{\rho_{sb}}{m}\right) \propto R^3 \Rightarrow$$

$$A = A_{sb}\left(\frac{R}{R_1}\right)^3 = \mathrm{IFAR}\left(\frac{R}{R_1}\right)^3$$

$$\Delta = -\frac{3}{2}\frac{{C_s}^2}{\ddot{R}} \quad \text{(p36)} \Rightarrow \quad \ddot{R} = -\frac{3}{2}\frac{{C_s}^2}{\Delta} = -\frac{3}{2}\left(\frac{2}{5}\frac{\rho_{sb}R^2}{m}\right)\left(\frac{5}{3}\frac{p_A}{\rho_{sb}}\right)$$

$$\dot{R}\frac{d\dot{R}}{\mathrm{dt}} = -\frac{p_A}{m}R^2\dot{R} \qquad \frac{1}{2}\int d\dot{R}^2 = -\frac{p_A}{m}\int R^2\mathrm{dR} \qquad \dot{R}^2 = \frac{2}{3}\frac{p_A {R_1}^3}{m}\left[1-\left(\frac{R}{R_1}\right)^3\right]$$

$$Mach^2 = \frac{\dot{R}^2}{{C_s}^2} = \frac{2}{3}\frac{p_A {R_1}^3}{m}\frac{3}{5}\frac{\rho_{sb}}{p_A}\left[1-\left(\frac{R}{R_1}\right)^3\right] = \frac{2}{5}\frac{{R_1}^3\rho_{sb}}{m}\left[1-\left(\frac{R}{R_1}\right)^3\right]$$

$$Mach = Mach_{max}\left[1-\left(\frac{R}{R_1}\right)^3\right]^{1/2}$$

$$Mach_{\max}{}^2 = \frac{2}{5}\frac{{R_1}^3\rho_{sb}}{m} = \frac{2}{5}\frac{5}{2}\frac{{R_1}^3\rho_{sb}}{\rho_{sb}{R_1}^2\Delta_{sb}} = \frac{R_1}{\Delta_{sb}} = A_{sb}$$

The model breaks down when A~Mach

$\boldsymbol{A \sim \mathrm{Mach}}$

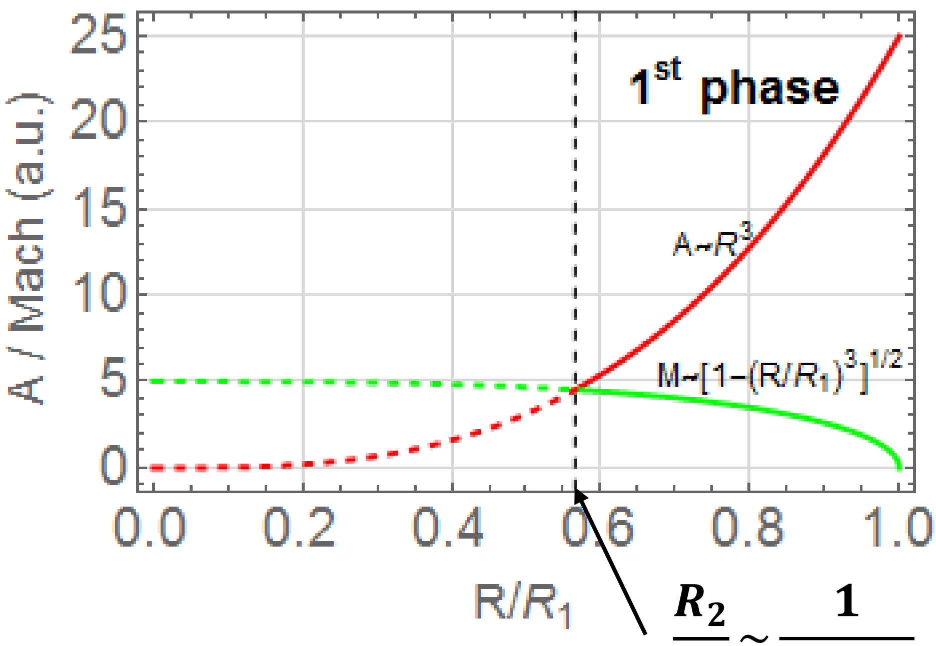
$$A_{\rm sb}\left(\frac{R}{R_1}\right)^3 \sim Mach_{\rm max}\left[1-\left(\frac{R}{R_1}\right)^3\right]^{1/2} = \sqrt{A_{\rm sb}}\left[1-\left(\frac{R}{R_1}\right)^3\right]^{1/2}$$

$$A_{\rm sb}\left(\frac{R}{R_1}\right)^6 \sim 1-\left(\frac{R}{R_1}\right)^3 \;\Rightarrow\; A_{\rm sb}\left(\frac{R}{R_1}\right)^6 + \left(\frac{R}{R_1}\right)^3 - 1 \sim 0$$

$$\left(\frac{R}{R_1}\right)^3 \sim \frac{-1 \pm \sqrt{1+4A_{\rm sb}}}{2A_{\rm sb}} \sim \frac{-1 \pm 2\sqrt{A_{\rm sb}}}{2A_{\rm sb}} \sim \frac{1}{\sqrt{A_{\rm sb}}} \qquad \because \sqrt{A_{\rm sb}} >> 1$$

$$\frac{R}{R_1} \sim \frac{1}{{A_{\rm sb}}^{1/6}} \ll 1 \qquad A = A_{\rm sb}\left(\frac{R}{R_1}\right)^3 \sim \sqrt{A_{\rm sb}} \gg 1$$

Summary of phase 1 (acceleration phase)



$$\frac{R_2}{R_1} \sim \frac{1}{{A_{sb}}^{1/6}}$$

$$\frac{1}{{A_{sb}}^{1/6}} < \frac{R}{R_1} \leq 1$$

$$A = A_{sb}\left(\frac{R}{R_1}\right)^3 = \mathrm{IFAR}\left(\frac{R}{R_1}\right)^3$$

$$Mach = Mach_{max}\left[1 - \left(\frac{R}{R_1}\right)^3\right]^{1/2}$$

$$Mach_2 \simeq \mathrm{Mach_{max}}\left(1 - \frac{1}{\sqrt{A_{sb}}}\right)^{1/2} \simeq \mathrm{Mach_{max}} = \sqrt{A_{sb}} \qquad A_2 \sim \sqrt{A_{sb}}$$

The 2nd phase starts when R < R_2

- *A* decreases as *R* decreases. Eventually, *A* < *Mach*
- *A* >> 1 is required for thin shell model
- Assuming that the laser is off (coasting phase) when $R/R_1 \sim {A_{sb}}^{1/6}$

$$\rho(\partial_t \vec{w} + \vec{w} \cdot \nabla \vec{w}) = -\nabla p \qquad t_{\mathrm{imp2}} \sim \frac{R_2}{u_{\mathrm{i}}}$$

$$\Rightarrow \frac{\dot{\Delta}}{t_{\mathrm{imp2}}} + \frac{\dot{\Delta}^2}{\Delta} \sim -\frac{p/\rho}{\Delta} \qquad \frac{\dot{\Delta}}{t_{\mathrm{imp2}}} = \frac{\dot{\Delta}}{\Delta}\frac{u_{\mathrm{i}}}{R_2/\Delta} = \frac{\dot{\Delta}}{\Delta}\frac{u_{\mathrm{i}}}{A}$$

$$\underbrace{\frac{\dot{\Delta} u_i}{A}}_{(1)} + \underbrace{\dot{\Delta}^2}_{(2)} \sim \underbrace{{C_s}^2}_{(3)}$$

