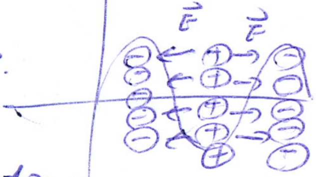
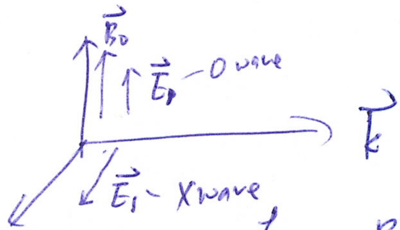
§4. Waves in Plasma.

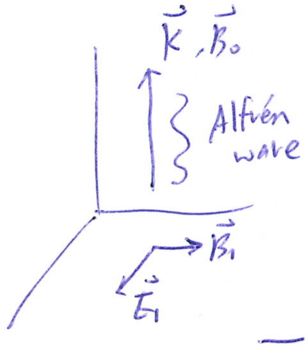
- Electrostatic wave: $\vec{B}_1 = 0$, $\vec{E}_1 \parallel \vec{k}$



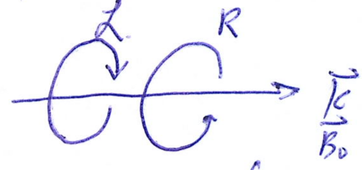
  - Electron waves – high freq. ions don't move.
    - $\vec{B}_0 = 0$, or $\vec{k} \parallel \vec{B}_0$ Plasma Oscillation
    - $\vec{k} \perp \vec{B}_0$ Upper hybrid Oscillation.
  - Ion waves – low freq., electrons move w/ ions.
    - $\vec{B}_0 = 0$, or $\vec{k} \parallel \vec{B}_0$ Acoustic waves
    - $\vec{k} \perp \vec{B}_0$ Electrostatic ion cyclotron wave (lower hybrid oscillations)

- Electromagnetic wave.
  - Electron waves – high freq.
    - $\vec{B}_0 = 0$ Light waves
    - $\vec{k} \perp \vec{B}_0$
      - $\vec{E}_1 \parallel \vec{B}_0$ O wave (Ordinary wave)
      - $\vec{E}_1 \perp \vec{B}_0$ X wave (extraordinary wave)

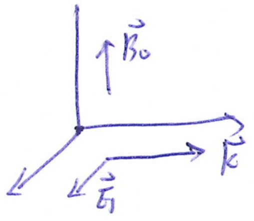




    - $\vec{k} \parallel \vec{B}_0$



      - R wave (whistler wave)
      - L wave.
  - Ion waves – low freq.
    - $\vec{B}_0 = 0$ X
    - $\vec{k} \parallel \vec{B}_0$ – Alfvén waves
    - $\vec{k} \perp \vec{B}_0$ – Magnetosonic waves



§4.1 representation of waves

When the oscillation amplitude is small, the waveform is generally sinusoidal.

$$n = \bar{n} \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$$

$\bar{n}$: amplitude.

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z \equiv kx \quad \text{for 1D}$$

$\vec{k}$: propagation const.

→ the real part of the expression is to be taken as the measurable quantity.

$$Re[n] = \bar{n}\cos(kx - \omega t)$$

A point of constant phase on the wave moves so that $\frac{d}{dt}(kx - \omega t) = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{k} \equiv v_\phi$ = phase velocity.

$$n = \bar{n}\exp[i(kx + \omega t)] \Rightarrow \text{toward } -\hat{x}.$$

Assuming the phase of n is zero,

$$E = \bar{E}\cos(kx - \omega t + \delta) \quad \text{or} \quad E = \bar{E}\exp[i(kx - \omega t + \delta)]$$

δ: phase.

$$\equiv \bar{E} e^{i\delta} \exp[i(kx - \omega t)]$$

$$\equiv \bar{E}_c \exp[i(kx - \omega t)]$$

$\bar{E}_c$: complex amplitude.

$$\tan\delta = \frac{Im(\bar{E}_c)}{Re(\bar{E}_c)}$$

⇒ Any oscillating quantity

$$g_1 = \bar{g}_1 \exp[i(kx - \omega t)]$$

$\bar{g}_1$: complex.

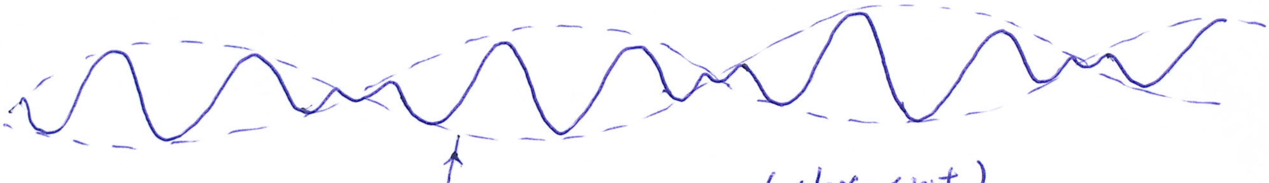
§ 4.2 Group velocity.

- phase velocity – can exceed c.
  - ↳ const amplitude cannot carry information.
- The "modulation information" does not travel at the phase velocity but at the group velocity. ($<c$)
- Modulated wave formed by adding ("beating") two waves.

$$\begin{cases} E_1 = E_0 \cos[(k+\Delta k)x - (\omega+\Delta\omega)t] \\ E_2 = E_0 \cos[(k-\Delta k)x - (\omega-\Delta\omega)t] \end{cases}$$

let $a = kx - \omega t$, $\quad b = \Delta k x - \Delta \omega t$

$$\begin{aligned} E_1 + E_2 &= E_0 \cos(a+b) + E_0 \cos(a-b) \\ &= E_0 [\cos a \cos b - \cancel{\sin a \sin b} + \cos a \cos b + \cancel{\sin a \sin b}] \\ &= 2E_0 \cos a \cos b. \\ &= 2E_0 \cos(kx - \omega t) \cos(\Delta k x - \Delta \omega t) \end{aligned}$$



