

# **PULSED POWER SYSTEM**

## **脈衝功率系統**



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**Institute of Space and Plasma Sciences, National Cheng Kung University**

**2024 Fall Semester**

**Thursday 9:10-12:00**

**Lecture 1**

**<http://capst.ncku.edu.tw/PGS/index.php/teaching/>**

**Online courses:**

**<https://nckucc.webex.com/nckucc/j.php?MTID=mf87b10f22c1e36d5c4b2337e60d8a847>**

# Grading

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- **Homework (Quizzes) – 30 %**
  - Design of each component of a pulsed-power system
- **Final presentations – 70 %**
  - Design of a pulsed-power system – 35 %.
  - Applications of pulsed-power system – 35 %.

• **The goal is you can design a pulsed-power system after taking this course.**

• **No class on 9/19!!!**

# Reference

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- **Foundations of pulsed power technology, by Jane Lehr & Pralhad Ron**
- **Pulsed power systems, by H. Bluhm**
- **Pulsed power, by Gennady A. Mesyats**
- **J. C. Martin on pulsed power, edited by T. H. Martin, A. H. Guenther, and M. Kristiansen**
- **Pulse power formulary, by Richard J. Adler**
  
- **Circuit analysis, by Cunningham and Stuller**

# Outlines

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- **Introduction to pulsed-power system**
- **Review of circuit analysis**
- **Static and dynamic breakdown strength of dielectric materials**
  - **Gas – Townsend discharge (avalanche breakdown), Paschen's curve**
  - **Liquid**
  - **Solid**
- **Energy storage**
  - **Pulse discharge capacitors**
  - **Marx generators**
  - **Inductive energy storage**

# Outlines

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- **Switches**
  - Closing switches
  - Opening switches
- **Pulse-forming lines**
  - Blumlein line
  - Pulse-forming network
  - Pulse compressor
- **Pulse transmission and transformation**
  - Self-magnetic insulation
  - Pulse transformer
  - Voltage multiplier
- **Other techniques**
  - H-bridge pulse generator
  - Pulse-width modulation (PWM)
  - Fast high-voltage pulse generator

# Outlines

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- **Power and voltage adding**
  - Marx generator
  - LC generator
  - Line pulse transformers
  - Induction voltage adder (IVA)
  - Linear induction accelerator (LIA)
  - Linear transformer driver (LTD)
- **Diagnostics**
  - Voltage measurement
  - Current measurement
- **Applications of pulsed-power system**

# Outlines

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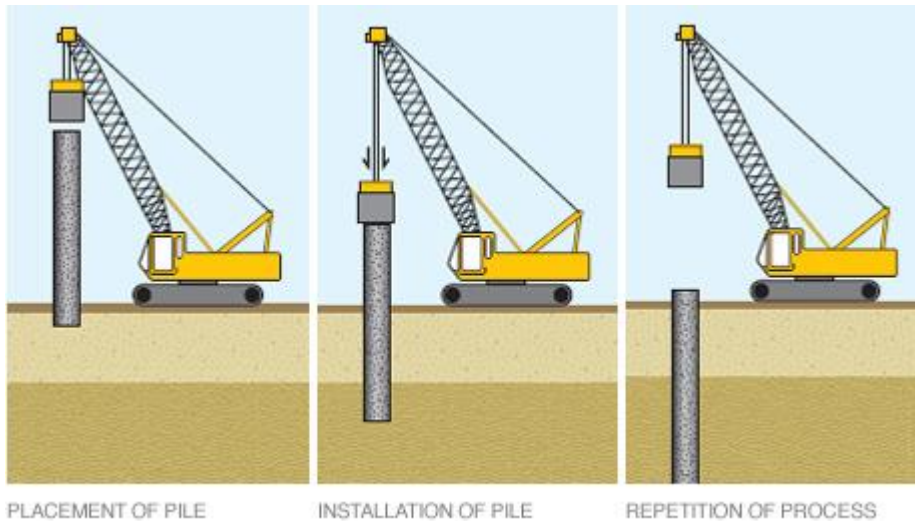
- **Introduction to pulsed-power system**
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  - Gas – Townsend discharge (avalanche breakdown), Paschen's curve
  - Liquid
  - Solid
- Energy storage
  - Pulse discharge capacitors
  - Marx generators
  - Inductive energy storage

# Pulsed-power system release the stored energy in a short period of time to provide high power output



- Pulsed power is a scheme where stored energy is discharged as electrical energy into a load in a short pulse or short pulses with a controllable repetition rate.
- Driven piles - prefabricated steel, wood or concrete piles are driven into the ground using impact hammers

- Driven piles



- Hammer





# Example of short pulses with a controllable repetition rate



# In general, a pulsed-power system provides a power in the order of 1 GW



- The highest energy and power that have been achieved in a simple pulse are in the order of 100 MJ & few hundreds TW, respectively.

	General cases	Our system
Energy per pulse	1 ~ 10 MJ	1 kJ
Peak power	1 MW ~ 100 TW	0.6 GW
Peak voltage	1 kV ~ 10 MV	20 kV
Peak current	1 kA ~ 10 MA	135 kA
Pulse width	0.1 ns ~ 10 us	1 us

# Physiological Effects of an Electric Shock

## Effects of Electrical Current\* on the Body<sup>3</sup>

Current	Reaction
1 milliamp	Just a faint tingle.
5 milliamps	Slight shock felt. Disturbing, but not painful. Most people can "let go." However, strong involuntary movements can cause injuries.
6–25 milliamps (women)† 9–30 milliamps (men)	Painful shock. Muscular control is lost. This is the range where "freezing currents" start. It may not be possible to "let go."
50–150 milliamps	Extremely painful shock, respiratory arrest (breathing stops), severe muscle contractions. Flexor muscles may cause holding on; extensor muscles may cause intense pushing away. Death is possible.
1,000–4,300 milliamps (1–4.3 amps)	Ventricular fibrillation (heart pumping action not rhythmic) occurs. Muscles contract; nerve damage occurs. Death is likely.
10,000 milliamps (10 amps)	Cardiac arrest and severe burns occur. Death is probable.
15,000 milliamps (15 amps)	Lowest overcurrent at which a typical fuse or circuit breaker opens a circuit!

\*Effects are for voltages less than about 600 volts. Higher voltages also cause severe burns.

†Differences in muscle and fat content affect the severity of shock.

- **Electric shock victims suffering from ventricular fibrillation will die if they do not receive prompt, emergency medical attention.**

# A pulse is characterized by its shapes



- The shape of a pulse is characterized by:

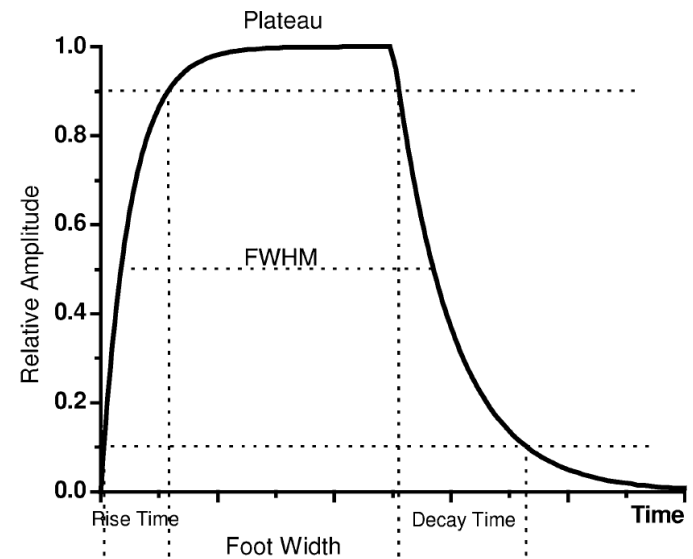
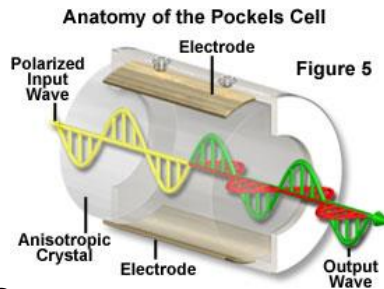
- Rise time: from 10 % to 90 % of the plateau
- Fall time: from 90 % to 10 % of the plateau

(Rise & fall time depend on the evolution of the “load impedance,” which in most cases varies with time.)

- Duration:

- FWHM
- Width of 90% of the peak amplitude

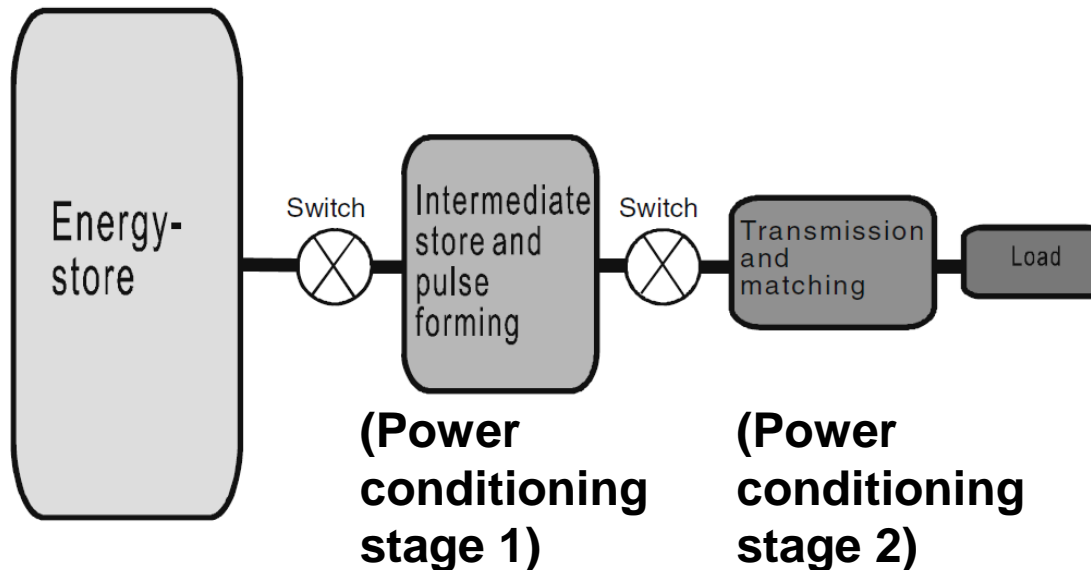
- Flatness of the plateau region: important for some applications such as for driving a Pockel’s cell.



# A pulsed-power system has an energy bank that is charged slowly and store the energy for some time

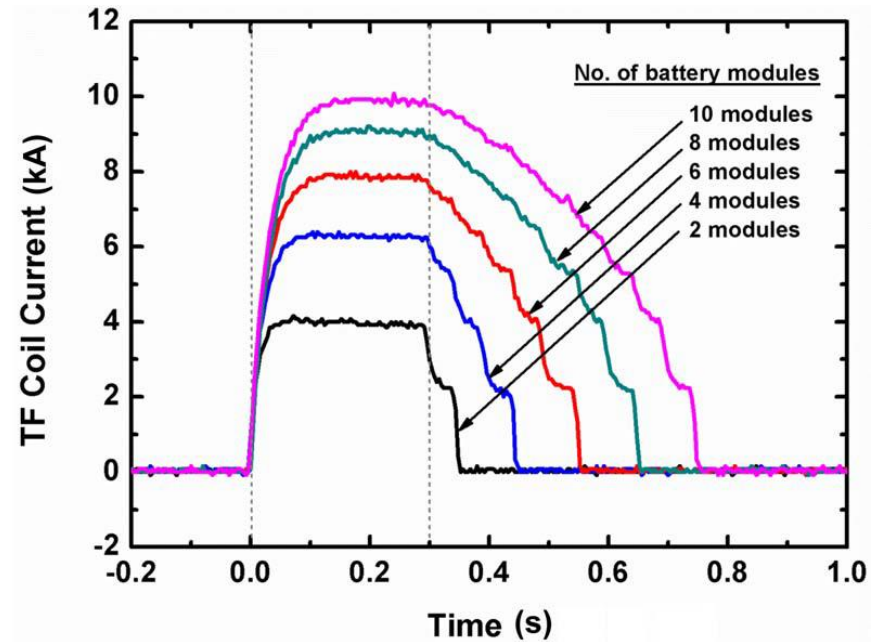
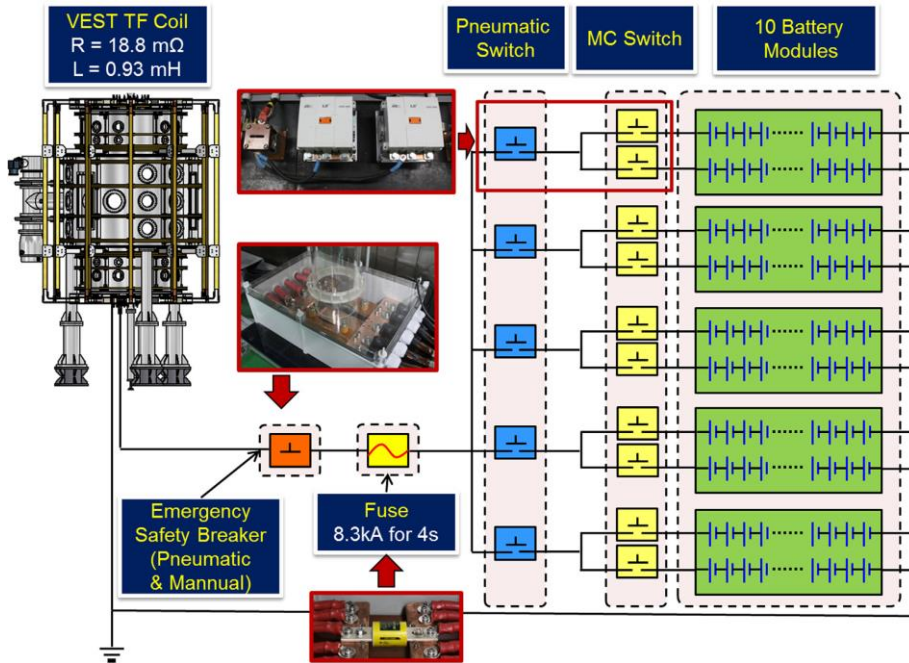


- A generator scheme for the production of high-power electrical pulses is always based on an energy store that is charged slowly at a relatively low charging power and is discharged rapidly by activating a switch.
- To achieve the desired power magnification factor and to shape the pulse, the above process can be repeated several time.