- There are two possibilities:
  - Case 1: (3) << (1) and/or (2)
  - Case 2: (3) ~ (1) and/or (2)

The shell thickness does not change in the 2nd phase (coasting phase)

$$\underbrace{\frac{\dot{\Delta} u_i}{A}}_{(1)} + \underbrace{\dot{\Delta}^2}_{(2)} \sim \underbrace{{C_s}^2}_{(3)}$$

- Case 1: (3) << (1) and/or (2)

$$\dot{\Delta}\left(\frac{u_i}{A} + \dot{\Delta}\right) \sim 0 \qquad \dot{\Delta} \sim 0 \quad \text{or} \quad \Delta \equiv \Delta_2 = \text{constant}$$

- Case 2: (3) ~ (1) and/or (2) and $A << Mach$
  - (3) ~ (1)

$$\frac{\dot{\Delta} u_i}{A} \sim {C_s}^2 \;\Rightarrow\; \dot{\Delta} \sim \frac{C_s A}{u_i / C_s} = \frac{C_s A}{\text{Mach}}$$

$$\frac{\delta\Delta}{\Delta} \sim \frac{\dot{\Delta} t_{\text{imp2}}}{\Delta} \sim \frac{1}{\Delta} \frac{C_s A}{\text{Mach}} \frac{R_2}{u_i} \sim \frac{A^2}{\text{Mach}^2} << 1$$

  - (3) ~ (2)

$$\dot{\Delta}^2 \sim {C_s}^2 \qquad \frac{\delta\Delta}{\Delta} \sim \frac{\dot{\Delta} t_{\text{imp2}}}{\Delta} \sim \frac{C_s}{\Delta} \frac{R_2}{u_i} \sim \frac{A}{\text{Mach}} << 1$$

Change of shell thickness is small!

$$\frac{R_2}{R_1} \sim \frac{1}{A_{\text{sb}}{}^{1/6}}$$

$$A_2 \sim \sqrt{A_{\text{sb}}}$$

$$\Delta \equiv \Delta_2 = \text{constant} = \Delta_2 \frac{R_2}{R_2} = \frac{R_2}{A_2} \frac{R_1}{R_1} = \frac{1}{{A_{sb}}^{1/6}} \frac{R_1}{\sqrt{A_{sb}}} \sim \frac{R_1}{{A_{sb}}^{2/3}}$$

To verify that *A* << *Mach*

- Comparison of *A* and *Mach* (with constant Δ_2):

$$A \approx \frac{R}{\Delta_2}\frac{R_2}{R_2} = A_2\left(\frac{R}{R_2}\right) \qquad Mach \sim \frac{u_i}{C_s} \sim \frac{u_i}{\sqrt{p/\rho}} \sim \frac{u_i}{\sqrt{\alpha\rho^{2/3}}} = \frac{u_i}{\alpha^{1/2}\rho^{1/3}}$$

$$m \sim \bar{\rho}R^2\Delta \simeq \bar{\rho}R^2\Delta_2 \qquad \Rightarrow\ \bar{\rho} \simeq \frac{m}{R^2\Delta_2}$$

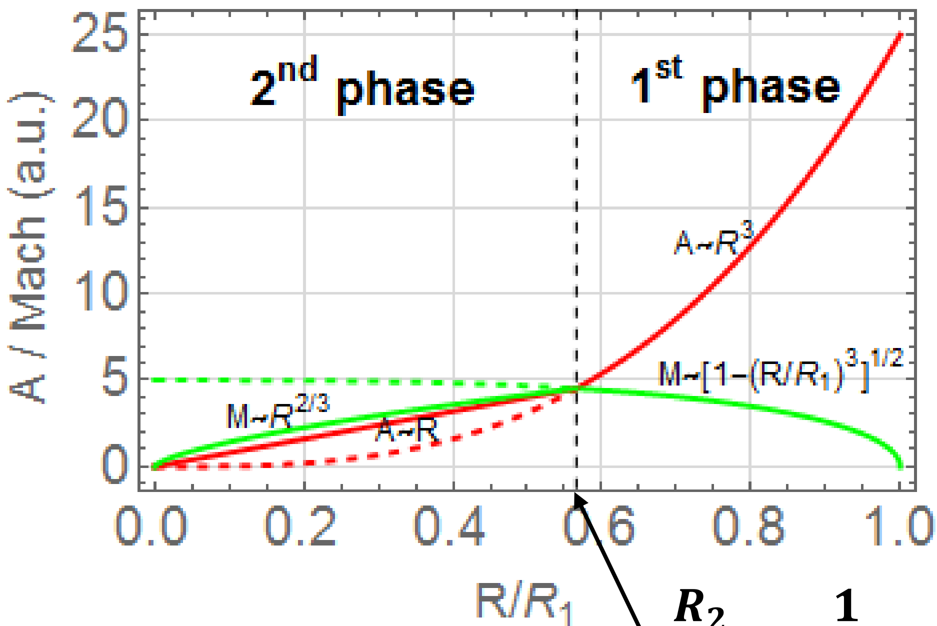
$$Mach \sim \frac{u_i}{\alpha^{1/2}}\left(\frac{R^2\Delta_2}{m}\right)^{1/3} = \frac{u_i}{\alpha^{1/2}}\left(\frac{\Delta_2 {R_2}^2}{m}\right)^{1/3}\left(\frac{R}{R_2}\right)^{2/3} = Mach_2\left(\frac{R}{R_2}\right)^{2/3}$$

$$\text{where } Mach_2 = Mach(R = R_2) = \frac{u_i}{\alpha^{1/2}}\left(\frac{{R_2}^2\Delta_2}{m}\right)^{1/3} \sim A_2 \sim \sqrt{A_{\mathrm{sb}}}$$

$$\frac{A}{\mathrm{Mach}} \sim \frac{A_2\left(\frac{R}{R_2}\right)}{\mathrm{Mach}_2\left(\frac{R}{R_2}\right)^{2/3}} \sim \left(\frac{R}{R_2}\right)^{1/3} \ll 1$$

- Requirement for thin shell model: $A \gg 1 \ \Rightarrow\ A_2\left(\frac{R}{R_2}\right) \gg 1 \ \Rightarrow\ \frac{R}{R_2} \gg \frac{1}{A_2} \sim \frac{1}{\sqrt{A_{\mathrm{sb}}}}$