Envelope → $\cos(\Delta k x - \Delta \omega t)$

↳ travel w/ speed $\frac{\Delta \omega}{\Delta k}$

Group velocity $V_g \equiv \frac{d\omega}{dk} < c$

3.4.3 Plasma oscillations

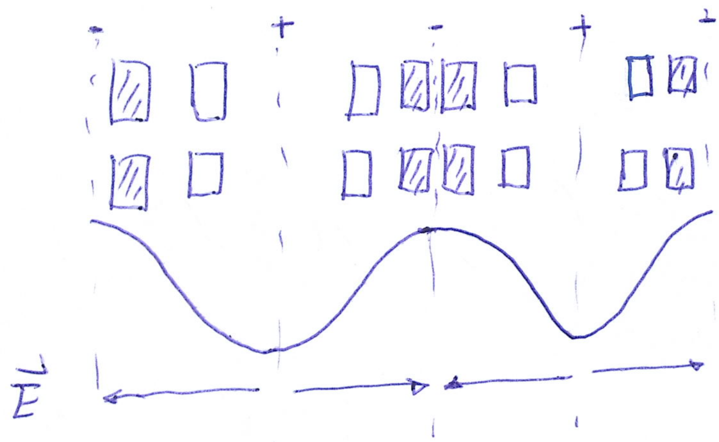
- If the ~~plasma~~ electrons in a plasma are displaced from a uniform background of ions, electric fields will be built up in such a direction as to restore the neutrality of the plasma by pulling the electrons back to their original positions.

$\because$ their <u>inertia</u>, the electrons will <u>overshoot</u> and oscillate around their equilibrium positions.

$\longrightarrow$ plasma frequency.

* ion $\rightarrow$ massive $\rightarrow$ considered as fixed

* The resulting charge bouncing causes a spatially periodic E field, which tends to restore the electrons to their neutral positions.



Assumption:

① $B = 0$

② $kT = 0$ ← cold

③ ion are fixed with uniform distribution in space

④ plasma is infinite

⑤ e^- move in $\hat{x}$ only

$\nabla = \frac{\partial}{\partial x}\hat{x}$ $\quad \vec{E} = E\hat{x}$ $\quad \underbrace{\nabla \times \vec{E}}_{\frac{\partial B}{\partial t}} = 0$, $E = -\nabla\phi$

$\Rightarrow$ Electrostatic oscillation.

$$m n_e \left[\frac{\partial \vec{V_e}}{\partial t} + (\vec{V_e}\cdot\nabla)\vec{V_e} \right] = -e n_e \vec{E}$$

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \vec{V_e}) = 0$$

poisson's eq: $\quad \epsilon_0 \nabla\cdot\vec{E} = \epsilon_0 \frac{\partial E}{\partial x} = e(n_i - n_e)$

* Linearization: amplitude of oscillation is small.

Separate the dependent variables into two parts:

- equilibrium part, w/ "0"
- perturbation part w/ "1"

i.e.

$$\begin{cases} n_e = n_0 + n_1 \\ \vec{V_e} = \vec{V_0} + \vec{V_1} \\ \vec{E} = \vec{E_0} + \vec{E_1} \end{cases}$$

Assuming that a uniform neutral plasma at rest:

$$\nabla n_0 = V_0 = E_0 = 0$$

$$\frac{\partial n_0}{\partial t} = \frac{\partial V_0}{\partial t} = \frac{\partial E_0}{\partial t} = 0$$

$$m \cancel{(n_0+n_1)} \left[\frac{\partial(\cancel{\vec{V_0}} + \vec{V_1})}{\partial t} + (\cancel{\vec{V_0}} + \vec{V_1})\cdot\nabla(\cancel{\vec{V_0}} + \vec{V_1}) \right] = -e \cancel{(n_0+n_1)} (\cancel{\vec{E_0}} + \vec{E_1})$$

$$m \left[\frac{\partial \vec{V_1}}{\partial t} + \underbrace{\cancel{(\vec{V_1}\cdot\nabla)\vec{V_1}}}_{\text{2nd order (quadratic)}} \right] = -e\vec{E_1} \Rightarrow \underline{m\frac{\partial V_1}{\partial t} = -eE_1}$$

- The linear theory is valid as long as $|\vec{v}_1|$ is small enough that such quadratic terms are indeed negligible.

$$\frac{\partial(\cancel{n_0}+n_1)}{\partial t} + \nabla\cdot\left[(n_0+n_1)(\cancel{\vec{v}_0}+\vec{v}_1)\right] = 0$$

$$\frac{\partial n_1}{\partial t} + \nabla\cdot\left(n_0\vec{v}_1 + \underbrace{\cancel{n_1\vec{v}_1}}_{2^{nd}\ \text{order}}\right) = 0$$

$$\frac{\partial n_1}{\partial t} + n_0\nabla\cdot\vec{v}_1 + \cancel{\vec{v}_1\cdot\nabla n_0} = 0$$

$$\frac{\partial n_1}{\partial t} + n_0\nabla\cdot\vec{v}_1 = 0 \Rightarrow \underline{\frac{\partial n_1}{\partial t} + n_0\frac{\partial v_1}{\partial x} = 0}$$

In poisson's Eq: $n_{i0} = n_{e0}$, $n_{i1} = 0 \leftarrow$ fixed ion

$$\epsilon_0\nabla\cdot(\cancel{\vec{E}_0}+\vec{E}_1) = e(\cancel{n_{i0}}+\cancel{n_{i1}}-\cancel{n_{e0}}-n_{e1})$$

$$\epsilon_0\nabla\cdot\vec{E}_1 = -e n_{e1} \Rightarrow \underline{\epsilon_0\frac{\partial E_1}{\partial x} = -e n_{e1}}$$