- The energy can be stored either chemically (battery), mechanically, or electrically.

# Batteries can be used for energy storage

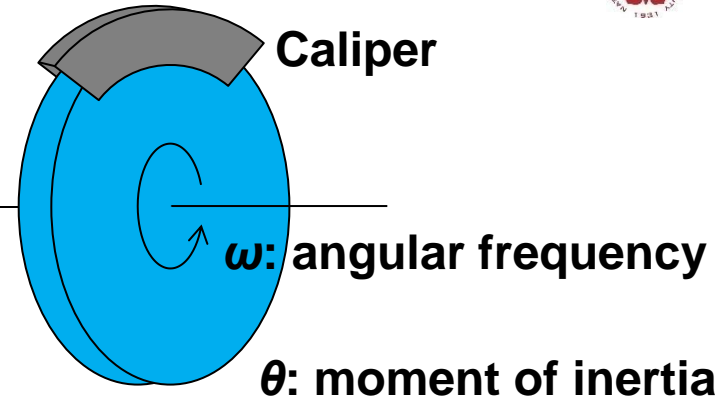




# A flywheel can store energy mechanically



- Mechanical energy:  $E_{\text{kin}} = \frac{1}{2} \theta \omega^2$
- For a massive cylinder:  
 $\theta = \frac{1}{2} Mr^2 \quad \rightarrow \quad E_{\text{kin}} = \frac{1}{2} \left( \frac{1}{2} Mr^2 \right) \omega^2$



- Stored energy density:

$$E_M = \frac{E_{\text{kin}}}{M} = \frac{1}{4} r^2 \omega^2$$

- The ultimate energy density is limited by the tensile strength of the material used to construct the flywheel.

$$\Sigma \sim \rho \omega_{\text{max}}^2 r^2$$

- For a AISI 302 stainless steel cylinder with a radius of 1 m:

$$\Sigma = 860 \text{MPa}$$

$$\rho = 8190 \text{ kg/m}^3$$

$$\omega_{\text{max}} \sim 300 (\text{sec}^{-1})$$

$$E_M \sim 2 \times 10^4 \text{ J/kg} \sim 1.6 \times 10^8 \text{ J/m}^3$$

- The problem with mechanical storage is to release the energy in a sufficiently short time. Several electrical compression stages are needed in combination with the mechanical storage to achieve the desired power level.

# Electrical energy can be stored either capacitively in an electric field or inductively in a magnetic field



• Electric field:  $E_e = \frac{1}{2} \epsilon_r \epsilon_0 E^2$

– Oil-impregnated paper:  $\epsilon_r = 6$   $E_{\text{break}} = 0.78 \times 10^8 \text{ V/m}$

$$E_e = \frac{1}{2} \times 6 \times 8.85 \times 10^{-12} \times (0.78 \times 10^8)^2 \sim 160 \text{ kJ/m}^3$$

– With the finite packing density

$$E_e' = \frac{1}{2} \times E_e \sim 8 \times 10^4 \text{ J/m}^3$$

Electrode @ V

Dielectric

Electrode @ V=0

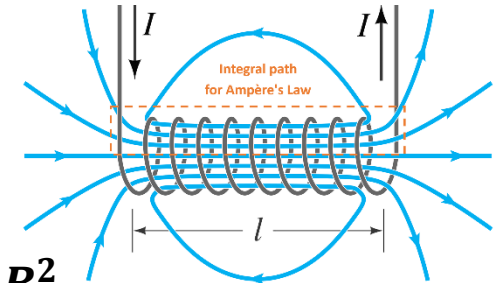




# Electrical energy can be stored either capacitively in an electric field or inductively in a magnetic field



- **Magnetic field:**  $E_B = \frac{1}{2} \frac{B^2}{\mu_r \mu_0}$ 
  - The maximum energy density is limited by the onset of melting at the conductor surface or by the mechanical strength of the storage inductor.
  - **Coil heating:**  $c_v \rho T = \frac{1}{2\mu_0} B^2 \theta \sim \frac{B^2}{2\mu_0}$ 
    - $c_v$  : Heat capacitor per unit mass
    - $\rho$  : Mass density
    - $T$  : Surface temperature
    - $\theta$  : A factor depending on the form of the pulse
- **Copper:**
  - $B \times l = \mu_0 N I$
  - $B = \mu_0 n I \sim I$
  - $P = I^2 R \sim B^2 R$
- **Copper:**
  - $c_v = 0.385 \text{ J/g} - k$
  - $T_{\text{melting}} = 1085^\circ \text{C}$
  - $\rho = 8960 \text{ kg/m}^3$
  - $\Sigma_{\text{copper}} = 70 \text{ MPa}$
- **The coil needs to hold the magnetic pressure:**  $\frac{B^2}{2\mu_0} = P_B \leq \Sigma$ 
  - $B \sim 13 \text{ T}$



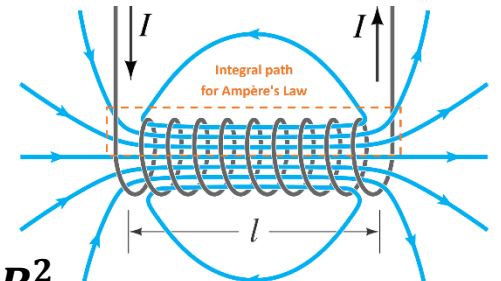
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  - The coil needs to hold the magnetic pressure:  $\frac{B^2}{2\mu_0} = P_B \leq \Sigma$

$$B \sim 100 \text{ T}$$



$$B \sim 13 \text{ T} \rightarrow E_B \sim 7 \times 10^7 \text{ J/m}^3$$

# More energy can be stored in a magnetic field



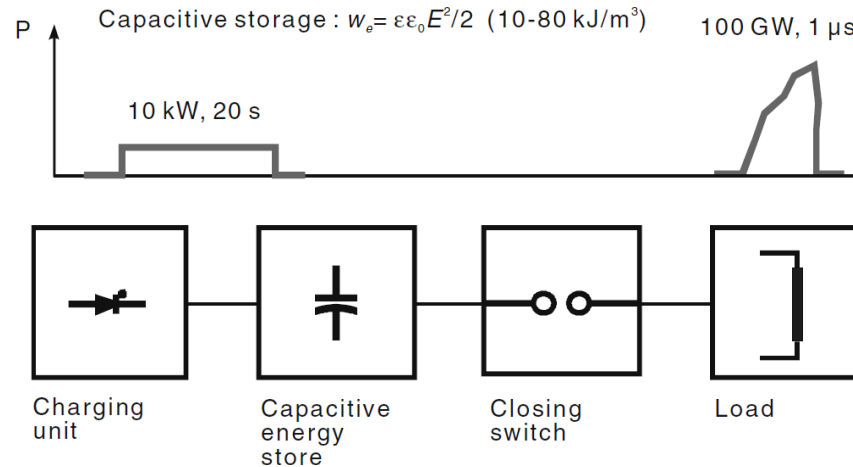
Mechanical Energy	Electrical energy	Magnetic energy
$1.6 \times 10^8 \text{ J/m}^3$	$8 \times 10^4 \text{ J/m}^3$	$7 \times 10^7 \text{ J/m}^3$

- The energy density stored in a magnetic field can be about 2~3 orders of magnitude higher than that storable in a electric field!
- Capacitive storage:
  - Requires one or more closing switches which remain open during charging and hold the charging voltage.
  - Power multiplication is done by current amplification.
- Inductive storage:
  - Requires an opening switch which is closed during charge-up, carrying a large current at this stage.
  - Power multiplication is done by voltage amplification.
- Opening switches are harder to operate than closing switches. They are generally slower leading to a lower power output.

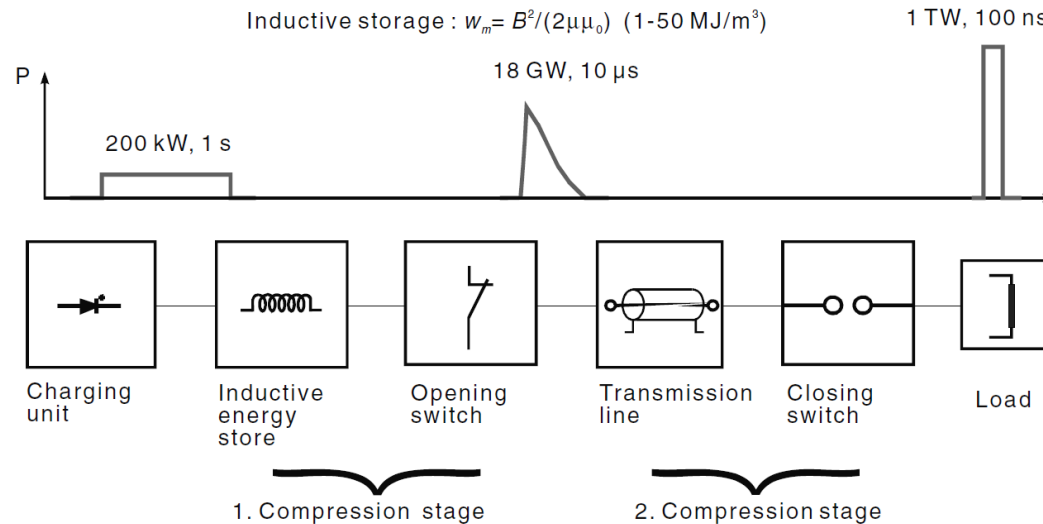
# It is more complicated to use inductive storage



- Capacitive storage:

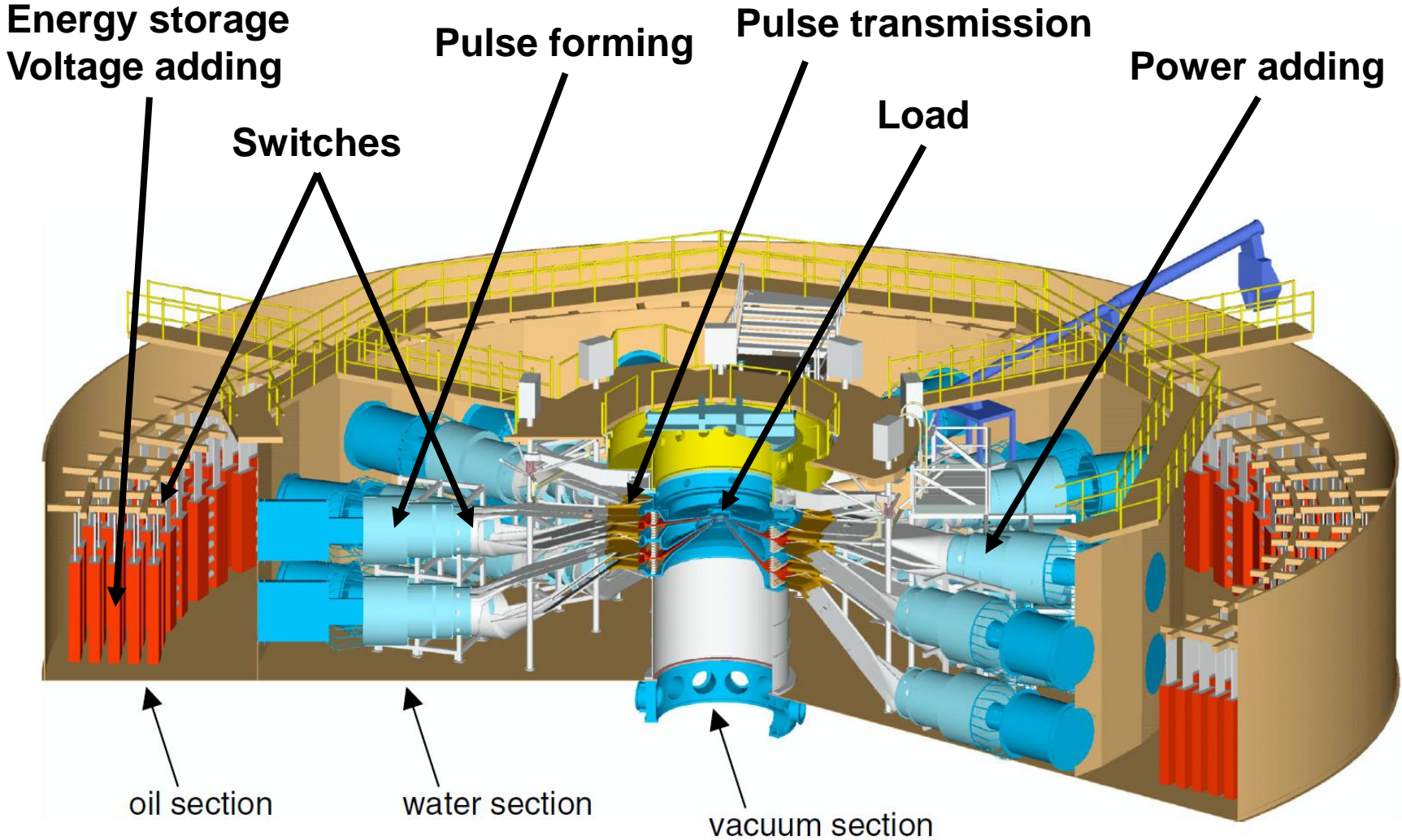


- Inductive storage:



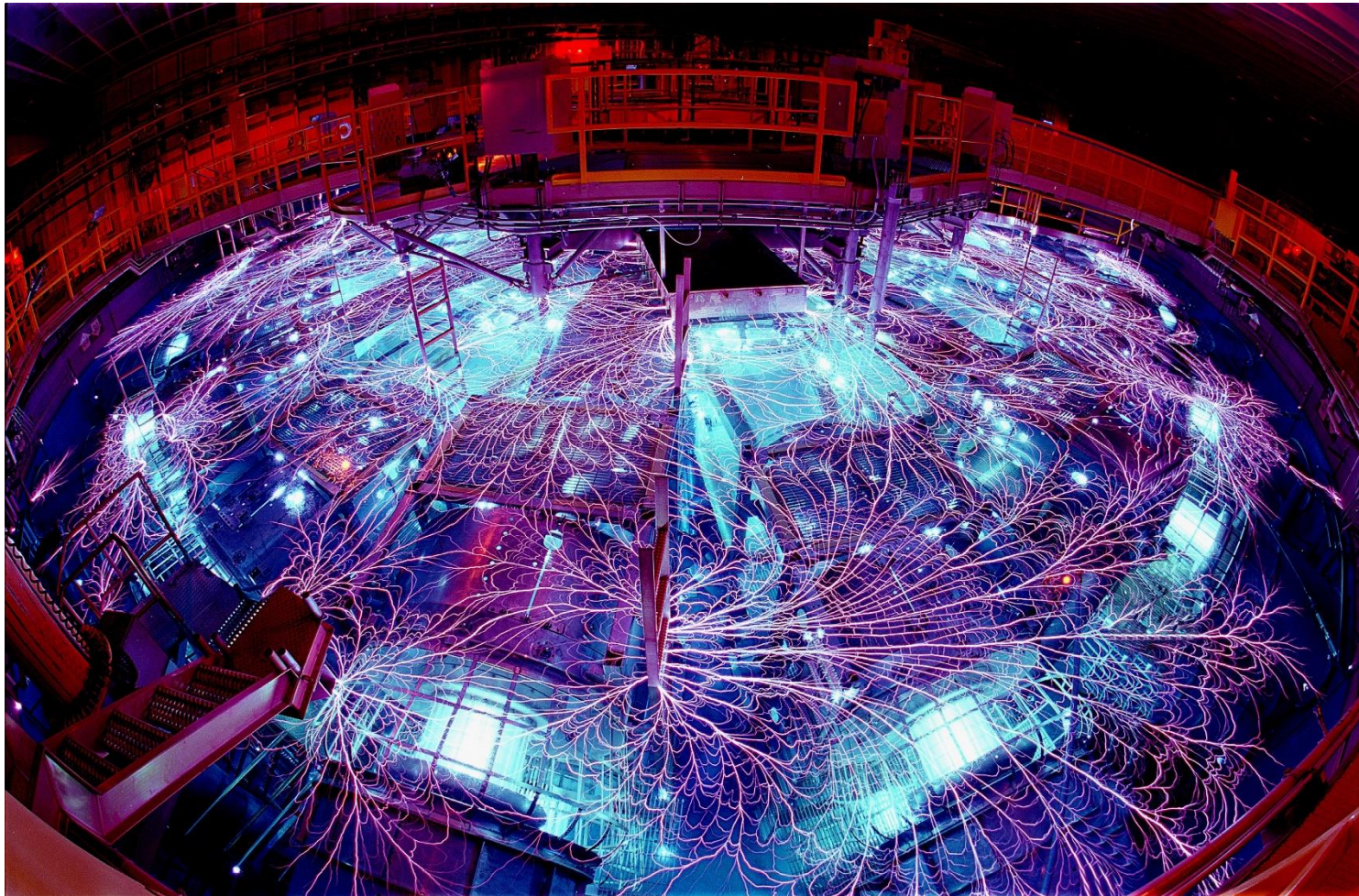
**Capacitive storage is more common and easier to operate.**

# A pulsed power machine at Sandia National Laboratories delivers a 20 MA, 3 MV, and 55-TW pulse



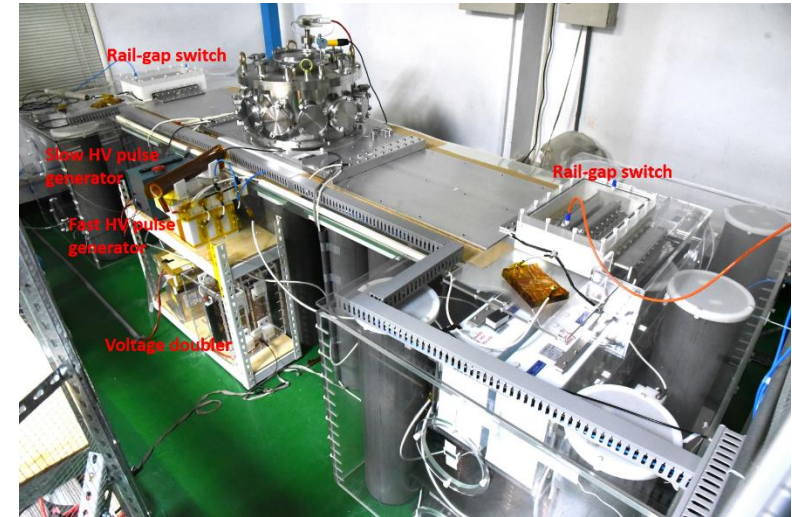
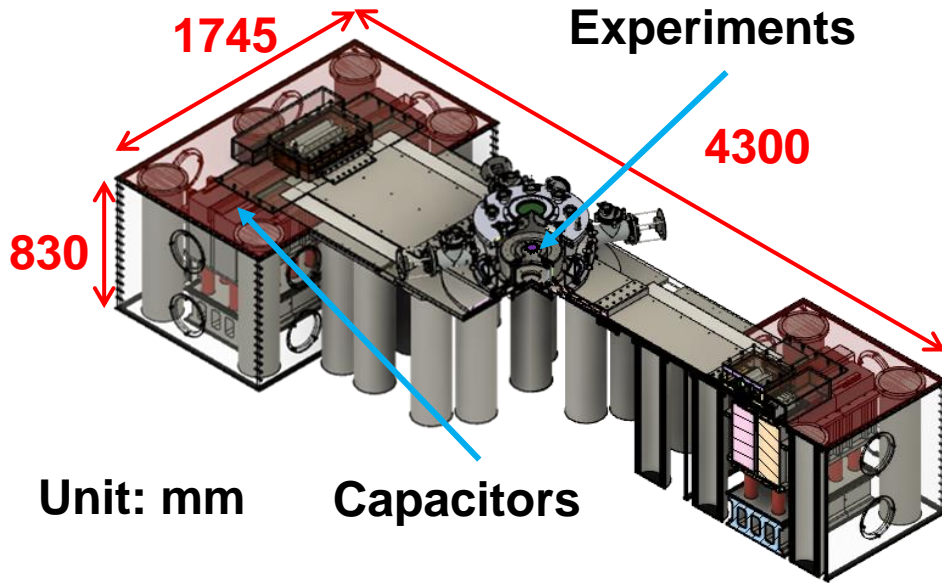


# Arcing may happen during the discharge

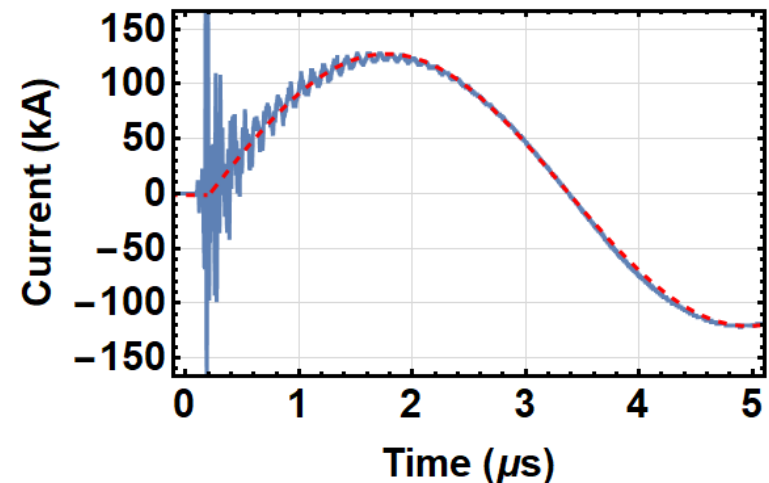




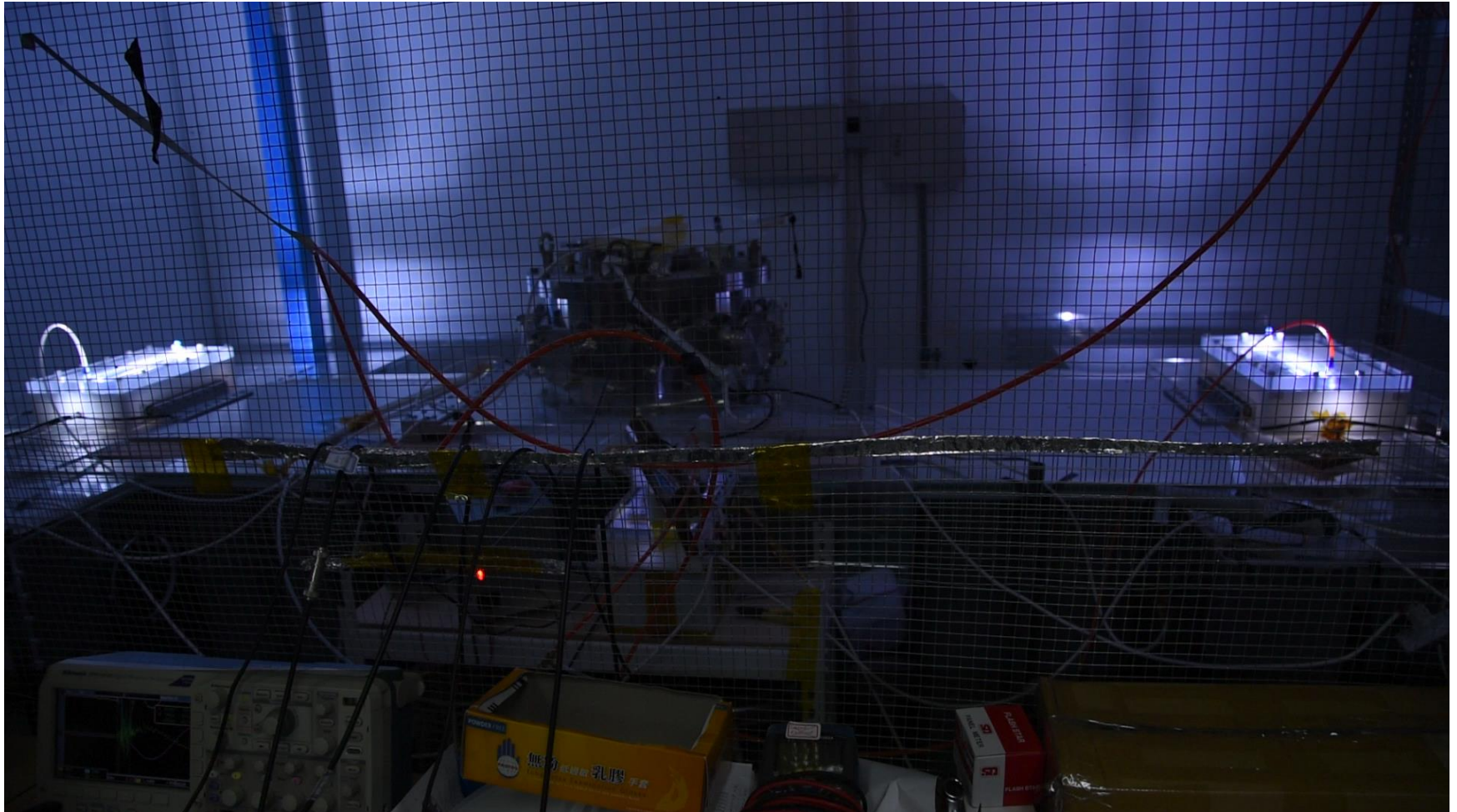
# The EUV light source will be developed using the pulsed-power system we built from scratch



Capacitance ( $\mu\text{F}$ )	5
$V_{\text{charge}}$ (kV)	20 (50)
Energy (kJ)	1 (6.25)
Inductance (nH)	$204 \pm 4$
Rise time (quarter period, ns)	$1592 \pm 3$
$I_{\text{peak}}$ (kA)	$135 \pm 1$ (~340)



# Discharge test in Pulsed-Power Generator for Space Science (PGS) laboratory

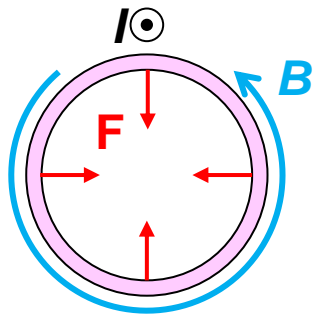
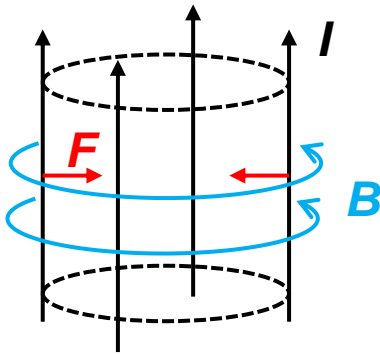




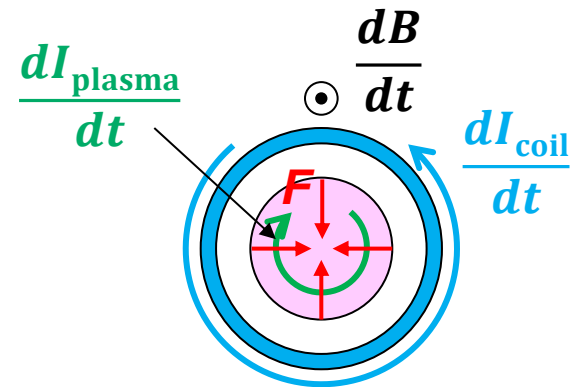
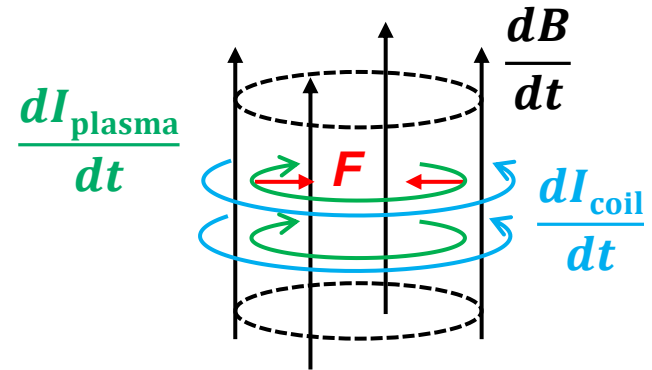
# Plasma can be compressed by using $\mathbf{j} \times \mathbf{B}$ force