Summary of phase 2 (coasting phase)



$$\frac{R_2}{R_1} \sim \frac{1}{{A_{\rm sb}}^{1/6}}$$

$$1 < A < \sqrt{A_{\rm sb}} \qquad A < \text{Mach}$$

$$\frac{1}{\sqrt{A_{\rm sb}}} \sim \frac{1}{A_2} < \frac{R}{R_2} < 1$$

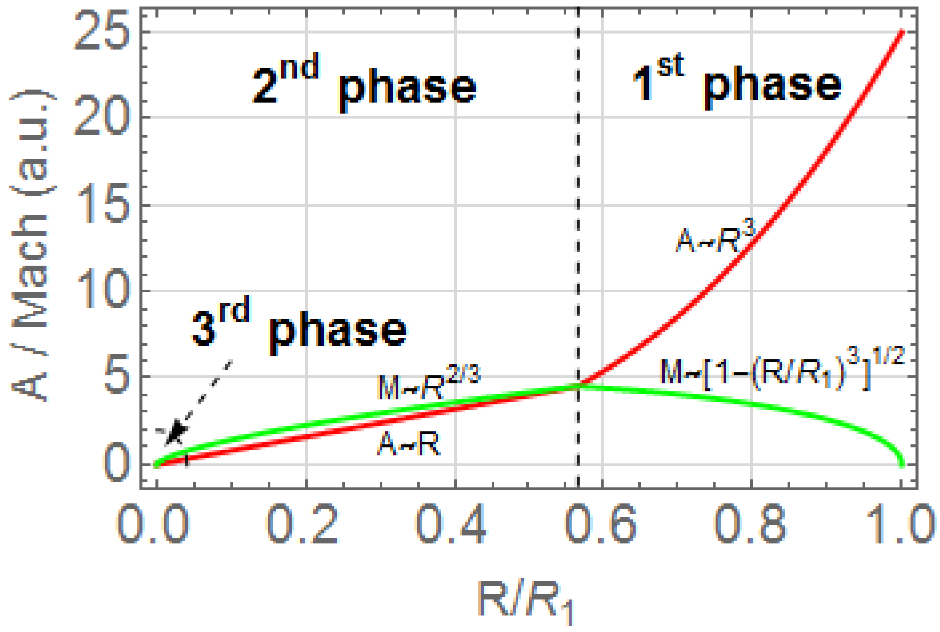
$$A = A_2\left(\frac{R}{R_2}\right) \sim \sqrt{A_{\rm sb}}\left(\frac{R}{R_2}\right)$$

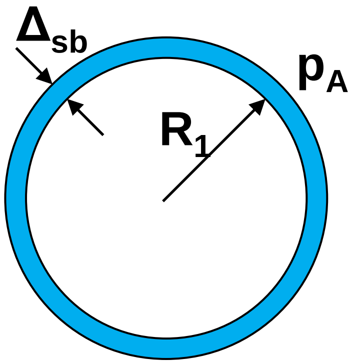
$$Mach \sim \text{Mach}_2\left(\frac{R}{R_2}\right)^{2/3} \sim \sqrt{A_{\rm sb}}\left(\frac{R}{R_2}\right)^{2/3}$$

$$Mach_2 = \text{Mach}_{\max} \simeq A_2 = \sqrt{A_{\rm sb}}$$

$$\Delta \simeq \text{constant} = \Delta_2 \sim \frac{R_1}{{A_{\rm sb}}^{2/3}} \qquad \bar{\rho} \simeq \rho_2\left(\frac{R_2}{R}\right)^2 \sim \rho_{\rm sb}\left(\frac{R_2}{R}\right)^2$$

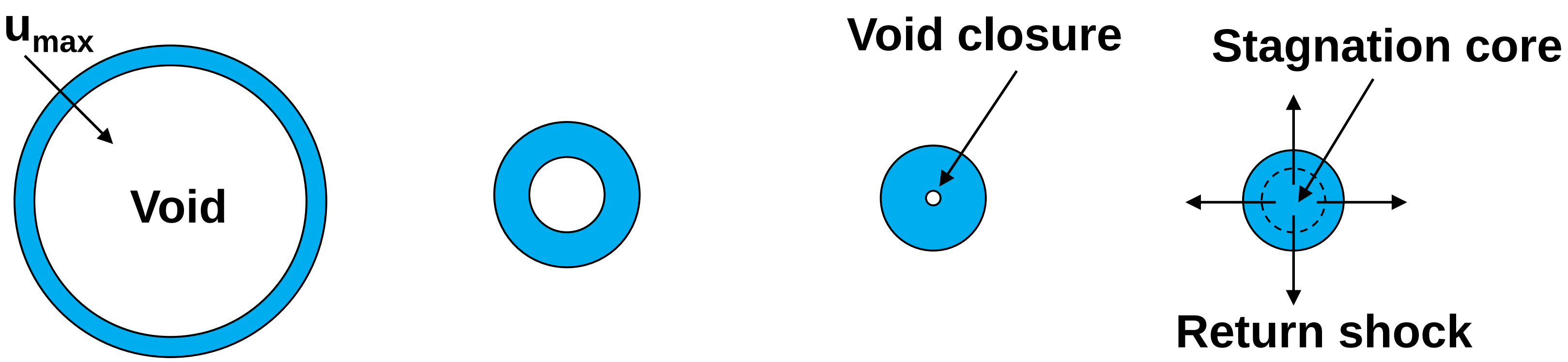
How about the 3rd phase where A~1 ?





- 1st phase: acceleration
- 2nd phase: coasting
- 3rd phase: stagnation

The thin shell model breaks down when A~1



- When A~1 => Δ~R, the "void" inside the shell closes and a "return shock" propagating outward is generated due to the collision of the shell with itself
- The density is compressed by a factor no more than 4 even if the strong shock is generated

$\rho_{st} \sim \rho_3$ where ρ_3 is the density right before the void closure

The stagnated density scales with square of the maximum Mach number

$$\rho_3 \sim \rho_2 \left(\frac{R_2}{R_3}\right)^2 \sim \rho_{\rm sb} \left(\frac{R_2}{R_3}\right)^2$$

$$\bar{\rho} \simeq \rho_2 \left(\frac{R_2}{R}\right)^2 \quad \text{(p40)}$$

$$A = A_3 \sim 1 \Rightarrow \frac{R_3}{\Delta_3} \sim \frac{R_3}{\Delta_2} \sim 1 \Rightarrow R_3 \sim \Delta_2$$

$$\rho_{\rm st} \sim \rho_3 \sim \rho_{\rm sb} \left(\frac{R_2}{\Delta_2}\right)^2 \sim \rho_{\rm sb} {A_2}^2 \sim \rho_{\rm sb} {Mach_2}^2 \sim \rho_{\rm sb} {Mach_{\rm max}}^2$$

$$\frac{\rho_{\rm st}}{\rho_{\rm sb}} \sim {Mach_{\rm max}}^2$$

← **Density compression scaling law.**

The stagnated pressure scales to the 4th power of the maximum Mach number

- **Conservation of energy at stagnation:**

$$p_{st}{R_{st}}^{3}\sim m{u_{max}}^{2} \qquad R_{st}\sim R_3\sim \Delta_3\sim \Delta_2 \quad \Rightarrow \quad p_{st}{\Delta_2}^{3}\sim \underline{m}{u_{max}}^{2}\sim \underline{\rho_2 {R_2}^{2}\Delta_2}{u_{max}}^{2}$$

$$p_{st}\sim \rho_2\left(\frac{R_2}{\Delta_2}\right)^2 {u_{max}}^{2} = \rho_2 {A_2}^{2}{u_{max}}^{2}\sim p_2\frac{{Mach_2}^{2}{u_{max}}^{2}}{p_2/\rho_2}\sim p_A {Mach_2}^{4}\sim p_A {Mach_{max}}^{4}$$