Sinusoidally assumption:

$$\vec{v}_1 = v_1 e^{i(kx-\omega t)}\hat{x}$$

$$n_1 = n_1 e^{i(kx-\omega t)}$$

$$\vec{E}_1 = E_1 e^{i(kx-\omega t)}\hat{x}$$

$$\Rightarrow \frac{\partial}{\partial t} \rightarrow -i\omega \quad ; \quad \nabla \Rightarrow ik\hat{x}$$

$$m\frac{\partial\vec{v}_1}{\partial t} = -e\vec{E}_1 \Rightarrow -i\omega m v_1 = -eE_1$$

$$\frac{\partial n_1}{\partial t} + n_0\nabla\cdot\vec{v}_1 = 0 \Rightarrow -i\omega n_1 + ikn_0 v_1 = 0 \rightarrow n_1 = \frac{k}{\omega}n_0 v_1$$

$$\epsilon_0\nabla\cdot\vec{E}_1 = -e n_{e1} \Rightarrow ik\epsilon_0 E_1 = -e n_1 \rightarrow E_1 = \frac{-e}{ik\epsilon_0}n_1$$

$$\cancel{+i}\omega m \cancel{v_1} = -eE_1 = -e\frac{-e}{i\cancel{k}\epsilon_0}\cdot\underbrace{\frac{\cancel{k}}{\omega} n_0 v_1}_{n_1}$$

$$= \cancel{+i}\frac{n_0 e^2}{\epsilon_0 \omega}\cancel{v_1}$$

$$\Rightarrow \quad \underline{\omega^2 = \frac{n_0 e^2}{m\epsilon_0}}$$

$$\text{plasma frequency} \equiv \omega_p = \left(\frac{n_0 e^2}{\epsilon_0 m}\right)^{1/2} \ \text{rad/sec}$$

$\frac{\omega_p}{2\pi} = f_p \simeq 9\sqrt{n_{(m^{-3})}}$ ← usually very high.

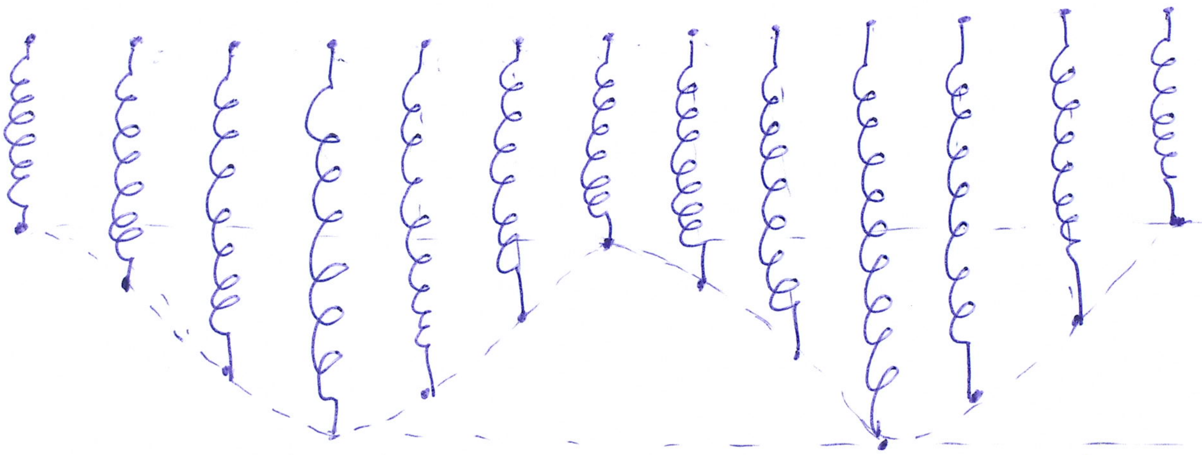
Ex: $n = 10^{18}\ m^{-3}$, $f_p \simeq 9\sqrt{10^{18}} = 9\times10^9\ sec^{-1} = \underline{9\ GHz}$

f_p : microwave range.

electron cyclotron frequency $f_{ce} \simeq 28\ GHz/Tesla$. $\underline{\left(\omega_{ce} = \frac{eB}{m}\right)}$

for $B \simeq 0.32T$, $f_{ce} \simeq f_p$ @ $n = 10^{18}\ m^{-3}$

$\frac{d\omega}{dk} = 0$ → group velocity is zero
The disturbance does not propagate



- The frequency will be fixed by the springs, but the wavelength can be arbitrary.
  The two undisturbed balls at the ends will not be affected, and the initial disturbance does not propagate.
- As long as electrons do not collide with ions or with each other, they can be pictured as independent oscillators moving horizontally.

§ 4.4 Electron plasma wave.

Thermal motion $\longrightarrow$ causes plasma oscillation to propagate.

- Electron streaming into adjacent layers of plasma w/ their thermal velocities will carry information about what is happening in the oscillating region
  $\longrightarrow$ plasma oscillation $\Rightarrow$ plasma wave.

$$\begin{cases} m n_e \left[\dfrac{\partial \vec{V_e}}{\partial t} + (\vec{V_e} \cdot \nabla) \vec{V_e} \right] = - e n_1 \vec{E} - \nabla P_e \\ \dfrac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V_e}) = 0 \\ \nabla P_e = 3 K T_e \nabla n_e \end{cases}$$

$\Leftarrow$ 1D, isothermal $\gamma = \dfrac{1+2}{1} = 3$ $\quad \left(\dfrac{2+N}{N} \right)$

$\nabla p = \gamma K T \nabla n$

$$m(n_0+\cancel{n_1})\left[\frac{\partial(\vec{v}_0+\vec{v}_1)}{\partial t}+\cancel{(\vec{v}_0+\vec{v}_1)\nabla(\vec{v}_0+\vec{v}_1)}\right]=-e(n_0+\cancel{n_1})(\vec{E}_0+\vec{E}_1)-\nabla(\vec{p}_0+\vec{p}_1)$$