- Z pinches



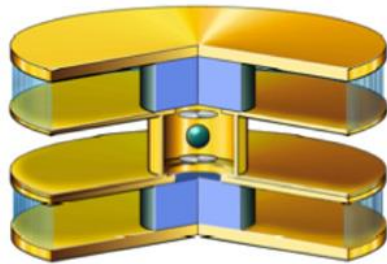
- Theta pinches



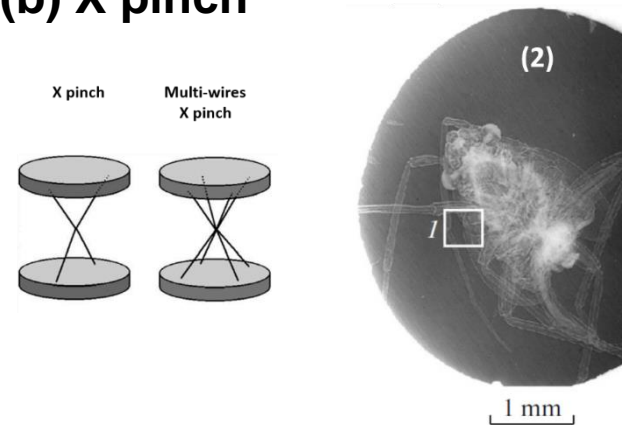
# There are multiple applications using pulsed power machine



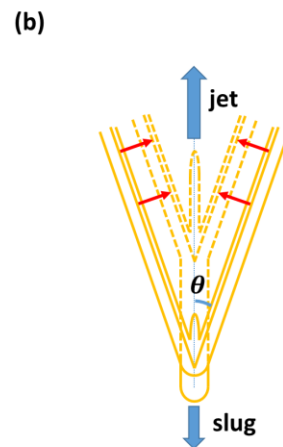
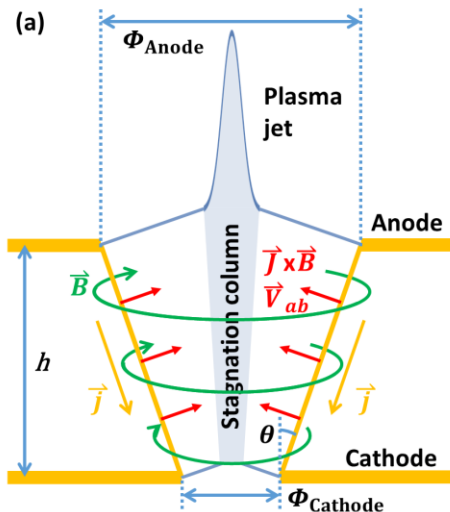
(a) Inertial confinement fusion



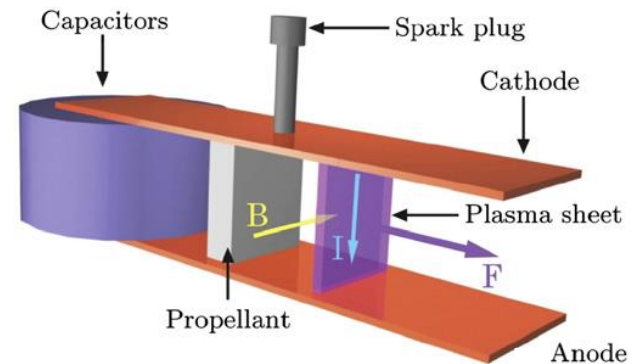
(b) X pinch



(c) Laboratory astrophysics



(d) Pulsed plasma thruster

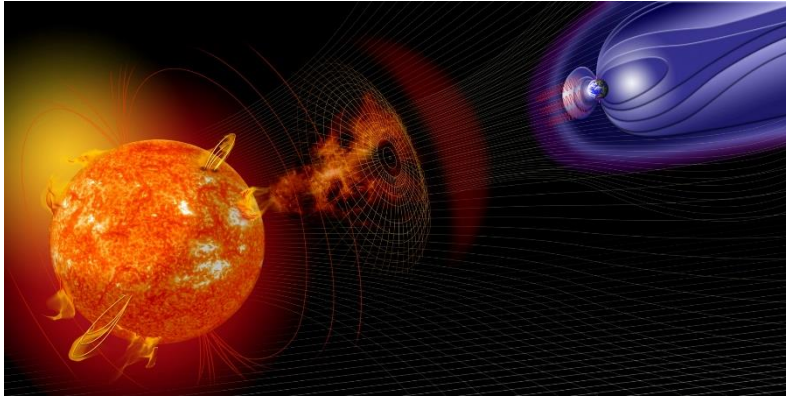


M. G. Haines, Plasma Phys. Control. Fusion 53, 093001 (2011)  
 G. Birkhoff, etc., J. of Appl. Phys. 19, 563 (1948)  
 J. Bio. Sci. and Eng. 8, 747 (2015)  
 Plasma Phys. Report, 42, 226, (2016)  
 Acta Astronautica 67, 440 (2010)

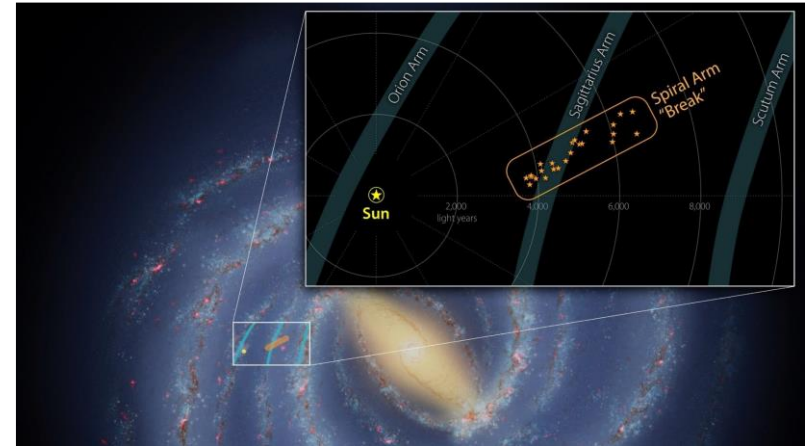
# The pulsed-power system can be used to study laboratory astrophysics and space sciences



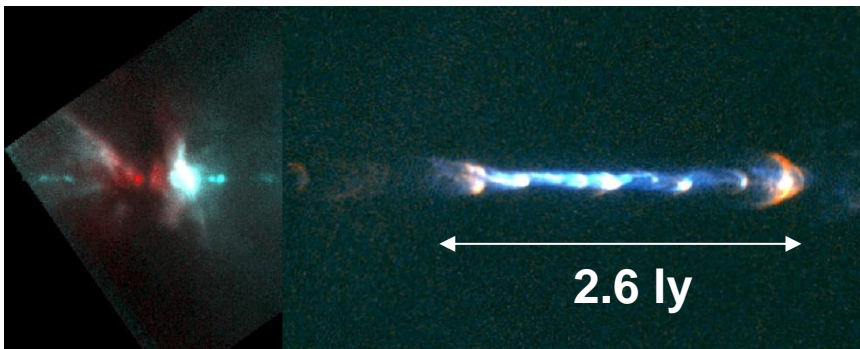
- Solar wind



- Milky way's spiral arms



- Plasma jets e.g., Herbig-Haro object 111

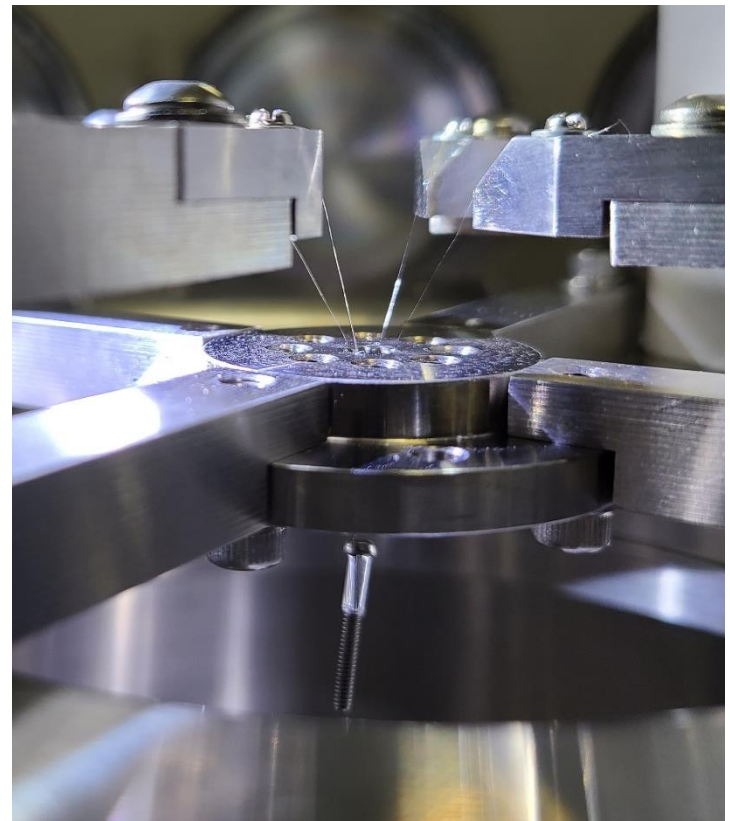


- [https://www.nasa.gov/mission\\_pages/sunearth/spaceweather/index.html](https://www.nasa.gov/mission_pages/sunearth/spaceweather/index.html)
- Jet Propulsion Laboratory [NASA/JPL] Astronomers Find a 'Break' in One of the Milky Way's Spiral Arms (Aug 17, 2021)
- B. Reipurth and J. Bally, Herbig-Haro Flows: Probes of Early Stellar Evolution. *Rev. Astron. Astrophys.*, 39:403-455, September 2001

# The conical-wire array we used in our experiments



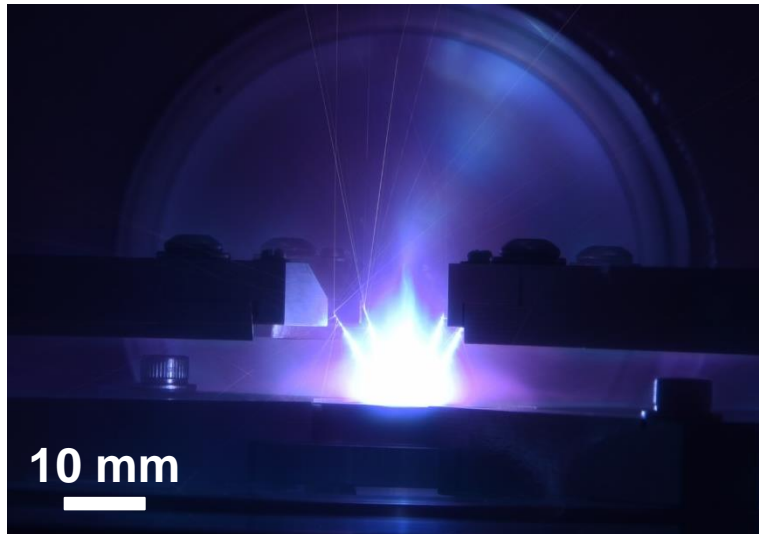
- **Material : Tungsten**
- **Number of wires : 4**
- **Diameter : 0.02 mm**



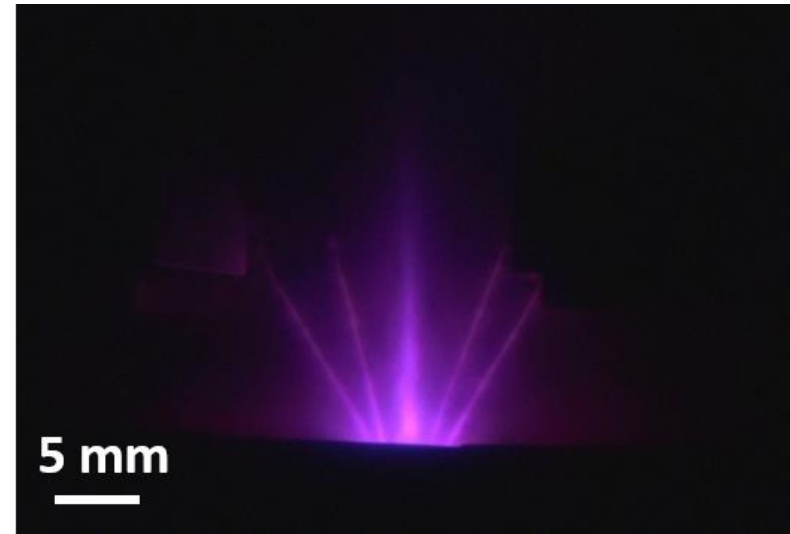
# Plasma jets were captured by time-integrated camera in the visible light



- One layer of cellophane with 8 % transmission.



- Three layers of cellophane with 1 % transmission.



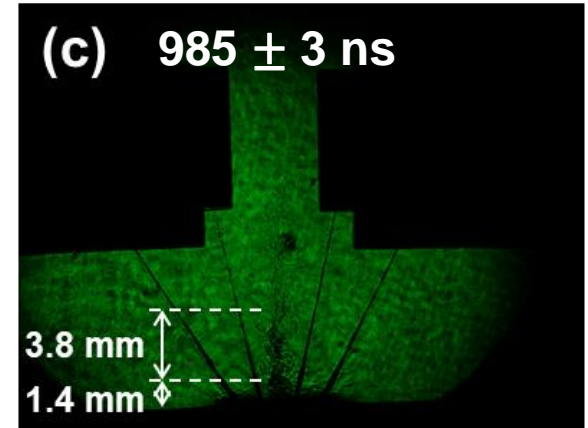
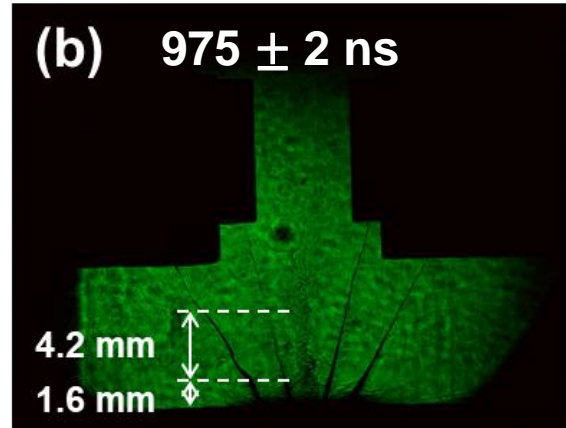
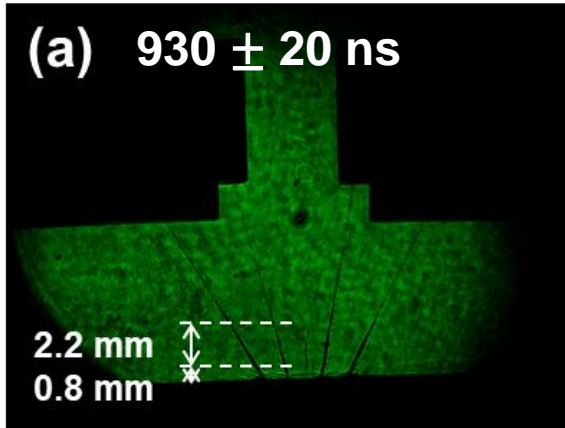
- Both images were taken by using Nikon D750 camera with D/# equals to 22 and ISO= 50 (effective). Exposure times were both 30 sec for being synchronized with the driver.