$$Mach_2 = Mach_{max}\simeq A_2 = \sqrt{A_{sb}}$$

$$\boxed{\frac{p_{st}}{p_A}\sim {Mach_{max}}^{4}}$$

$$\alpha_{st}\sim\frac{p_{st}}{{\rho_{st}}^{5/3}}\sim\frac{p_A {Mach_{max}}^{4}}{{\rho_{sb}}^{5/3}{Mach_{max}}^{10/3}} = \alpha_{sb}{Mach_{max}}^{2/3}$$

$$\frac{\rho_{st}}{\rho_{sb}}\sim {Mach_{max}}^{2}$$

$$\boxed{\frac{\alpha_{st}}{\alpha_{sb}}\sim {Mach_{max}}^{2/3}}$$

Scaling of the areal density of the compressed core

$$\rho_{\rm st} R_{\rm st} \sim \rho_{\rm st} \Delta_2 \sim \left(\frac{p_{\rm st}}{\alpha_{\rm st}}\right)^{3/5} \frac{\Delta_2}{R_2} \frac{R_2}{R_1} R_1 \sim \left(\frac{p_A\, {\rm Mach_{max}}^4}{\alpha_{\rm sb}\, {\rm Mach_{max}}^{2/3}}\right)^{3/5} \frac{1}{A_2} \frac{1}{{A_{\rm sb}}^{1/6}} R_1$$

$A_2 \sim {\rm Mach_{max}}$ $\quad$ $A_{\rm sb} \sim {\rm Mach_{max}}^2$

$$\rho_{\rm st} R_{\rm st} \sim \left(\frac{p_A}{\alpha_{\rm sb}}\right)^{3/5} {\rm Mach_{max}}^2 \frac{1}{\rm Mach_{max}} \frac{1}{{\rm Mach_{max}}^{1/2}} R_1$$

$$\sim \left(\frac{p_A}{\alpha_{\rm sb}}\right)^{3/5} {\rm Mach_{max}}^{2/3} R_1 \sim \left(\frac{p_A}{\alpha_{\rm sb}}\right)^{3/5} \frac{{u_{\rm max}}^{2/3}}{(p_A/\rho_{\rm sb})^{1/3}} \frac{{p_0}^{1/3} R_1}{{p_0}^{1/3}}$$

$$\sim \left(\frac{p_A}{\alpha_{\rm sb}}\right)^{3/5} \frac{{u_{\rm max}}^{2/3}}{({p_A}^{2/5} {\alpha_{\rm sb}}^{3/5})^{1/3}} \frac{\left(p_A {R_1}^3\right)^{1/3}}{{p_A}^{1/3}} \sim \frac{{p_A}^{2/15}}{{\alpha_{\rm sb}}^{4/5}} {u_{\rm max}}^{2/3} {E_k}^{1/3}$$

$E_k \sim E_{\rm las} \Rightarrow$

$$\rho_{\rm st} R_{\rm st} \sim \frac{{p_A}^{2/15} {u_{\rm max}}^{2/3} {E_{\rm las}}^{1/3}}{{\alpha_{\rm sb}}^{4/5}}$$

$E_k \sim p_A {R_1}^3$

Amplification of areal density

$$\rho_{st}R_{st} \sim \rho_{st}{}^{2/3}\left(\rho_{st}R_{st}{}^{3}\right)^{1/3} \sim \rho_{sb}{}^{2/3} Mach_{max}{}^{4/3} Mass^{1/3}$$

$$\sim \frac{\rho_{sb}{}^{2/3}}{\rho_1{}^{2/3}} Mach_{max}{}^{4/3} \rho_1{}^{2/3}\left(\rho_1 R_1{}^2 \Delta_1\right)^{1/3}$$

$$\rho_{st}R_{st} \sim (\rho_1 \Delta_1) Mach_{max}{}^{4/3} A_1{}^{2/3}\left(\frac{\rho_{sb}}{\rho_1}\right)^{2/3}$$

$$\frac{\rho_{sb}}{\rho_1} = 4\left(\frac{I_{max}}{I_{foot}}\right)^{2/5}$$

$$(\rho R)_{st} \sim (\rho_1 \Delta_1)\mathrm{IFAR}^{2/3} A_1{}^{2/3}\left(\frac{I_{max}}{I_{foot}}\right)^{4/15}$$

$$E_{las} = 4\pi R_1{}^2 I_{max} t_{imp} \approx 4\pi R_1{}^2 I_{max}\frac{R_1}{u_{max}}$$

$$E_{las} \approx \frac{4\pi R_1{}^3 I_{max}}{u_{max}}$$

Summary

$$A_{\mathrm{sb}} = \mathrm{IFAR} = 4A_1\left(\frac{I_{\max}}{I_{\mathrm{foot}}}\right)^{2/5} \qquad u_{\max,\mathrm{cm}/s} \approx 10^7\sqrt{0.7A_1\alpha^{3/5}I_{15,\max}{}^{4/15}\left(\frac{I_{\max}}{I_{\mathrm{foot}}}\right)^{2/5}}$$

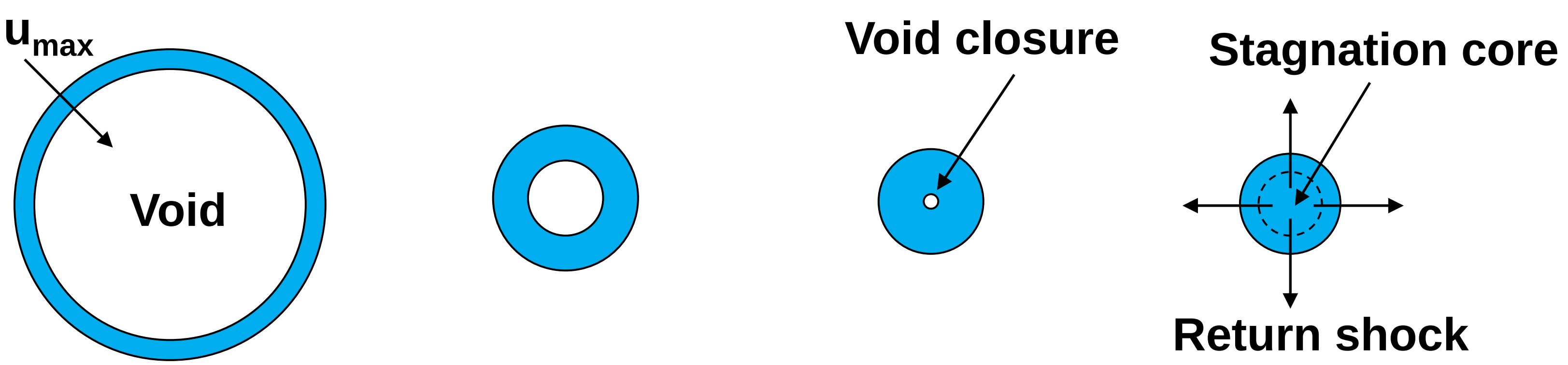
$$\rho_{\mathrm{st}} \sim \rho_{\mathrm{sb}}\mathrm{Mach}_{\max}{}^2 \sim \rho_1\mathrm{IFAR}\left(\frac{I_{\max}}{I_{\mathrm{foot}}}\right)^{2/5}$$