2nd order   2nd order   2nd order   P83

$$\Rightarrow\quad m n_0\frac{\partial v_1}{\partial t}=-en_0E_1-\nabla p_1=-en_0E_1-3kT_e\nabla(\cancel{n_0}+n_1)$$

$$\Rightarrow\quad \underline{-i\omega m n_0 v_1=-en_0E_1-ik\,3\cdot kT_e n_1}$$

$$\frac{\partial n_e}{\partial t}+\nabla\cdot(n_e\vec{v}_e)=0\;\Rightarrow\;\frac{\partial n_1}{\partial t}+n_0\frac{\partial v_1}{\partial x}=0$$

$$\Rightarrow\;\underline{-i\omega n_1+ikn_0v_1=0}\;\Rightarrow\; n_1=\frac{k}{\omega}n_0v_1$$

$$\epsilon_0\nabla\cdot\vec{E}_1=-en_{e1}\;\Rightarrow\;\underline{ik\epsilon_0E_1=-en_1}\;\Rightarrow\;E_1=\frac{-e}{ik\epsilon_0}n_1$$

$$\Rightarrow\;\cancel{i}\omega m\cancel{n_0v_1}=en_0\left(\frac{-e}{ik\epsilon_0}\right)n_1+ik\,3kT\,n_1$$

$$=\left[+\cancel{i}\frac{n_0e^2}{k\epsilon_0}+\cancel{i}k3kT\right]\left(\frac{k}{\omega}\cancel{n_0v_1}\right)$$

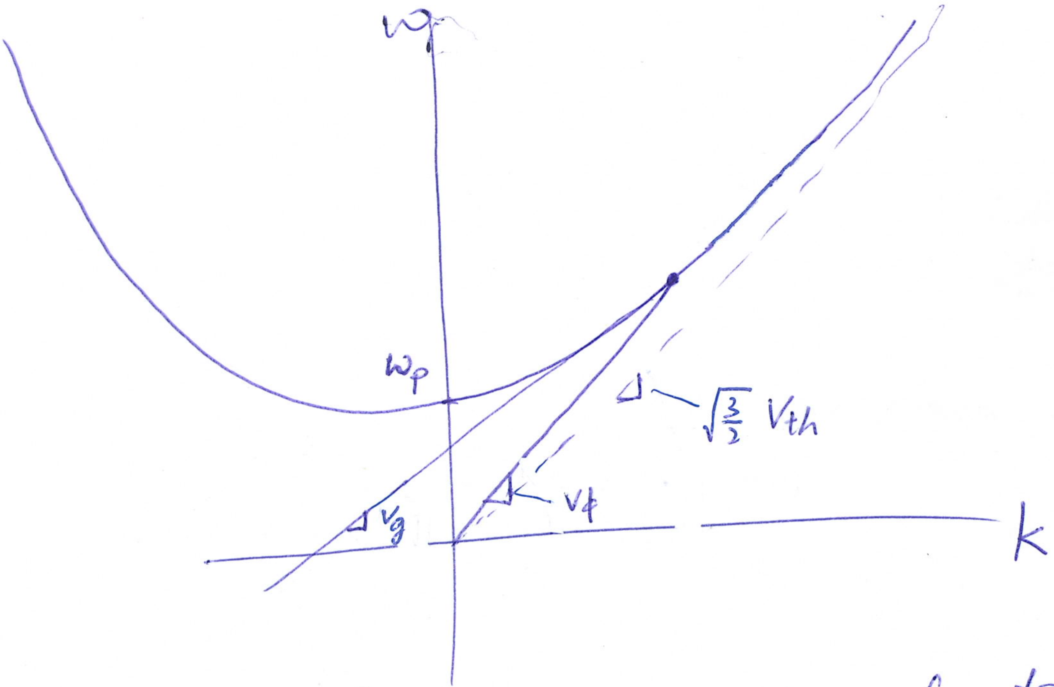
$$\omega^2=\frac{k}{m}\left(\frac{n_0e^2}{k\epsilon_0}+3kTk\right)$$

$$=\frac{n_0e^2}{\epsilon_0 m}+\boxed{\frac{3kT}{m}}k^2\;\longrightarrow\;\frac{3}{2}v_{th}^2$$

$$=\omega_p^2+\frac{3}{2}v_{th}^2k^2\qquad \because\; \underline{v_{th}^2=\frac{2kT_e}{m}}$$

$$\frac{d}{dk}:\quad\frac{d\omega^2}{dk}=2\omega\cdot\frac{d\omega}{dk}=3v_{th}^2k.$$