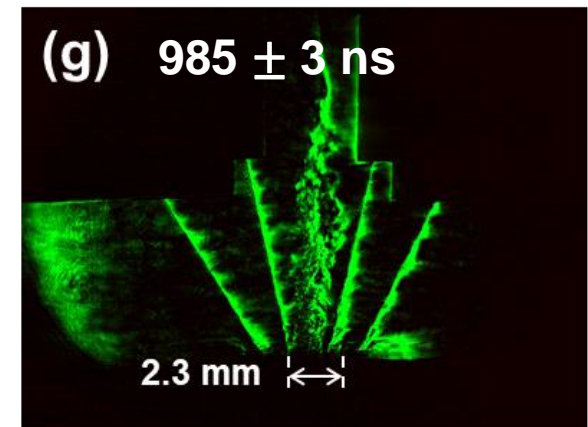
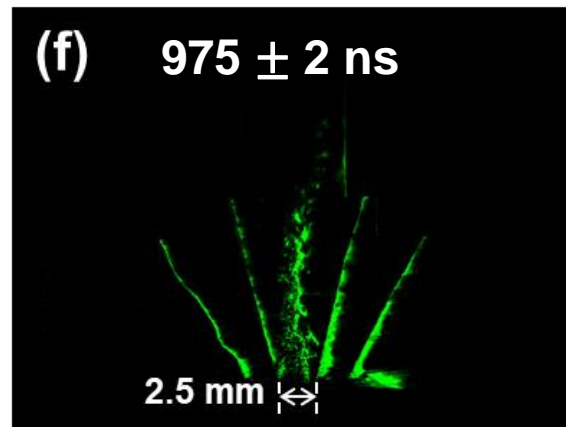
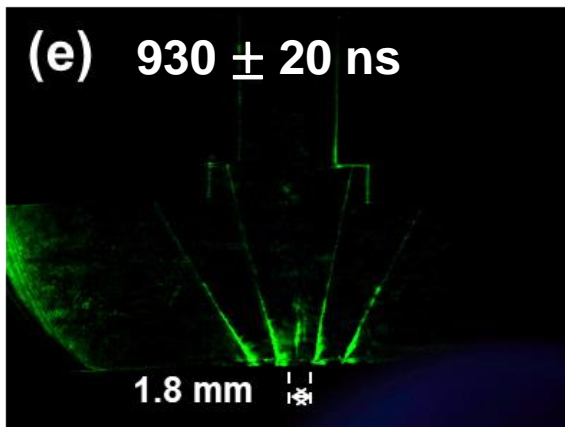
# Time-resolved images show how the plasma plume was generated



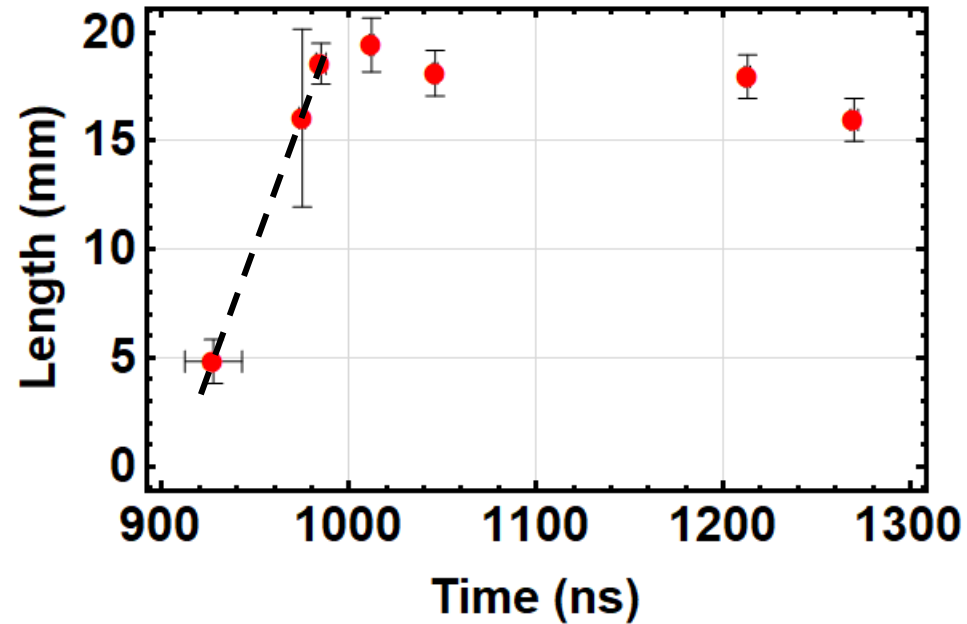
- Shadowgraph images:



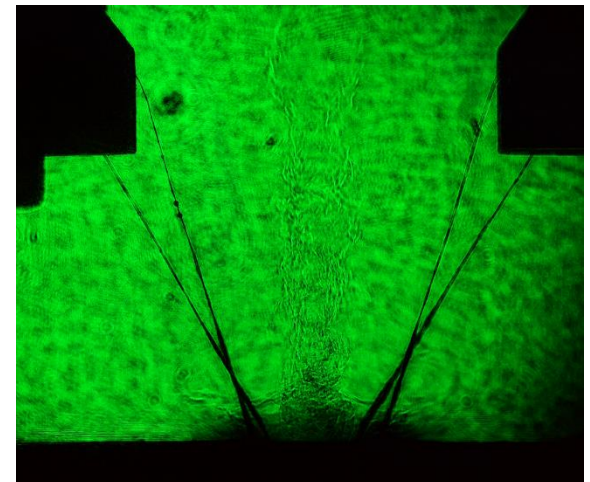
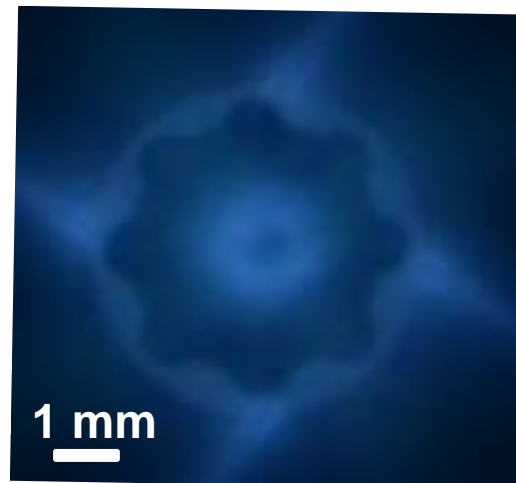
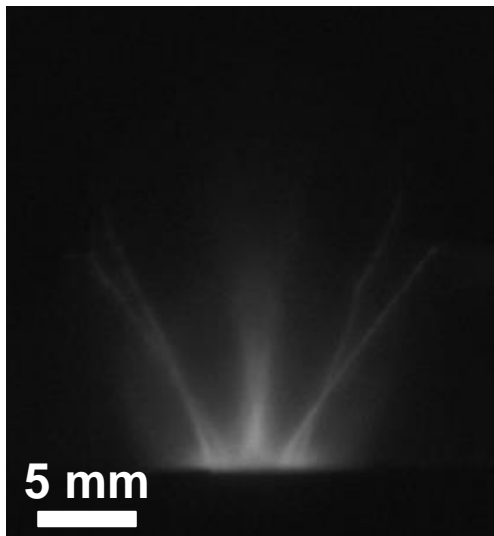
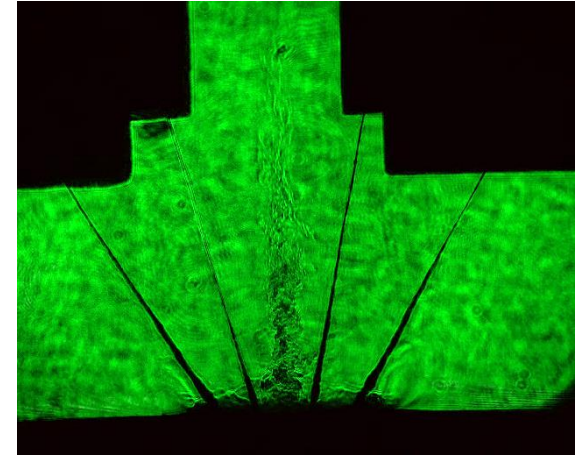
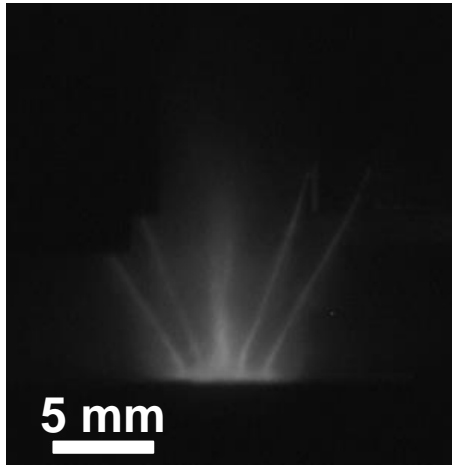
- Schlieren images:



# A speed of $170 \pm 70$ km/sec was estimated



# A “tornado” is generated by the twisted conical-wire array

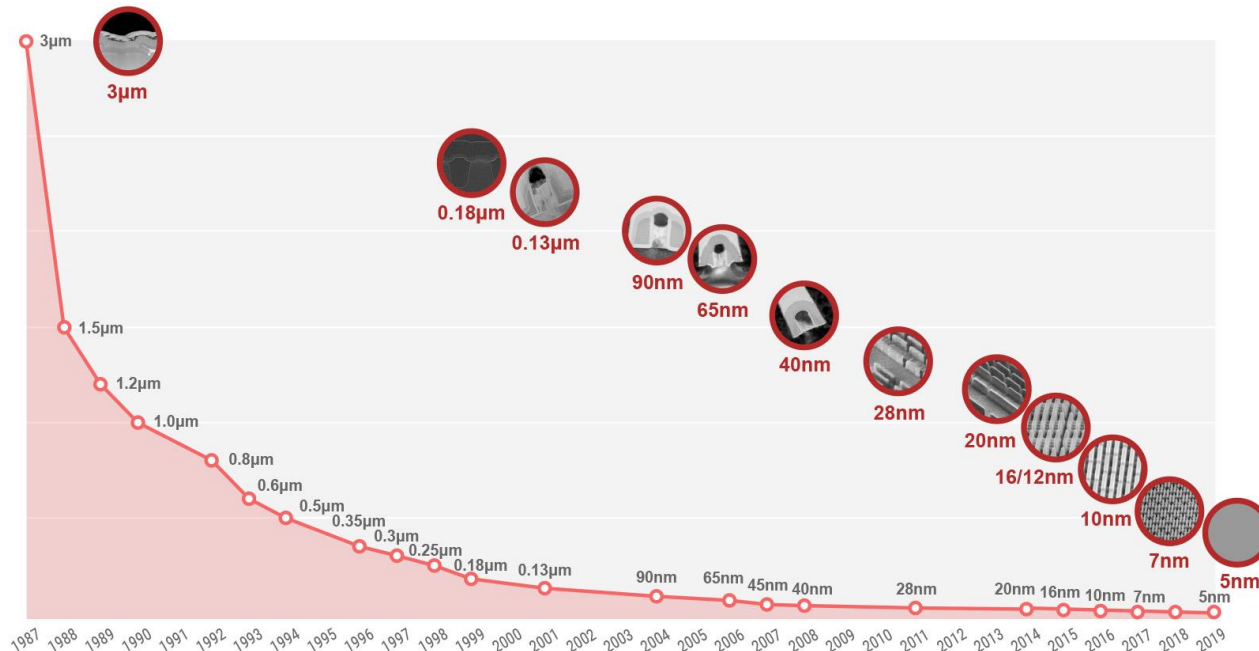




# Ultraviolet lithography (EUVL) is one of the key technologies in semiconductor manufacturing nowadays



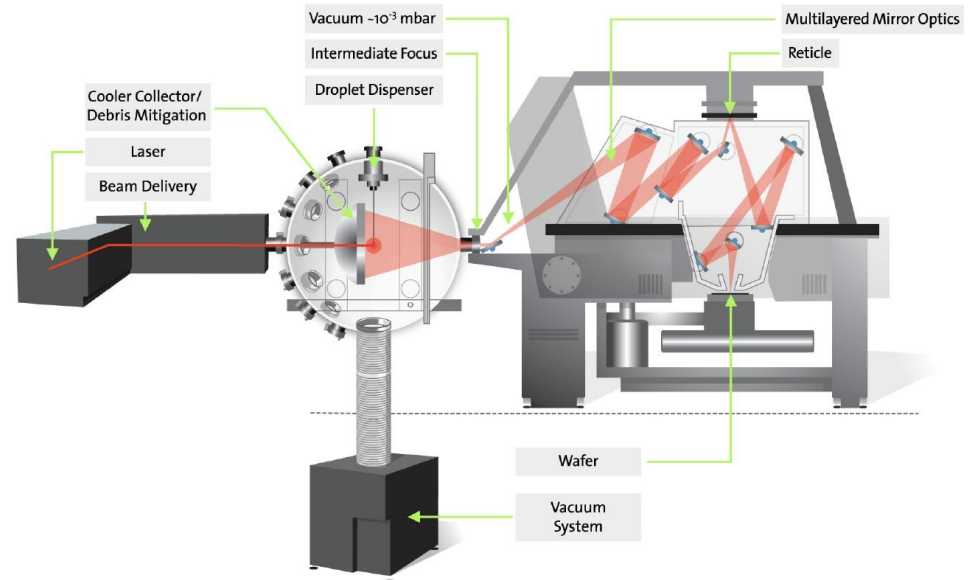
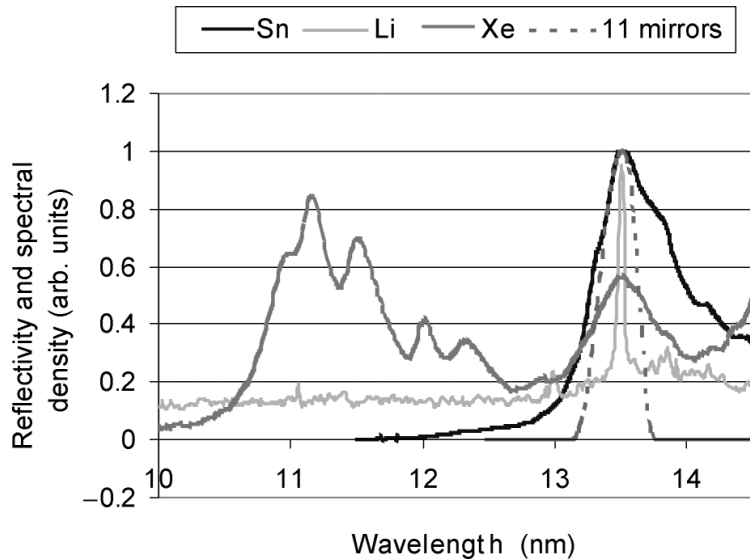
- The process technology of Taiwan Semiconductor Manufacturing Company Limited (TSMC):