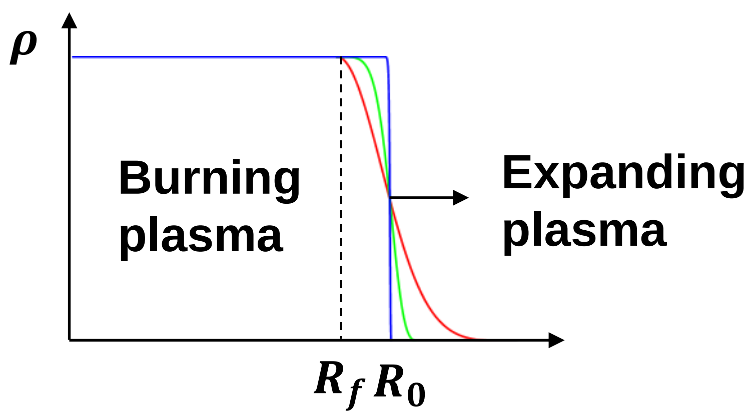
$$p_{\mathrm{st}} \sim p_A\mathrm{Mach}_{\max}{}^4 \sim p_A\mathrm{IFAR}^2$$

$$\alpha_{\mathrm{st}} \sim \alpha_{\mathrm{sb}}\mathrm{Mach}_{\max}{}^{2/3} \sim \alpha_{\mathrm{sb}}\mathrm{IFAR}^{1/3}$$

$$(\rho R)_{\mathrm{st}} \sim (\rho_1\Delta_1)\mathrm{IFAR}^{2/3}A_1{}^{2/3}\left(\frac{I_{\max}}{I_{\mathrm{foot}}}\right)^{4/15}$$

Calculation of the burn-up fraction



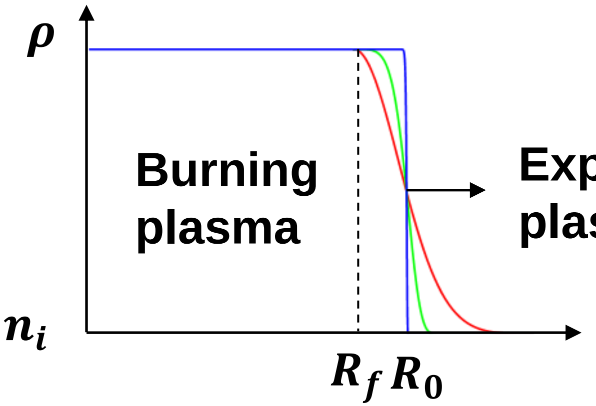


$$R_f = R_0 - C_s t$$

$$\frac{\partial n_i}{\partial t} = -\nabla \cdot (n_i v) - \frac{{n_i}^2}{4} < \sigma v > \times 2$$

Calculation of the burn-up fraction - continue

$$4\pi\int_0^{R_f} r^2 dr\left(\frac{\partial n_i}{\partial t} = -\nabla\cdot(n_i v) - \frac{{n_i}^2}{2}\langle\sigma v\rangle\right)$$



$$4\pi\int_0^{R_f} r^2\frac{\partial n_i}{\partial t}dr = 4\pi\partial_t\int_0^{R_f} r^2 n_i dr - 4\pi\dot{R}_f {R_f}^2 n_i$$

$$= -n_i v 4\pi {R_f}^2 - \frac{{n_i}^2}{2}\langle\sigma v\rangle V_f$$

(neglect)

(Leibniz integral rule)

$$N_f \equiv \frac{4\pi}{3}{R_f}^3 n_i \equiv V_f n_i$$

$$d_t N_f - 3N_f\frac{\dot{R}_f}{R_f} = -\frac{{N_f}^2}{V_f}\frac{\langle\sigma v\rangle}{2}$$

$$\frac{d_t N_f}{{N_f}^2} - \frac{3\dot{R}_f}{N_f R_f} = -\frac{\langle\sigma v\rangle}{2V_f}$$

$$d_t\frac{1}{N_f} + \frac{3\dot{R}_f}{N_f R_f} = \frac{\langle\sigma v\rangle}{2V_f}$$

$${R_f}^3 d_t\frac{1}{N_f} + 3{R_f}^2\frac{\dot{R}_f}{N_f}$$

$$= \frac{d}{\mathrm{dt}}\left(\frac{{R_f}^3}{N_f}\right) = \frac{\langle\sigma v\rangle}{2V_f}{R_f}^3$$

Calculation of the burn-up fraction - continue

$$\frac{d}{\mathrm{dt}}\left(\frac{{R_f}^3}{N_f}\right) = \frac{\langle\sigma v\rangle}{2V_f}{R_f}^3$$

$$\frac{{R_f}^3}{N_f} = \int_0^t \frac{\langle\sigma v\rangle}{2V_f}{R_f}^3 dt + \frac{{R_0}^3}{N_0}$$

$$R_f = R_0 - C_s t \qquad dt = -\frac{\mathrm{dR}_f}{C_s} \qquad V_f = \frac{4\pi}{3}{R_f}^3$$

$$\frac{{R_f}^3}{N_f} = -\int_{R_0}^{R_f} \frac{\langle\sigma v\rangle}{2\times 4\,\pi/3}\frac{dR_f}{C_s} + \frac{{R_0}^3}{N_0}$$

$$n_0 = \frac{N_0}{V_0}$$

$$\frac{{R_f}^3}{N_f} = -\int_{R_0}^{R_f} \frac{\langle\sigma v\rangle}{2C_s}\frac{3}{4\pi}dR_f + \frac{{R_0}^3}{N_0}$$

$$\frac{V_f}{N_f} = \frac{V_0}{N_0}\left[1 + \frac{\langle\sigma v\rangle}{2C_s}n_0 R_0\left(1 - \frac{R_f}{R_0}\right)\right]$$

$$\frac{{R_f}^3}{N_f} = \frac{\langle\sigma v\rangle}{2C_s}\frac{3}{4\pi}\left(R_0 - R_f\right) + \frac{{R_0}^3}{N_0}$$

$$\xi \equiv \frac{\langle\sigma v\rangle}{2C_s}n_0 R_0$$

$$\frac{V_f}{N_f} = \frac{\langle\sigma v\rangle}{2C_s}\mathrm{R}_0\left(1 - \frac{R_f}{R_0}\right) + \frac{V_0}{N_0}$$

$$\frac{V_f}{N_f} = \frac{1}{n_0}\left[1 + \xi\left(1 - \frac{R_f}{R_0}\right)\right]$$

Calculation of the burn-up fraction - continue

$$\frac{V_f}{N_f} = \frac{1}{n_0}\left[1 + \xi\left(1 - \frac{R_f}{R_0}\right)\right] \qquad n_i = \frac{N_f}{V_f}$$