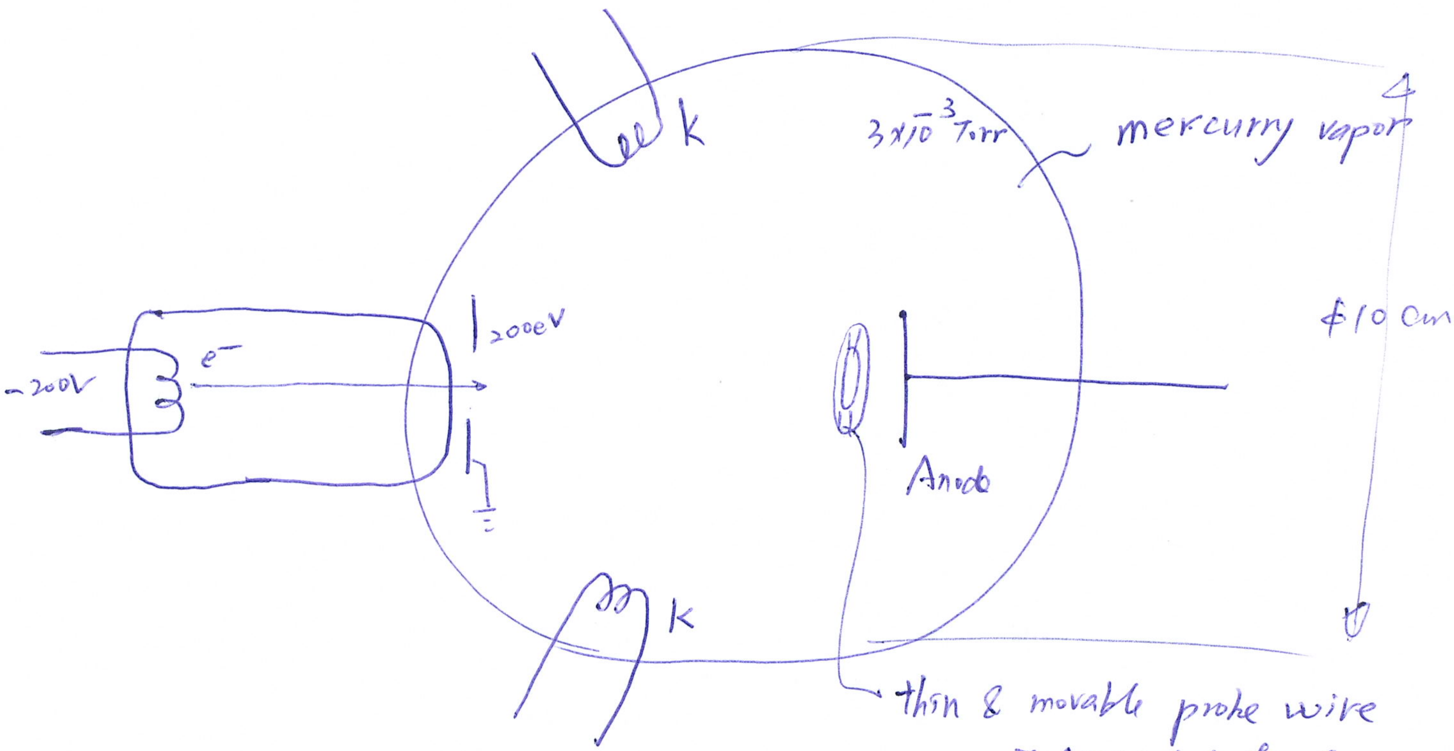
$$\Rightarrow\quad v_g\equiv\frac{d\omega}{dk}=\frac{3}{2}v_{th}^2\underbrace{\frac{k}{\omega}}_{1/v_\phi}=\frac{3}{2}\frac{v_{th}^2}{v_\phi}<c$$



* At large k (small λ), information travels essentially at the thermal velocity.

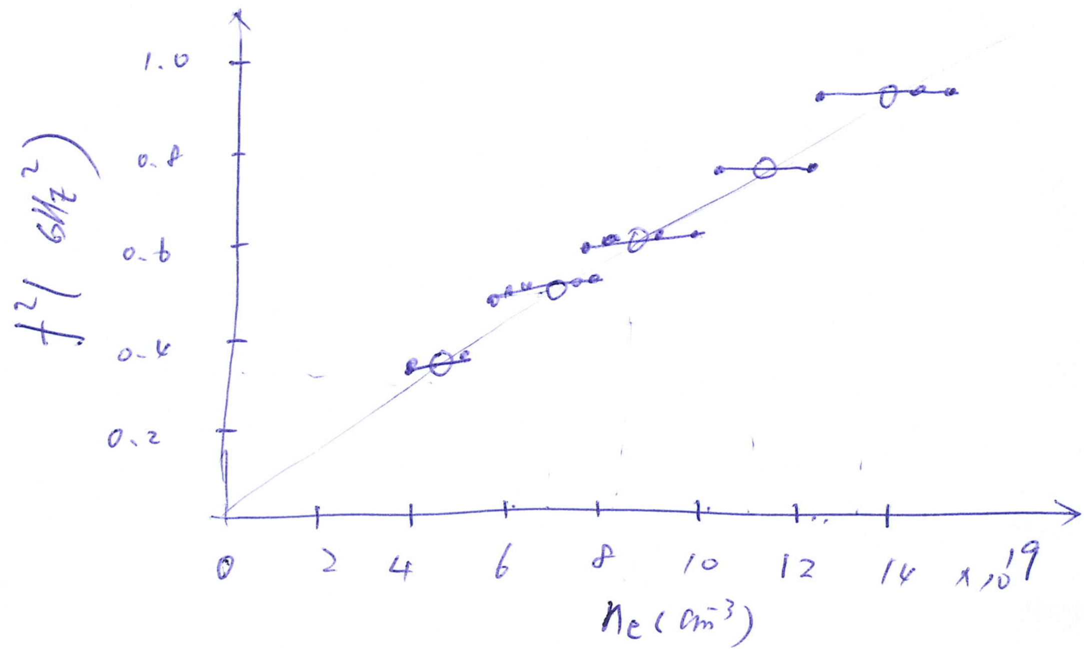
* At small k (large λ), - - - - - - more slowly than v_{th} even though v_ϕ is greater than v_{th}.
  ∵ density gradient is small at large λ, thermal motions carry very little net momentum into adjacent layers.

* A simple way to excite plasma waves would be to apply an oscillating potential to a grid or a series of grids in a plasma.
  → GHz oscillator is hard back in the day
  → use an electron beam
  → once the plasma oscillate arise, they will bunch the electrons, and the oscillations will grow by a positive feedback mechanism.



thin & movable probe wire is connected to a radio receiver → f measurement.

Anode & K → electric discharge.



* Only standing waves were observed.

$f^2 \propto$ discharge current $\propto n^2$

Note that $f_p \simeq 9\sqrt{n}$

$\Rightarrow$ plasma oscillation, not wave!!

No traveling wave may be because.

the beam was so thin that thermal motions carried electrons out of the beam, thus dissipating the oscillating energy.

The electron bunching was accomplished not in the plasma but in the oscillating sheaths at the ends of the plasma column.

7.45 Sound waves

Neglecting viscosity.