- Optical diffraction needs to be taken into account.
- Shorter wavelength is preferred.

• Light source with a center wavelength of 13.5 nm is used.

# EUV light is generated from laser-produced plasma (LPP)



- $\lambda = 13.5 \text{ nm} \pm 1\%$  is required.
- At  $T=35\text{-}40 \text{ eV}$  ( $\sim 450,000 \text{ K}$ ), in-band emission occurs.
- Xenon:
  - $4p^6 4d^8 \rightarrow 4p^6 4d^7 5p$   
from single ion stage  $\text{Xe}^{10+}$
  - UTA @ 11 nm

- Tin:
  - $4p^6 4d^N \rightarrow 4p^5 4d^{N+1} + 4p^6 4d^{N-1} 4f$   
( $1 \leq N \leq 6$ ) in ions ranging from  $\text{Sn}^{8+}$  to  $\text{Sn}^{12+}$
  - UTA @ 13.5 nm
- UTA: unresolved transition array

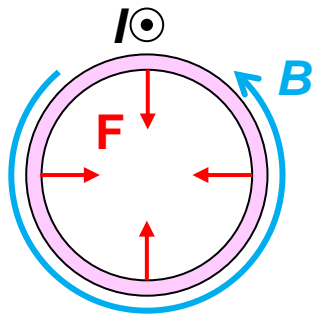
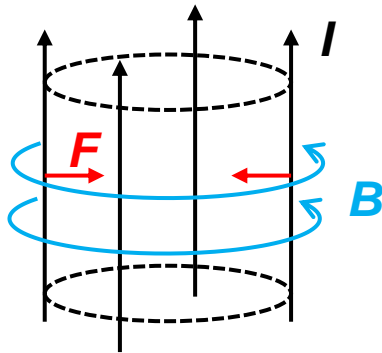
V. Bakshi, EUV sources for lithography

R. S. Abhari, etc., J. Micro/Nanolithography, MEMS, and MOEMS, 11, 021114 (2012)

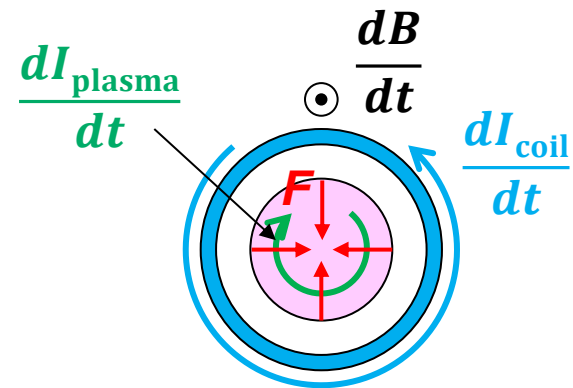
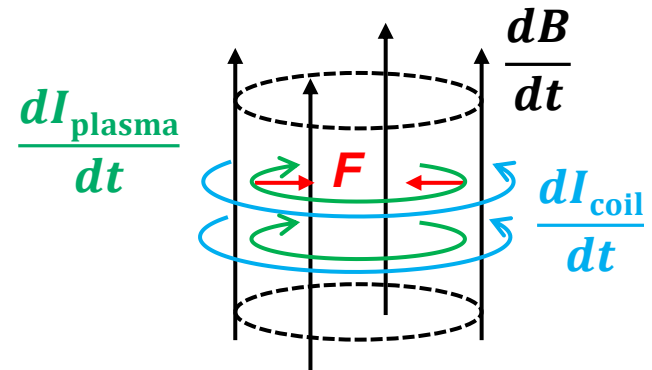
# Plasma can be compressed by using $\mathbf{j} \times \mathbf{B}$ force



- Z pinches



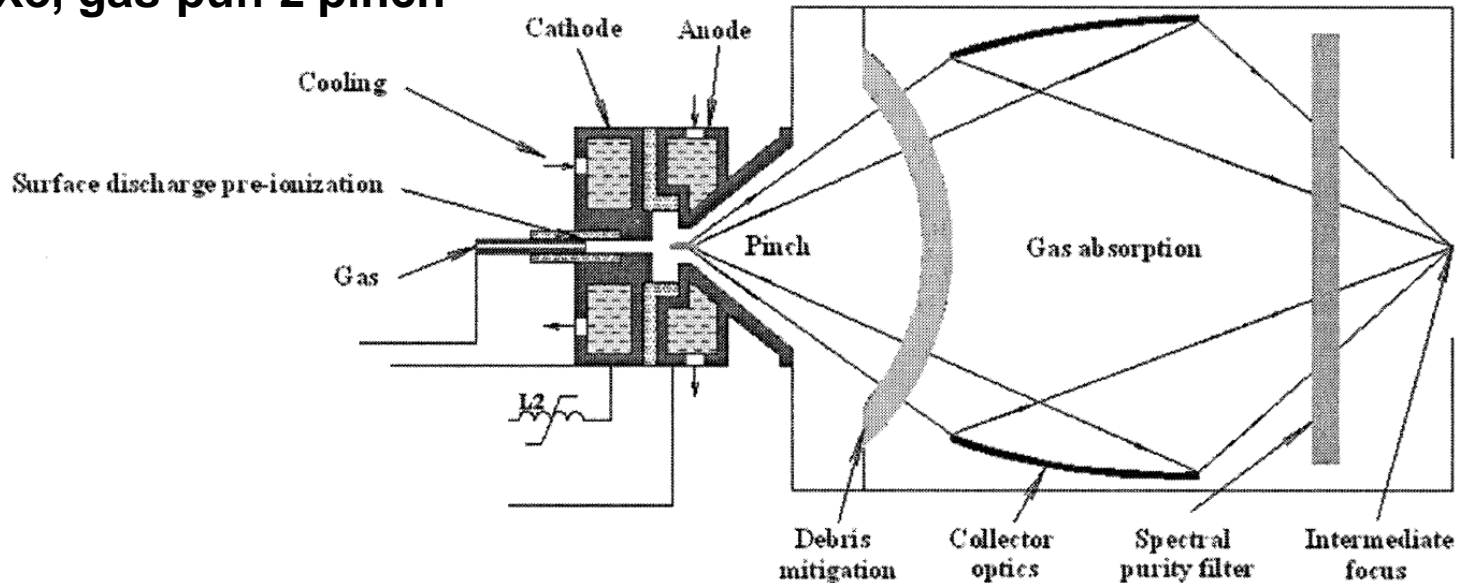
- Theta pinches



# Discharge produced plasma (DPP) can generate EUV light for EUV lithography



- DPP – Xe, gas-puff z pinch



- Electrodes are damaged significantly due to the heat and sputtering by ions.

	Laser-produced plasma (LPP)	Discharge-produced plasma (DPP)
Pros	Commercial system available.	High conversion efficiency.
Cons	Low conversion efficiency.	Short system life time due to electrode erosion.

# Outlines

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- Introduction to pulsed-power system
- **Review of circuit analysis**
- Static and dynamic breakdown strength of dielectric materials
  - Gas – Townsend discharge (avalanche breakdown), Paschen's curve
  - Liquid
  - Solid
- Energy storage
  - Pulse discharge capacitors
  - Marx generators
  - Inductive energy storage

# Kirchhoff's current law



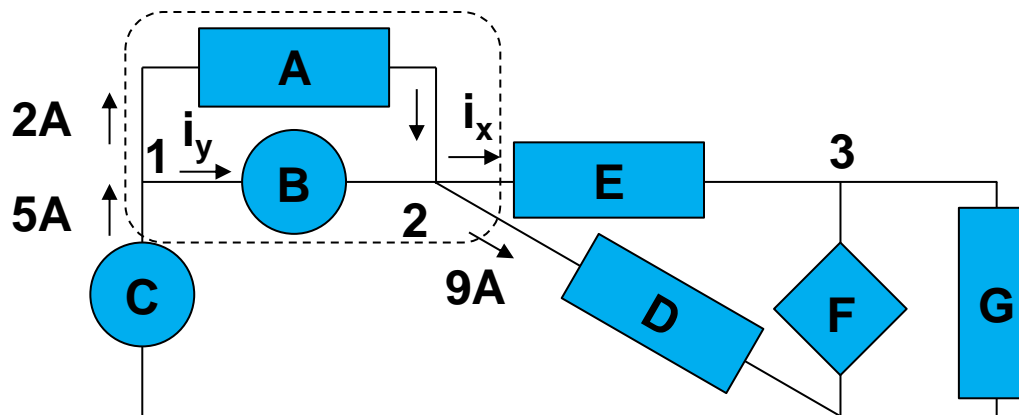
- At any instant in time, the algebraic sum of all currents leaving any closed surface is zero.

$$i_1 + i_2 + \dots + i_N = 0$$

or in abbreviated notation:

$$\sum_{k=1}^N i_k = 0$$

where  $i_k$  is the  $k^{\text{th}}$  current of the  $N$  currents leaving the closed surfaces.



$$i_y + 2 - 5 = 0$$

$$i_y = 3(\text{A})$$

$$-5 + i_x + 9 = 0$$

$$i_x = -4(\text{A})$$

$$2 + i_y - i_x - 9 = 0$$

# Kirchhoff's voltage law



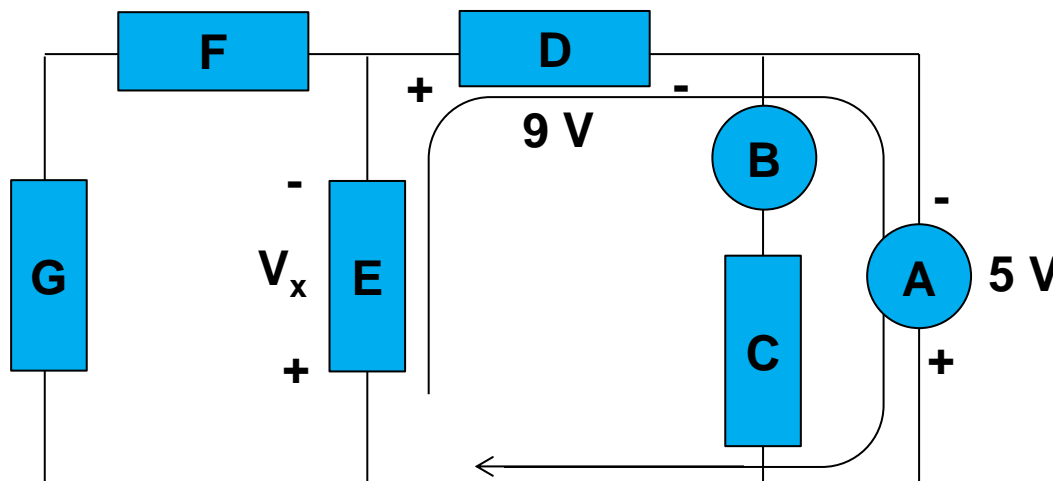
- At any instant in time, the algebraic sum of all voltage drops taken around any closed path is zero.

$$V_1 + V_2 + \dots + V_N = 0$$

or in abbreviated notation:

$$\sum_{k=1}^N V_k = 0$$

where  $V_k$  is the voltage drop, taken in the direction of the path along the  $k^{\text{th}}$  segment of the  $N$  segments in the closed path.