#Burned ions $= \int_0^t \frac{\langle\sigma v\rangle}{2} n_i^2 V_f dt = \int_0^t \frac{\langle\sigma v\rangle}{2} \frac{N_f^{\,2}}{V_f} dt = -\int_{R_0}^{R_f} \frac{\langle\sigma v\rangle}{2}\left(\frac{N_f}{V_f}\right)^2 V_f \frac{dR_f}{C_s}$

$$= \int_{R_f}^{R_0} \frac{\langle\sigma v\rangle}{2} \frac{n_0^{\,2}}{\left[1 + \xi\left(1 - \frac{R_f}{R_0}\right)\right]^2} \left(\frac{R_f}{R_0}\right)^3 V_0 R_0 \frac{dR_f/R_0}{C_s}$$

$$= \int \frac{\langle\sigma v\rangle}{2} \frac{n_0^{\,2}}{[1 + \xi(1 - x)]^2} x^3 V_0 \frac{R_0}{C_s} dx = N_0 \xi \int_0^1 \frac{x^3 dx}{[1 + \xi(1 - x)]^2}$$

$$= N_0 \frac{\xi[6 + \xi(9 + 2\xi)] - 6(1 + \xi)^2 Ln[1 + \xi]}{2\xi^3}$$

#Burn-up Fraction

$$\Theta(\xi) = \frac{\xi[6 + \xi(9 + 2\xi)] - 6(1 + \xi)^2 \mathrm{Ln}[1 + \xi]}{2\xi^3}$$

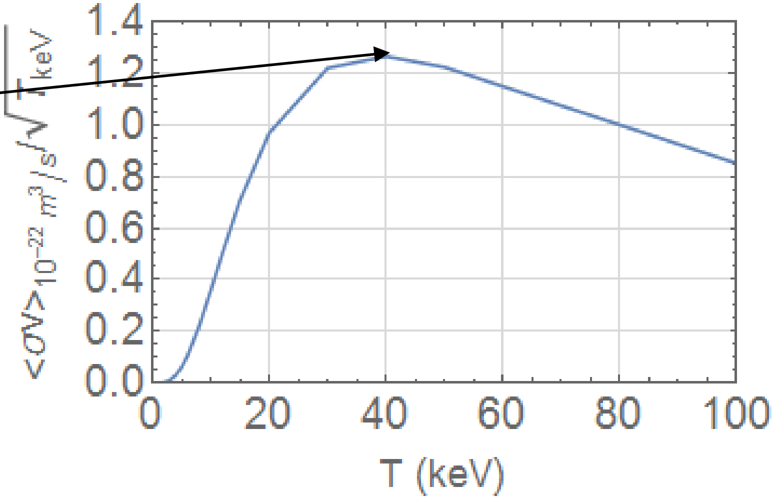
Calculation of the burn-up fraction - continue

$$C_s = \sqrt{\frac{T_e + T_i}{m_i}} = \sqrt{\frac{2T}{m_i}} \qquad \rho = n_0 m_i \quad m_i = \frac{m_D + m_T}{2} = 2.5 \times 1.67 \times 10^{-27}\mathrm{kg}$$

$$\xi = \frac{\langle \sigma v \rangle}{\sqrt{T}} (\rho R_0) \frac{1}{2\sqrt{2m_i}} = \frac{\langle \sigma v \rangle_{m^3/s}}{\sqrt{T_{keV} \times 1.6 \times 10^{-16}}} \frac{(\rho R_0)_{g/cm^2} \times 10}{2\sqrt{5 \times 1.67 \times 10^{-27}}}$$

$$\xi \simeq \frac{1.25 \times 10^{-22}}{\sqrt{1.6 \times 10^{-16}}} \frac{10(\rho R_0)_{g/cm^2}}{2\sqrt{5 \times 1.67 \times 10^{-27}}} = 0.54(\rho R_0)_{g/cm^2}$$

$$\left.\frac{\langle \sigma v \rangle}{\sqrt{T_{keV}}}\right|_{\max} = 1.25 \times 10^{-22} \quad @\ T = 40\ \mathrm{keV}$$



H.-S. Bosch and G.M. HaleNucl. Fusion ***32*** (1992) 611

Smallest areal density (ρR)

#Burned-up Fraction $$\Theta(\xi) = \frac{\xi[6+\xi(9+2\xi)] - 6(1+\xi)^2 \mathrm{Ln}[1+\xi]}{2\xi^3}$$

$$\lim_{\xi\to 0} \Theta(\xi) = \frac{\xi}{4} \qquad \lim_{\xi\to\infty} \Theta(\xi) = 1 \qquad \Theta(\xi) \approx \frac{\xi}{4+\xi} \qquad \xi \simeq 0.54(\rho R_0)_{g/cm^2}$$

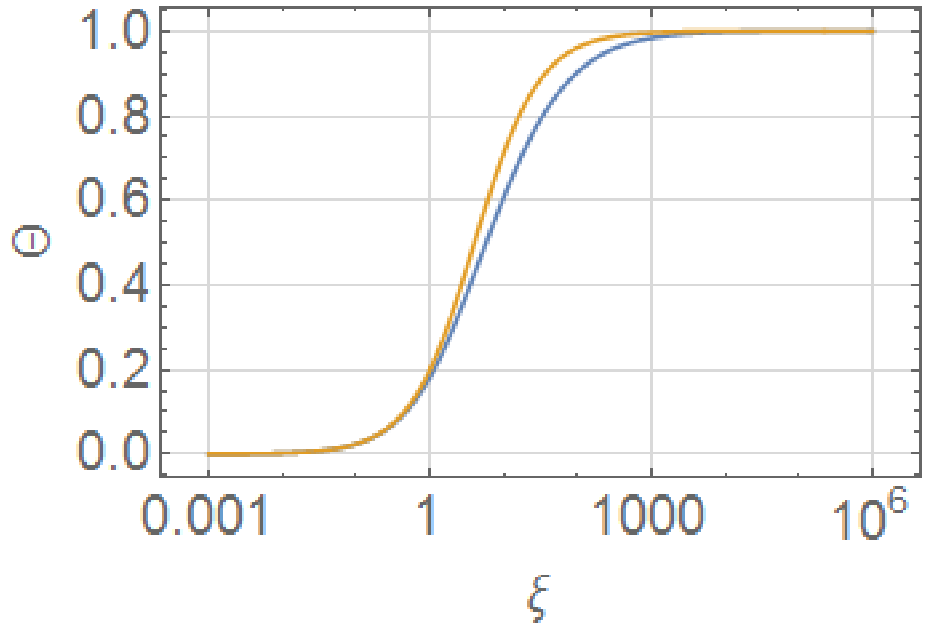
$$\Theta(\xi) \approx \frac{0.54\rho R}{4+0.54\rho R}$$

$$\Theta(\xi) \approx \frac{(\rho R)_{g/cm^2}}{7+(\rho R)_{g/cm^2}}$$

Large *ρR* is needed to have high burn-up fraction.