Navier-Stokes eq.:

$$\begin{cases} \rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=-\nabla p=-\frac{\gamma p}{\rho}\nabla\rho \\ \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \end{cases}$$

2nd order.

$$\Rightarrow (\rho_0+\cancel{\rho_1})\left[\frac{\partial(\cancel{\vec{v}_0}+\vec{v}_1)}{\partial t}+\cancel{(\vec{v}_0+\vec{v}_1)\nabla(\vec{v}_0+\vec{v}_1)}\right]=-\frac{\gamma(p_0+\cancel{p_1})}{\rho_0+\cancel{\rho_1}}\nabla(\cancel{\rho_0}+\rho_1)$$

2nd order   2nd order   2nd order

$$\Rightarrow \rho_0\frac{\partial\vec{v}_1}{\partial t}=-\frac{\gamma p_0}{\rho_0}\nabla\rho_1 \Rightarrow -i\omega\rho_0 v_1=-ik\frac{\gamma p_0}{\rho_0}\rho_1$$

$$\frac{\partial(\cancel{\rho_0}+\rho_1)}{\partial t}+\nabla\cdot\left[(\rho_0+\cancel{\rho_1})(\cancel{\vec{v}_0}+\vec{v}_1)\right]=0$$

2nd order

$$\frac{\partial\rho_1}{\partial t}+\nabla(\rho_0\vec{v}_1)=0 \Rightarrow \frac{\partial\rho_1}{\partial t}+\rho_0\nabla\vec{v}_1=0 \Rightarrow -i\omega\rho_1+ik\rho_0 v_1=0$$

$$+\cancel{i}\omega\rho_0 v_1 = +\cancel{i}k\,\gamma\frac{p_0}{\rho_0}\rho_1$$

$$-\cancel{i}\omega\rho_1 + \cancel{i}k\rho_0 v_1 = 0 \quad \Rightarrow \quad \rho_1 = \frac{k}{\omega}\rho_0 v_1$$

$$\Rightarrow \quad \omega\rho_0 \cancel{v_1} = k\gamma\frac{p_0}{\rho_0}\left(\frac{k}{\omega}\rho_0 v_1\right)$$

$$= \frac{k^2\gamma p_0 \cancel{v_1}}{\omega}$$

$$\Rightarrow \quad \frac{\omega}{k} = \sqrt{\gamma\frac{p_0}{\rho_0}} = \left(\frac{\gamma KT}{M}\right)^{1/2} \equiv C_s$$

C_s : sound speed of neutral gas.

analogous phenomenon to

ion acoustic wave or

ion wave

→ carry by pressure gradient.

§ 4.6 Ion Waves.

* Ordinary sound waves – it would not occur without collisions.
* Acoustic waves – it can occur through the intermediary of an electric field.
  – massive ions will be involved.
  → low-frequency oscillations
  ⇒ plasma approximation
  ⇒ $n_i = n_e = n$. Poisson's eq. is not used.

$\underline{B=0}$, ion fluid eq.:

$$Mn\left[\frac{\partial V_i}{\partial t} + (V_i\cdot\nabla)V_i\right] = en\vec{E} - \nabla p = -en\nabla\phi - \gamma_i K T_i \nabla n$$

→ Linearizing ~~& plane waves. $\frac{\partial}{\partial t}\to -i\omega$, $\nabla\to ik$~~

$$M(n_0+\cancel{n_1})\left[\frac{\partial(\cancel{\vec{V}_{i0}}+\vec{V}_{i1})}{\partial t} + (\cancel{\vec{V}_{i0}}+\cancel{\vec{V}_{i1}})\cdot\nabla(\cancel{\vec{V}_{i0}}+\vec{V}_{i1})\right] = -e(n_0+\cancel{n_1})\nabla(\cancel{\phi_0}+\phi_1) - \gamma_i K(T_{i0}+\cancel{T_{i1}})\nabla(\cancel{n_0}+n_1)$$

2nd order 2nd order 2nd order 2nd order

$$\cancel{Mn_0\left[\frac{\partial\vec{V}_{i1}}{\partial t} + \vec{V}_{i0}\cdot\nabla\vec{V}_{i1}\right] = -en_0\nabla\phi_1 - \gamma_i T_{i0}\nabla n_1}$$

$$Mn_0\frac{\partial\vec{V}_{i1}}{\partial t} = -en_0\nabla\phi_1 - \gamma_i K T_{i0}\nabla n_1$$

→ plane waves, $\frac{\partial}{\partial t}\to -i\omega$, $\nabla\to ik$

$$\Rightarrow\ -\cancel{i}\omega M n_0 \vec{V}_{i1} = -\cancel{i}k e n_0\phi_1 - ik\gamma_i K T_{i0} n_1$$

$$\Rightarrow\ \omega M n_0 V_{i1} = k e n_0 \phi_1 + k\gamma_i K T_i n_1$$

For electron, $m \equiv 0$ ← very light. The balance of forces on electrons:

$$n_e = n = n_0\exp\left(\frac{e\phi_1}{KT_e}\right) \simeq n_0\left(1+\frac{e\phi_1}{KT_e}+\cdots\right) = n_0 + n_1 + \cdots$$

$$\underline{n_1 = n_0\frac{e\phi_1}{KT_e}}$$

$\phi = \phi_0 + \phi_1 = \phi_1$, $\because E_0 = 0$

Continuity of ion:

$$\frac{\partial n}{\partial t} + \nabla \cdot (\vec{v}_i n) = 0$$

$$\Rightarrow \frac{\partial (\cancel{n_0} + n_1)}{\partial t} + \nabla \cdot \left[(\cancel{\vec{v}_{i0}} + \vec{v}_{i1})(n_0 + \cancel{n_1}) \right] = 0$$