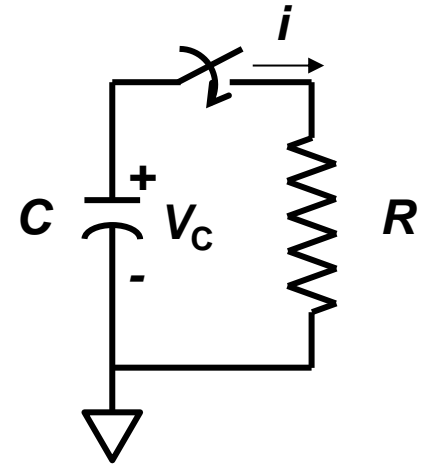
$$-V_x - 9 + 5 = 0$$

$$V_x = -4 \text{ (V)}$$

# Source-free RC circuit



- Assuming that the capacitor is fully charged to  $V_0$ .
- At  $t=0^+$ , the switch is closed.



$$V_C - iR = 0 \quad i = \frac{dQ}{dt} = -C \frac{dV_C}{dt}$$

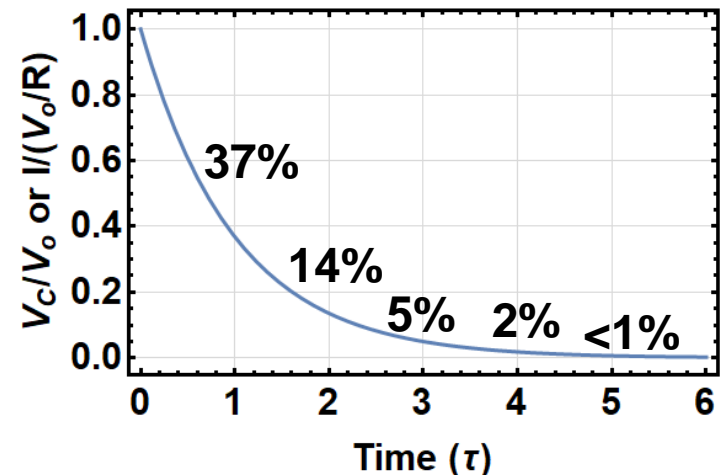
$$V_C + RC \frac{dV_C}{dt} = 0 \quad \frac{dV_C}{dt} + \frac{1}{RC} V_C = 0$$

$$\int_{V_0}^{V_C(t)} \frac{1}{V_C} dV_C = -\frac{1}{RC} \int_0^t dt$$

$$\ln \left( \frac{V_C(t)}{V_0} \right) = -\frac{t}{RC}$$

$$V_C(t) = V_0 e^{-t/RC} \equiv V_0 e^{-t/\tau_C} \quad \tau_C \equiv RC$$

$$I(t) = -C \frac{dV_C}{dt} = -V_0 C \left( -\frac{1}{\tau_C} \right) e^{-t/\tau_C} = \frac{V_0}{R} e^{-t/\tau_C}$$





# Bleeder resistors dissipate energy in the capacitor for safety



- Example 1:

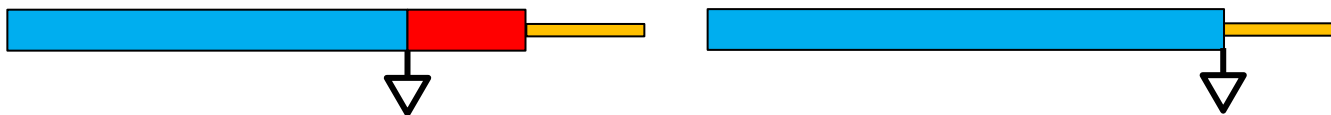
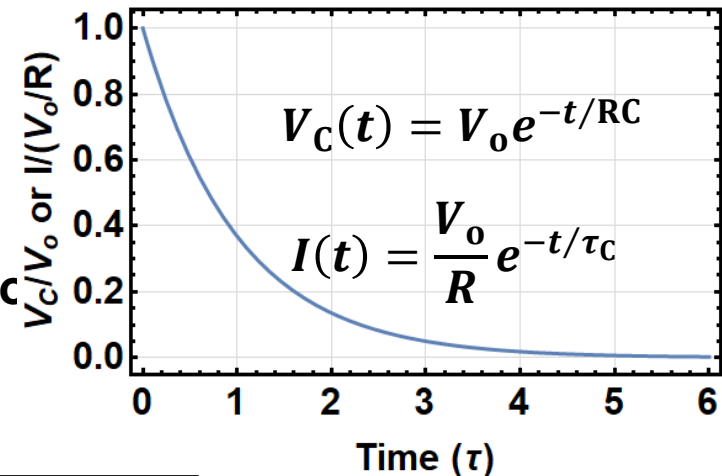
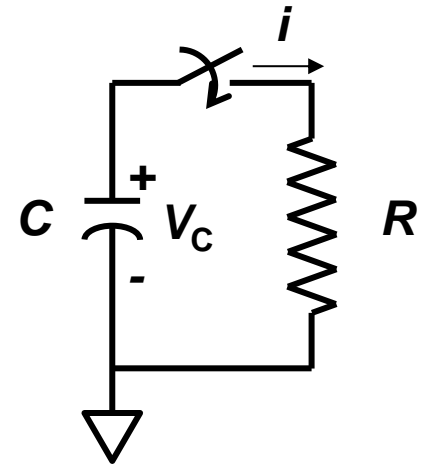
$V_o = 50 \text{ kV}$  ,  $C = 1 \text{ } \mu\text{F}$  ,  $V \leq 10 \text{ V}$  is safe.

If the bleeder resistor takes 15 mins to dissipate energy in the capacitor, then

$$10 = 50 \text{ k} \text{ Exp} \left( -\frac{15 \times 60}{R \times 10^{-6}} \right) \quad R = 10 \text{ M}\Omega$$

- Example 2: SOP for working on high voltage system.

- 1<sup>st</sup> chicken stick with a large resistors is needed to dissipate the energy in the capacitor slowly first.
- 2<sup>nd</sup> chicken stick that ground the capacitor is needed after most of the energy is dumped.



# Source-free RL circuit



- Assuming that the current is at steady state for  $t \leq 0$ ,  $I(0) = I_0$ .
- At  $t = 0^+$ , the switch is opened/closed.

$$-IR - V_L = 0 \qquad V_L = L \frac{dI}{dt}$$

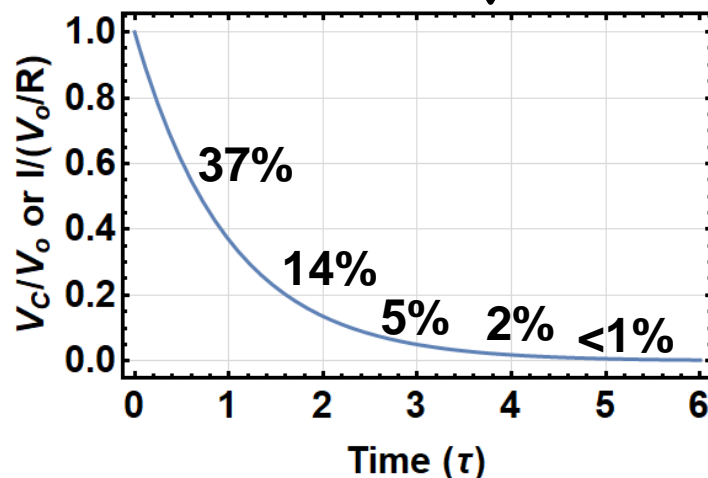
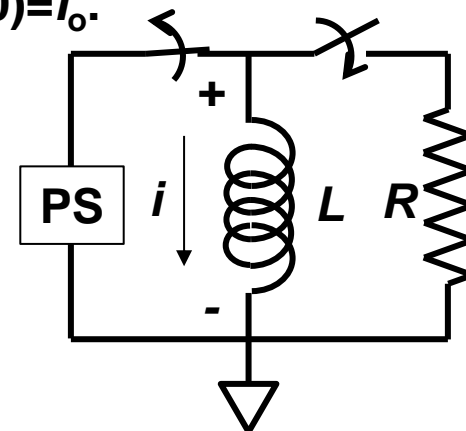
$$IR + L \frac{dI}{dt} = 0 \qquad \frac{dI}{dt} + \frac{R}{L} I = 0$$

$$\int_{I_0}^{I(t)} \frac{1}{I} dI = -\frac{R}{L} \int_0^t dt$$

$$\ln\left(\frac{I(t)}{I_0}\right) = -\frac{R}{L} t \equiv -\frac{t}{\tau_L} \qquad \tau_L \equiv \frac{L}{R}$$

$$I(t) = I_0 e^{-\frac{R}{L} t} \equiv I_0 e^{-t/\tau_L}$$

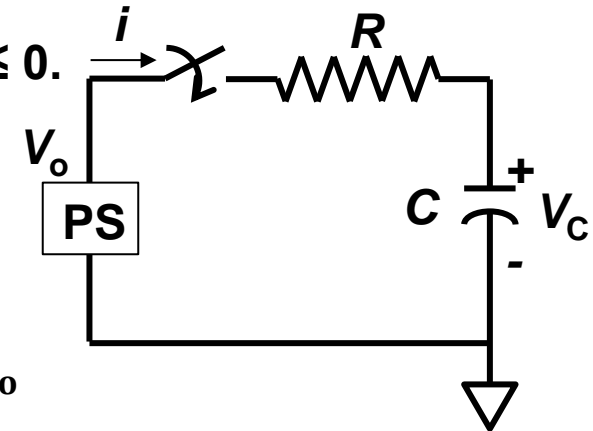
$$V(t) = L \frac{dI}{dt} = LI_0 \left(-\frac{1}{\tau_L}\right) e^{-t/\tau_L} = -RI_0 e^{-t/\tau_L}$$



# Charging of a capacitor



- Assuming that the capacitor is NOT charged at  $t \leq 0$ .
- At  $t=0^+$ , the switch is closed.



$$V_o - iR - V_c = 0 \quad i = \frac{dQ}{dt} = C \frac{dV_c}{dt}$$

$$V_o - RC \frac{dV_c}{dt} - V_c = 0 \quad \frac{dV_c}{dt} = -\frac{1}{RC} V_c + \frac{1}{RC} V_o$$

$$\frac{d(V_c - V_o)}{dt} = -\frac{1}{RC} (V_c - V_o) \quad V' \equiv V_c - V_o \quad V'(t=0) = -V_o$$

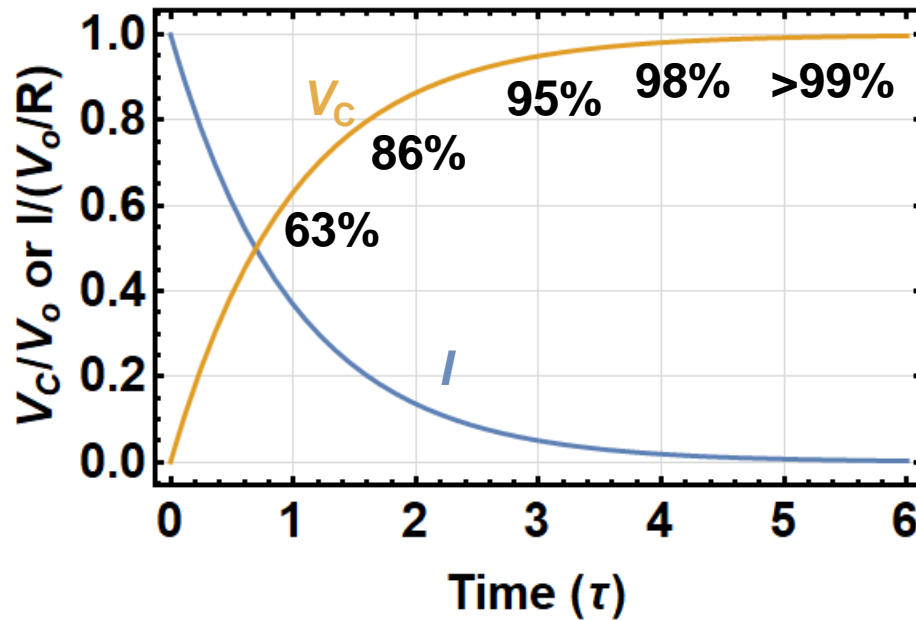
$$\frac{dV'}{dt} = -\frac{1}{RC} V' \quad \int_{-V_o}^{V'(t)} \frac{1}{V'} dV' = -\frac{1}{RC} \int_0^t dt \quad \ln\left(\frac{V'(t)}{-V_o}\right) = -\frac{t}{RC}$$

$$V'(t) = -V_o e^{-t/\tau_c} \quad V_c = V_o(1 - e^{-t/\tau_c})$$

$$\tau_c \equiv RC$$

$$I(t) = C \frac{dV_c}{dt} = -V_o \left(-\frac{1}{\tau_c}\right) e^{-t/\tau_c} = \frac{V_o}{R} e^{-t/\tau_c}$$

# The capacitor is almost fully charged after 5 time constant



$$V_C = V_0(1 - e^{-t/\tau_C})$$

$$I(t) = \frac{V_0}{R} e^{-t/\tau_C}$$

$$\tau_C \equiv RC$$

# LC oscillation



- Assuming that the capacitor is fully charged to  $V_0$ ,  $i(0)=0$ .
- At  $t=0^+$ , the switch is closed.

$$V_C - V_L = 0 \quad i = \frac{dQ}{dt} = -C \frac{dV_C}{dt} \quad V_L = L \frac{di}{dt} = -LC \frac{d^2V_C}{dt^2}$$

$$\frac{d^2V_C}{dt^2} + \frac{1}{LC} V_C = 0$$

$$V_C(t) = \alpha \sin(\omega t) + \beta \cos(\omega t) \quad \omega \equiv \frac{1}{\sqrt{LC}}$$