For energy applications:

$\Theta \gtrsim 0.3$ $\rho R \geq 3\ g/\mathrm{cm}^2$



Energy gain

Mass = M_0
Temp = T
DT
Volume = V_0

Fusion energy $= \dfrac{M_0}{2m_i}\epsilon_f \Theta$

$$\epsilon_f = 17.6\text{MeV}$$

Energy gain $= \dfrac{\text{Fusion Energy}}{\text{Input Energy}}$

- Input energy: the sphere is heated to the temperature T

Thermal energy in sphere: $\dfrac{3}{2}(n_{i0}T_i + n_{e0}T_e)V_0$

$$n_{i0} = n_{e0} \equiv n_0 \qquad T_e = T_i \Longrightarrow 3n_0\text{TV}_0 = 3\frac{M_0}{m_i}T$$

Set heating efficiency: $\eta = \dfrac{\text{Thermal Energy}}{\text{Input Energy}}$

$$\text{Gain} = \frac{\frac{M_0}{2m_i}\epsilon_f\Theta}{3\frac{M_0}{m_i}T/\eta} = \eta\frac{M_0}{2m_i}\frac{\epsilon_f\Theta}{3\frac{M_0}{m_i}T} = \frac{\eta}{6}\frac{\epsilon_f}{T}\Theta$$

$$\text{Gain} = \eta\, 293\left(\frac{10}{T_{\text{keV}}}\right)\Theta$$

The power to heat the plasma is enormous

- **Consider the small T limit:**

$$\Theta(\xi) \approx \frac{\xi}{4+\xi} \qquad \xi \equiv \frac{\langle \sigma v \rangle}{2C_s} n_0 R_0 = \frac{\langle \sigma v \rangle}{\sqrt{T}} (\rho R_0) \frac{1}{2\sqrt{2m_i}}$$

$$\langle \sigma v \rangle \sim T^4 \text{ for } T \to 0 \text{ , then } \quad \xi \sim T^{7/2} \quad \text{and} \quad Gain \sim T^{5/2} \to 0$$

- **Required input power:**

$$P_w = \frac{E_{\text{input}}}{\tau_{\text{input}}} \qquad \tau_{\text{input}} \ll \tau_{\text{burn}} = \frac{R}{C_s}$$

(Heat out before it runs away)

$$P_w = \frac{E_{\text{input}}}{\mu R / C_s} = \frac{E_{\text{thermal}}}{\eta \mu R / C_s} = 3 \frac{M_0}{m_i} \frac{T}{R} \frac{C_s}{\eta \mu} \qquad \tau_{\text{input}} = \mu \frac{R}{C_s} \quad \text{Ex: } \mu \sim 0.1$$

$$\frac{P_w}{M_0} = \frac{3}{m_i} \frac{T}{R} \frac{C_s}{\eta \mu} = \frac{3}{m_i} \frac{T}{R} \sqrt{\frac{2T}{m_i}} \frac{1}{\eta \mu}$$

$$\frac{P_w}{M_0} = 10^{18} \left(\frac{T_{\text{keV}}}{10} \right)^{3/2} \frac{0.1}{\mu} \frac{1}{R_{\text{cm}}} \frac{1}{\eta} \text{ Watts}/g$$

A clever way is needed to ignite a target

- **For T = 10 keV**

$$\xi \approx 0.18(\rho R) \qquad Gain|_{10\text{keV}} \approx 293\eta \frac{0.18\rho R}{4 + 0.18\rho R} \approx 293\eta \frac{\rho R_{g/cm^2}}{22 + \rho R_{g/cm^2}}$$

- **For T=40 keV**

$$\xi \approx 0.54(\rho R) \qquad Gain|_{40\text{keV}} \approx 73\eta \frac{\rho R_{g/cm^2}}{7 + \rho R_{g/cm^2}}$$

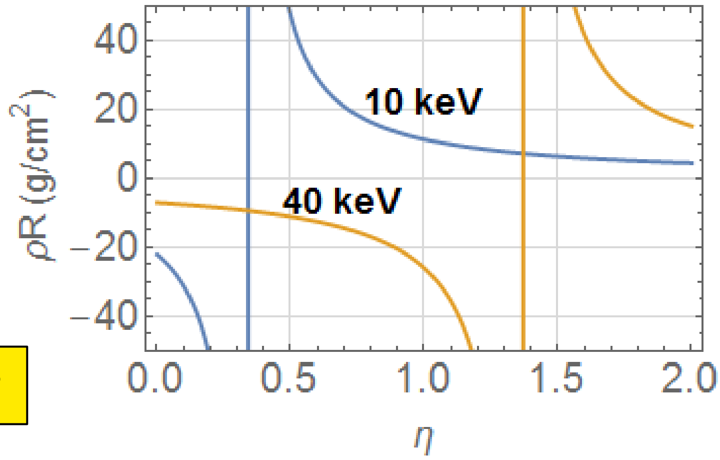
- **For Gains ≳ 100**
  - **T = 10 keV**

    $\rho R \gtrsim 22\, g/\text{cm}^2 \quad \eta > 1$

  - **T = 40 keV**

    $\eta > 1$

How do we get $\eta > 1$?



Requirement to ignite a target

- For T=10 keV and ρR ≳ 22 g/cm²

$$\rho R = \frac{4\pi}{3}\frac{\rho R^3}{4\,\pi R^2/3} = \frac{M_0}{\frac{4\pi}{3}R^2} = \frac{3}{4\pi}\frac{M_0}{R^2} \gtrsim 22\, g/\mathrm{cm}^2$$

$$\frac{M_0}{R^2} \gtrsim 92\, g/\mathrm{cm}^2$$

$$P_w\Big|_{10keV} = 10^{18}\left(\frac{T_{\mathrm{keV}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{M_0}{R_{\mathrm{cm}}}\frac{1}{\eta} = 10^{18}\frac{0.1}{\mu}\frac{1}{\eta}92R_{\mathrm{cm}} \quad \text{Watts}$$

$$P_w\Big|_{10keV} \approx 10^{20}\frac{0.1}{\mu}\frac{R_{\mathrm{cm}}}{\eta}\text{Watts}$$

- For T=40keV

$$\rho R \gtrsim 7 \;\Rightarrow\; \frac{M_0}{R^2} \gtrsim 30\, g/\mathrm{cm}^2$$

$$P_w\Big|_{40keV} \approx 2.4 \times 10^{20}\frac{0.1}{\mu}\frac{R_{\mathrm{cm}}}{\eta}\text{Watts}$$

- **Needed:**

  $R_{\mathrm{cm}} \ll 1$

  $\eta \gg 1$

  $\mu \gg 0.1$

Requirements to ignite a target

$$P_w\Big|_{10keV} \approx 10^{20}\,\frac{0.1}{\mu}\,\frac{R_{\mathrm{cm}}}{\eta}\,\mathrm{Watts}$$

- $R_{\mathrm{cm}} \ll 1$: sphere size in the order of 100's um
- $\eta \gg 1$: input energy amplification
- $\mu \gg 0.1$: energy delivery time decoupled from burn time. Need longer energy delivery time. Need to bring down power to ~10^{15} W

Math….#!@%$$#&^%$#

$$P_w = 10^{18}\frac{M_{0,g}}{\eta}\left(\frac{T_{\mathrm{keV}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{1}{R_{\mathrm{cm}}}\ \mathrm{Watts}/g$$

$$\tau_{\mathrm{input}} = \mu\frac{R}{C_s}\quad \mathrm{Ex:}\ \mu\sim 0.1\qquad \eta = \frac{\text{Thermal Energy}}{\text{Input Energy}}$$

$$\mathrm{Gain} = 293\eta\left(\frac{10}{T_{\mathrm{keV}}}\right)\Theta(\xi)\qquad \Theta(\xi)\approx\frac{\xi}{4+\xi}\qquad \xi = \frac{\langle\sigma v\rangle}{2m_i C_s}(\rho \mathrm{R}_0)$$

$$G_{\max} \equiv 293\eta\left(\frac{10}{T_{\mathrm{keV}}}\right)\qquad G = G_{\max}\frac{\xi}{4+\xi}\Rightarrow \xi = \frac{4G}{G_{\max}-G}$$

$$P_w = \frac{10^{18}}{\eta}\left(\frac{T_{\mathrm{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{4\pi}{3}\frac{\rho {R_0}^3}{R_0} = \frac{10^{18}}{\eta}\left(\frac{T_{\mathrm{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{4\pi}{3}(\rho R_0)R_0$$