2nd order

$$\Rightarrow \frac{\partial n_1}{\partial t} + \nabla \cdot (n_0 \vec{v}_{i1}) = 0$$

$$\Rightarrow -\cancel{i}\omega n_1 + \cancel{i}k n_0 v_{i1} = 0 \Rightarrow \underline{\omega n_1 = k n_0 v_{i1}}$$

$$\cancel{M n_0 \frac{\partial v_1}{\partial t} = -e n_0 \phi_1}$$

$$\begin{cases} \omega M n_0 v_{i1} = k e n_0 \phi_1 + k \gamma_i k T_i n_1 \\ n_1 = n_0 \frac{e\phi_1}{kT_e} \Rightarrow \phi_1 = \frac{n_1}{n_0} \frac{kT_e}{e} \\ \omega n_1 = k n_0 v_{i1} \Rightarrow n_1 = \frac{k}{\omega} n_0 v_{i1} \end{cases}$$

$$\Rightarrow \omega M \cancel{n_0} \cancel{v_{i1}} = k \cancel{e} \cancel{n_0} \frac{n_1}{\cancel{n_0}} \frac{kT_e}{\cancel{e}} + k \gamma_i k T_i n_1$$

$$\cancel{= k (kT_e + k\gamma T_e) n_1}$$

$$= k (kT_e + \gamma_i k T_i) n_1$$

$$= k (kT_e + \gamma_i k T_i) \frac{k}{\omega} \cancel{n_0} \cancel{v_{i1}}$$

$$\Rightarrow \omega^2 = k^2 \left(\frac{kT_e}{M} + \frac{\gamma_i k T_i}{M} \right) \qquad \underline{\frac{d\omega}{dk} \equiv v_g = v_s}$$

$$\frac{\omega}{k} = \left(\frac{kT_e}{M} + \frac{\gamma_i k T_i}{M} \right)^{1/2} \equiv v_s$$

dispersion relation for ion accoustic waves.

i.e., the sound speed ~~of~~ in a plasma.

For 1D: $\gamma_i = 3$.

$\gamma_e = 1$, ∵ it moves so fast relative to these waves so that electrons are isothermal

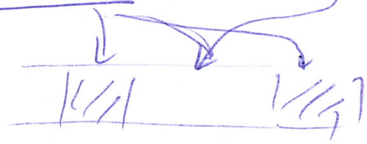
* plasma oscillations are basically "constant-frequency waves", with a correction due to thermal motions.

* Ion waves are "const.-velocity waves" and exist only when there are thermal motions.

$$V_g = V_s.$$

* [Electron plasma oscillations : ions remain fixed.
  [Ion accoustic waves : electrons follow ions and tend to shield out electric fields arising from the bounching of ions.

* The ions form regions of compression and rarefaction, like ordinary sound waves



$$\frac{\omega}{k} = \left(\frac{KT_e}{M} + \frac{\gamma_i K T_i}{M} \right)^{1/2}$$

The ion thermal motions spread out the ro...

field is shielded out by electrons, only a fraction ($\propto KT_e$) is available to act on the ion bounch

* Note that. $KT_i \to 0$. $V_s = \frac{\omega}{k} \to \left(\frac{KT_e}{M}\right)^{1/2}$, ion waves still exist!

~~This~~ This is not the same as neutral gas.

3.4.7 Validity of the plasma Approximation.

* In ion waves, we set $n_i = n_e$ while $\vec{E} \neq 0$

Poisson's eq. : $\epsilon_0 \nabla\cdot\vec{E} = e(n_i - n_e) = -\epsilon_0\nabla^2\phi$ what's wrong?!

Linearization $\longrightarrow$ $\epsilon_0 \nabla\cdot(\vec{E}_0 + \vec{E}_1) = e(n_{i0} + n_{i1} - n_{e0} - n_{e1}) = -\epsilon_0\nabla^2(\phi_0+\phi_1)$

$$\epsilon_0 \nabla\cdot\vec{E}_1 = e(n_{i1} - n_{e1}) = -\epsilon_0\nabla^2\phi_1 \Rightarrow k^2\epsilon_0\phi_1$$

Note that $n_e = n = n_0 \exp\left(\frac{e\phi}{kT_e}\right) = n_0 \exp\left[\frac{e(\phi_0+\phi_1)}{kT_e}\right]$

$$= n_0 \exp\left[\frac{e\phi_1}{kT_e}\right] \simeq n_0\left(1 + \frac{e\phi_1}{kT_e} + \cdots\right) \equiv n_{e0} + n_{e1}$$

$$n_{e1} = \frac{e\phi_1}{kT_e} n_0$$

$$e n_{i1} - e n_{e1} = e n_{i1} - \frac{e^2 n_0}{kT_e}\phi_1 = k^2\epsilon_0\phi_1$$

$$\Rightarrow e n_{i1} = \epsilon_0\phi_1\left(k^2 + \frac{n_0 e^2}{\epsilon_0 kT_e}\right) \qquad \lambda_D^2 \equiv \frac{\epsilon_0 kT_e}{ne^2}$$

$$= \epsilon_0\phi_1\left(k^2 + \frac{1}{\lambda_D^2}\right)$$

$$\Rightarrow \epsilon_0\phi_1(k^2\lambda_D^2 + 1) = e n_{i1}\lambda_D^2 \Rightarrow \phi_1 = \frac{e\lambda_D^2}{\epsilon_0(1+k^2\lambda_D^2)} n_{i1}$$

Note from continuity, $\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0 \Rightarrow n_{i1} = \frac{k}{\omega} n_0 v_{i1}$

$$\omega M \cancel{n_0} \cancel{v_{i1}} = k e n_0 \phi_1 + k\gamma_i kT_i n_{i1}$$

$$= k e n_0 \frac{e\lambda_D^2}{\epsilon_0(1+k^2\lambda_D^2)} n_{i1} + k\gamma_i kT_i n_{i1}$$

$$= k\left[\frac{e^2 n_0\lambda_D^2}{\epsilon_0(1+k^2\lambda_D^2)} + \gamma_i kT_i\right] n_{i1}$$

$$= \frac{k^2}{\omega}\left[\frac{n_0 e^2\epsilon_0^{-1}\lambda_D^2}{1+k^2\lambda_D^2} + \gamma_i kT_i\right] \cancel{n_0} \cancel{v_{i1}}$$

$$\omega^2 = \frac{k^2}{M}\left(\frac{n_0 e^2\epsilon_0^{-1}\lambda_D^2}{1+k^2\lambda_D^2} + \gamma_i kT_i\right) \qquad \because\ n e^2\epsilon_0^{-1}\lambda_D^2 \equiv kT_e$$

$$\frac{\omega}{k} = \left(\frac{kT_e}{M}\frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i kT_i}{M}\right)^{1/2}$$

$$\frac{\omega}{k} = \left(\frac{KT_e}{M} \boxed{\frac{1}{1+k^2\lambda_D^2}} + \frac{\gamma_i K T_i}{M} \right)^{1/2}$$

$$\Rightarrow \frac{\omega}{k} = \left(\frac{KT_e}{M} + \frac{\gamma_i K T_i}{M} \right)^{1/2}$$