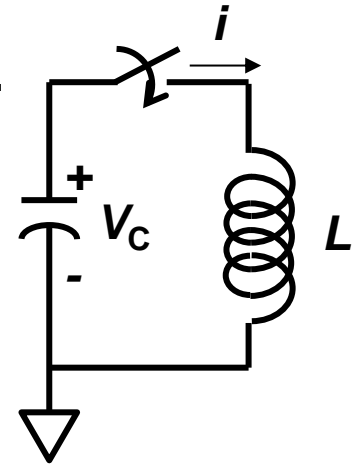
$$i = -C(\alpha \omega \cos(\omega t) - \beta \omega \sin(\omega t))$$

$$i(t=0) = 0 = -C\alpha\omega \quad \alpha = 0$$

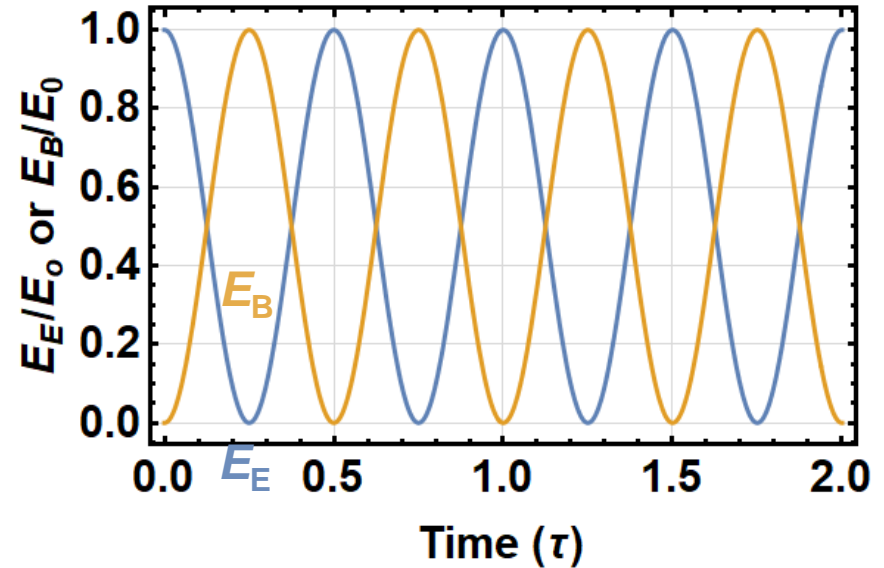
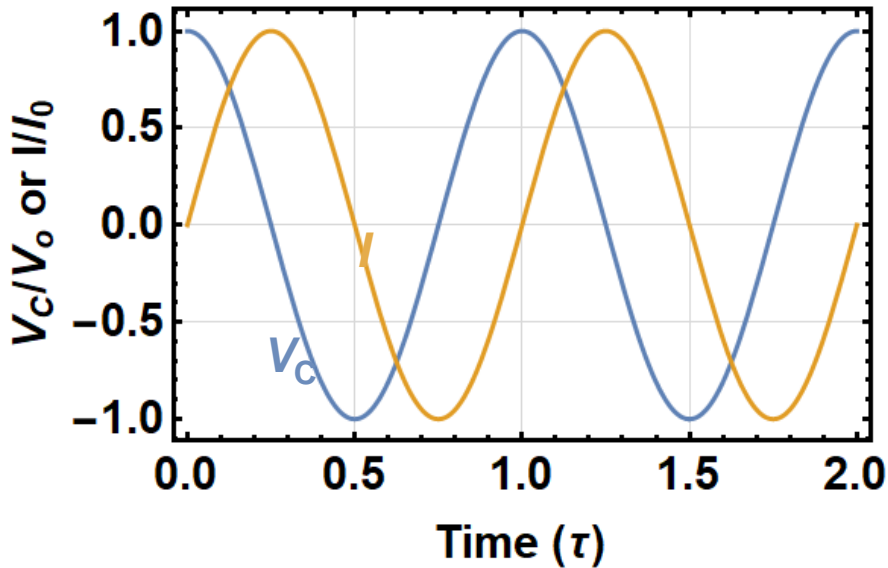
$$V_C(t=0) = V_0 = \beta$$

$$V_C = V_0 \cos(\omega t)$$

$$i = \frac{V_0}{\sqrt{L/C}} \sin(\omega t)$$



# Energy is oscillating between the capacitor and the inductor

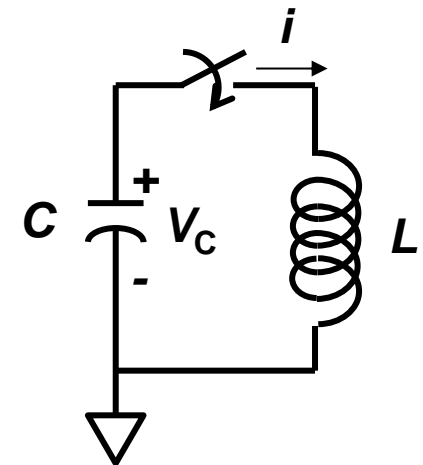


$$V_C = V_o \cos(\omega t) \quad \omega \equiv \frac{1}{\sqrt{LC}}$$

$$i = \frac{V_o}{\sqrt{L/C}} \sin(\omega t)$$

$$E_E = \frac{1}{2} C V_C^2$$

$$E_B = \frac{1}{2} L i^2$$





# Series RLC circuit



- Assuming that the capacitor is fully charged to  $V_0$ ,  $I(0)=0$ .
- At  $t=0^+$ , the switch is closed.

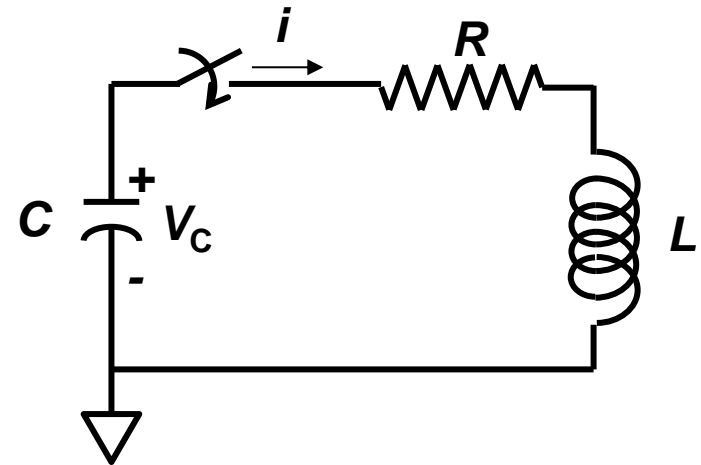
$$V_C - iR - V_L = 0 \quad i = \frac{dQ}{dt} = -C \frac{dV_C}{dt}$$

$$V_L = L \frac{di}{dt} = -LC \frac{d^2V_C}{dt^2}$$

$$\frac{d^2V_C}{dt^2} + \frac{R}{L} \frac{dV_C}{dt} + \frac{1}{LC} V_C = 0$$

$$D^2 + \frac{R}{L} D + \frac{1}{LC} = 0 \quad D = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$V_C = \exp\left(-\frac{R}{2L} t\right) \left[ \alpha \exp\left(\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} t\right) + \beta \exp\left(-\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} t\right) \right]$$



# Underdamped condition



$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$$

$$V_C = \exp\left(-\frac{R}{2L}t\right) \left[ \alpha \exp\left(i\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}t\right) + \beta \exp\left(-i\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}t\right) \right]$$

$$V_C = \exp\left(-\frac{R}{2L}t\right) \left[ \alpha \sin\left(\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}t\right) + \beta \cos\left(\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}t\right) \right]$$

$$V_C = \exp\left(-\frac{R}{2L}t\right) [\alpha \cos(\omega t) + \beta \sin(\omega t)] \quad \omega \equiv \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$V_C(0) = \alpha = V_o \quad V_C(t) = \exp\left(-\frac{R}{2L}t\right) [V_o \cos(\omega t) + \beta \sin(\omega t)]$$

$$i = -C \frac{dV_C}{dt} = \left[ -\frac{R}{2L} \exp\left(-\frac{R}{2L}t\right) (V_o \cos(\omega t) + \beta \sin(\omega t)) \right. \\ \left. + \exp\left(-\frac{R}{2L}t\right) (-V_o \omega \sin(\omega t) + \beta \omega \cos(\omega t)) \right]$$

# Underdamped condition



$$I(0) = -C \left( -\frac{R}{2L} V_o + \beta \omega \right) = 0 \quad \beta = \frac{R}{2L} \frac{V_o}{\omega} = V_o \frac{R/2L}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} = \frac{V_o}{\sqrt{\frac{4L}{R^2 C} - 1}}$$

$$I = -C \frac{dV_C}{dt} = \left[ -\frac{R}{2L} \exp\left(-\frac{R}{2L}t\right) \left( V_o \cos(\omega t) + \frac{R}{2L} \frac{V_o}{\omega} \sin(\omega t) \right) + \exp\left(-\frac{R}{2L}t\right) \left( -V_o \omega \sin(\omega t) + \frac{R}{2L} \frac{V_o}{\omega} \omega \cos(\omega t) \right) \right]$$

$$i(t) = \frac{V_o}{\sqrt{\frac{L}{C} - \left(\frac{R}{2}\right)^2}} \exp\left(-\frac{R}{2L}t\right) \sin(\omega t)$$

$$V_C(t) = V_o \exp\left(-\frac{R}{2L}t\right) \left[ \cos(\omega t) + \frac{R/2L}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \sin(\omega t) \right]$$

# Overdamped condition



$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$$

$$V_C = \exp\left(-\frac{R}{2L}t\right) \left[ \alpha \exp\left(\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}t\right) + \beta \exp\left(-\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}t\right) \right]$$

$$V_C = \exp\left(-\frac{R}{2L}t\right) [\alpha \exp(\gamma t) + \beta \exp(-\gamma t)] \quad \gamma \equiv \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$V_C(t=0) = V_0 = \alpha + \beta$$

$$i = -C \frac{dV_C}{dt} = -C \left\{ -\frac{R}{2L} \exp\left(-\frac{R}{2L}t\right) [\alpha \exp(\gamma t) + \beta \exp(-\gamma t)] \right. \\ \left. + \exp\left(-\frac{R}{2L}t\right) [\alpha \gamma \exp(\gamma t) - \beta \gamma \exp(-\gamma t)] \right\}$$

# Overdamped condition



$$i(0) = -C \left[ -\frac{R}{2L} (\alpha + \beta) + (\alpha\gamma - \beta\gamma) \right] = 0$$

$$\alpha \left( \gamma - \frac{R}{2L} \right) - \beta \left( \gamma + \frac{R}{2L} \right) = 0 \quad \alpha = \frac{\left( \gamma + \frac{R}{2L} \right)}{\left( \gamma - \frac{R}{2L} \right)} \beta \quad V_0 = \alpha + \beta$$

$$\beta \left[ 1 + \frac{\left( \gamma + \frac{R}{2L} \right)}{\left( \gamma - \frac{R}{2L} \right)} \right] = V_0 \quad \beta = \frac{V_0}{2} \left( 1 - \frac{R/2L}{\gamma} \right) \quad \alpha = \frac{V_0}{2} \left( 1 + \frac{R/2L}{\gamma} \right)$$

$$V_C = \frac{V_0}{2} \exp\left(-\frac{R}{2L}t\right) \left[ \left( 1 + \frac{R/2L}{\gamma} \right) \exp(\gamma t) + \left( 1 - \frac{R/2L}{\gamma} \right) \exp(-\gamma t) \right]$$

$$i = \frac{V_0}{2\gamma} \frac{1}{L} \exp\left(-\frac{R}{2L}t\right) [\exp(\gamma t) - \exp(-\gamma t)] \quad \gamma \equiv \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

# Critically damped condition



$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0 \quad R_{\text{cri}} = 2\sqrt{\frac{L}{C}}$$

$$V_C = (\alpha + \beta t)\exp\left(-\frac{R}{2L}t\right) \quad V_C = (V_o + \beta t)\exp\left(-\frac{R}{2L}t\right) \quad V_C(0) = V_o = \alpha$$

$$i = -C \frac{dV_C}{dt} = -C \left( \beta \exp\left(-\frac{R}{2L}t\right) - \frac{R}{2L} (V_o + \beta t) \exp\left(-\frac{R}{2L}t\right) \right)$$

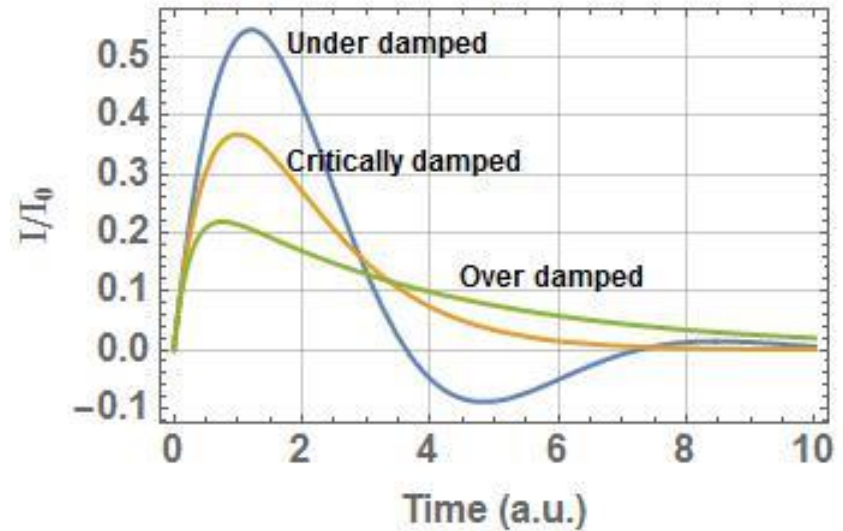
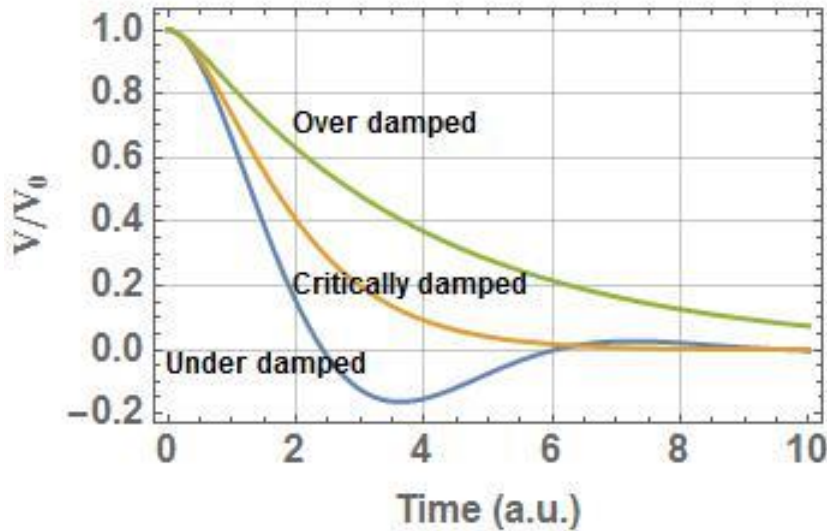
$$i(0) = -C \left( \beta - \frac{R}{2L} V_o \right) = 0 \quad \beta = \frac{R}{2L} V_o$$

$$V = V_o \left( 1 + \frac{R}{2L} t \right) \exp\left(-\frac{R}{2L} t\right)$$

$$i = \frac{V_o}{L} t \exp\left(-\frac{R}{2L} t\right)$$



# Varying R can move the discharge currents into different regime



$$R_{\text{cri}} = 2 \sqrt{\frac{L}{C}}$$