More math…!#$%%^&*&^(*&%)(#%!@$#%%^*&*%(

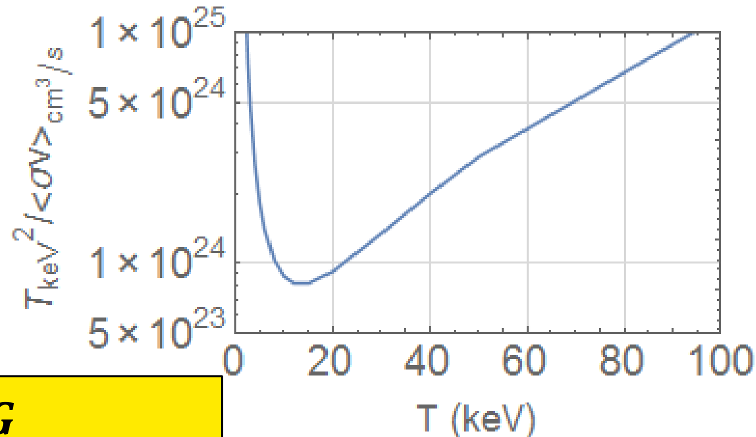
$$P_w = \frac{10^{18}}{\eta}\left(\frac{T_{\text{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{4\pi}{3}\frac{\rho {R_0}^3}{R_0} = \frac{10^{18}}{\eta}\left(\frac{T_{\text{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{4\pi}{3}(\rho R_0)R_0$$

$$= \frac{10^{18}}{\eta}\left(\frac{T_{\text{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{4\pi}{3}R_0\frac{2m_iC_s}{<\sigma v>}\xi \qquad \text{where} \quad \xi \equiv \frac{\langle\sigma v\rangle}{2m_iC_s}(\rho R)$$

$$= \frac{10^{18}}{\eta}\left(\frac{T_{\text{kev}}}{10}\right)^{3/2}\frac{0.1}{\mu}\frac{32\pi}{3}R_{0,\text{cm}}\frac{\sqrt{Tm_i}}{<\sigma v>}\frac{G}{G_{\max}-G} \qquad \text{where} \quad C_s = \sqrt{\frac{2T}{m_i}}$$

$$P_w = \frac{10^{18}}{\eta}\frac{{T_{\text{kev}}}^2}{\langle\sigma v\rangle_{\text{cm}^2/s}}\frac{0.1}{\mu}R_{0,\text{cm}}\frac{G}{G_{\max}-G} \quad \text{Watts}$$

$$\left.\frac{{T_{kev}}^2}{\langle\sigma v\rangle_{cm^2/s}}\right|_{min} = 8\times10^{23} \quad \text{for} \quad T = 14\text{keV}$$



$$\frac{G}{G_{\max}-G} \approx \frac{G}{G_{\max}}$$

$$P_w \approx \frac{7\times10^{19}}{\eta}\frac{0.1}{\mu}R_{0,\text{cm}}\frac{G}{G_{\max}}\text{Watts}$$

Need to lower the power by 5 orders of magnitude

$$P_w \approx \frac{7\times 10^{19}}{\eta}\frac{0.1}{\mu}R_{0,\mathrm{cm}}\frac{G}{G_{\max}}\,\mathrm{Watts}$$

- $\mu \uparrow$:
- $\eta \uparrow$: require the fuel ignition from a "spark." Ignite only a small portion of the DT plasma, i.e., $M_h \ll M_0$
- $R_0 \downarrow$: smaller system size

$$P_w = P_w(M_0)\frac{M_h}{M_0}$$

$$P_w^{\min} = \frac{7\times 10^{15}}{\eta_h}\left(\frac{M_h/M_0}{0.01}\right)\left(\frac{R_{0,\mu\mathrm{m}}}{100}\right)\left(\frac{0.1}{\mu}\right)\left(\frac{G}{G_{\max}}\right)\mathrm{Watts}$$

Effective increase in η

Target design using an 1MJ laser

$$P_w^{\min} = \frac{7\times 10^{15}}{\eta_h}\left(\frac{M_h/M_0}{0.01}\right)\left(\frac{R_{0,\mu\mathrm{m}}}{100}\right)\left(\frac{0.1}{\mu}\right)\left(\frac{G}{G_{\max}}\right)\mathrm{Watts}$$

- For the case of using a huge laser, ex: 1MJ.
- The ignition requires temperatures $T \gtrsim 5\mathrm{keV}$, then the energy required for ignition is

$$E_{\mathrm{ign}} \approx 3\frac{M_h}{m_i}\frac{T}{\eta_h}$$

$$M_h \approx \frac{m_i}{3}\frac{\eta_h E_{\mathrm{ign}}}{T}$$

$$M_{h,\mu\mathrm{g}} \approx 17\left(\frac{5}{T_{\mathrm{keV}}}\right)E_{\mathrm{igm,MJ}}\left(\frac{\eta_h}{0.01}\right) \qquad M_h \approx 20\mu\mathrm{g}$$

Target design using an 1MJ laser - continue

- **For "inefficient" heating mechanism $(\eta_h \approx 1\%)$, the mass that can be heated to $T \approx 5$ keV is in the order of $M_h \approx 20$ μg.**
- **If $M_h/M_0 \approx 0.01$, then $M_0 \approx 2$ mg .**
- **Assuming that the burned-up fraction** $\Theta \approx \frac{\rho R}{7+\rho R}$

for $\Theta \approx 30\% \rightarrow \rho R \approx 3\ g/cm^2$

$$M_0 = \frac{4\pi}{3}\rho R^3 = \frac{4\pi}{3}R^2(\rho R) \qquad R = \sqrt{\frac{4\pi}{3}\frac{M_0}{\rho R}} = 126\sqrt{\frac{M_{0,\mathrm{mg}}}{2}}\sqrt{\frac{3}{\rho R}}\ \mu\mathrm{m}$$

$$\rho = \frac{3M_0}{4\pi R^3} = 240\sqrt{\frac{M_{0,\mathrm{mg}}}{2}}\left(\frac{126}{R_{\mu\mathrm{m}}}\right)^3\ g/\mathrm{cm}^3 \qquad \rho_{\mathrm{DT}} = 0.25\ g/\mathrm{cm}^3$$

- **DT must be compressed ~1000 times**
- **The initial radius of a 2 mg sphere of *DT* is $R_{init} \simeq 2.6$ mm while the final radius $R_{final} \simeq 100\ \mu m$, the convergence ratios of 30 ~ 40 are required.**

Requirements of the density and size of the ignition mass

$$M_h \approx 20\mu g$$

$$\rho_h R_h \approx 0.3\ g/\mathrm{cm}^2$$ ← **To stop 3.5 MeV α particles**

$$R_h \simeq \sqrt{\frac{3}{4\pi}\frac{M_h}{\rho_h R_h}} \approx 40\mu m$$

$$\rho_h \approx \frac{(\rho_h R_h)}{R_h} = \frac{0.3}{40 * 10^{-4}} = 75\ g/\mathrm{cm}^3$$

Summary

- **Possible fuel assembly for 1MJ ICF driver**