An error of order $k^2\lambda_D^2 = \left(\frac{2\pi\lambda_D}{\lambda}\right)^2$

$\because$ λ_D is small.

$\therefore$ the approximation is valid for all

Except the shortest wavelength waves $(\lambda \simeq \lambda_D)$

§ 4.8 Comparison of ion and electron waves

ion accoustic wave:

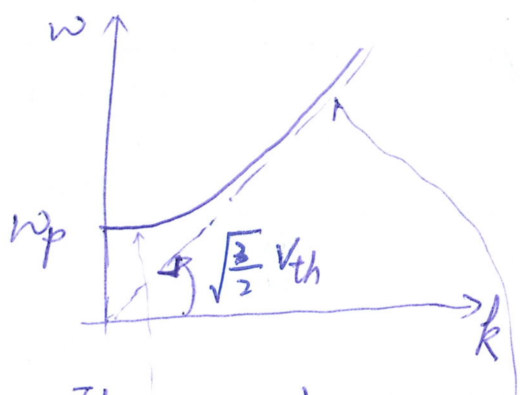
$$\omega^2 = \frac{k^2}{M}\left(\frac{n_0 e^2 \epsilon_0^{-1} \lambda_D^2}{1+k^2\lambda_D^2} + \gamma_i K T_i \right)$$

$$\xrightarrow[\text{for } k^2\lambda_D^2 \gg 1]{} \simeq \frac{k^2}{M}\left(\frac{n_0 e^2 \epsilon_0^{-1} \cancel{\lambda_D^2}}{k^2 \cancel{\lambda_D^2}} + \gamma_i K T_i \right)$$

$$\xrightarrow[T_i \to 0]{} \simeq \frac{k^2}{M}\frac{n_0 e^2}{\epsilon_0 k^2} = \frac{n_0 e^2}{\epsilon_0 M} \equiv \Omega_p^2$$

ion plasma freq.

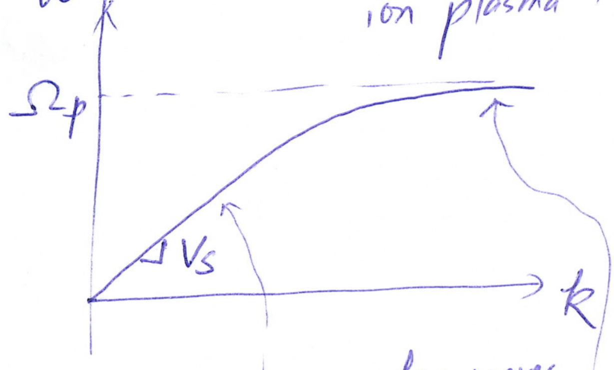
$\omega_p = \frac{n_0 e^2}{\epsilon_0 m}$ electron plasma freq.



Electron plasma wave

Const. freq. $\to$ Const. velocity

$k \uparrow$



ion acoustic waves

Const. velocity $\to$ Const. freq.

$k \uparrow$

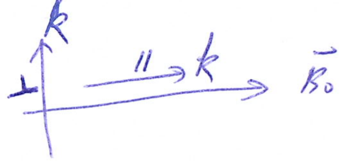
§ 4.9 Electrostatic electron oscillations perpendicular to B (⊥B)

$B \neq 0$,
simplest case: high freq., electrostatic, electron oscillations propagating $\perp \vec{B}$

Definition:

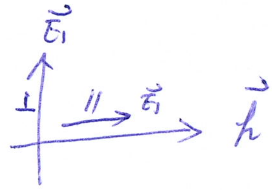
$\Rightarrow$ Parallel, $\rightarrow \vec{k} \parallel \vec{B}_0$

Perpendicular $\rightarrow \vec{k} \perp \vec{B}_0$



Longitudinal $\rightarrow \vec{k} \parallel \vec{E}_1$

Transverse $\rightarrow \vec{k} \perp \vec{E}_1$



Electrostatic $\rightarrow \vec{B}_1 = 0$

Electromagnetic $\rightarrow \vec{B}_1 = 0$

For Maxwell's Eq: $\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1$

$\Rightarrow \vec{k} \times \vec{E}_1 = -\dot{\vec{B}}_1$

$\Rightarrow$ Longitudinal : $\dot{\vec{B}}_1 = -\vec{k} \times \vec{E}_1 = 0$

Transversed : $\vec{B}_1 = -\vec{k} \times \vec{E}_1 \neq 0$

$\vec{B}_0 \neq 0$, electron oscillations $\perp \vec{B}_0$,
High-freq., electrostatic.

* Ions are heavy $\longrightarrow$ uniform background of positive charge.
* Neglect thermal motions $\rightarrow KT_e = 0$
* Equilibrium: Const. n_0, $\vec{B}_0$, $\vec{E}_0 = 0$, $\vec{V}_0 = 0$

p95

electron:

$$m\cancel{n_e}\left[\frac{\partial \vec{V}_e}{\partial t}+(\vec{V}_e\cdot\nabla)\vec{V}_e\right] = -e\cancel{n_e}(\vec{E}+\vec{v}\times\vec{B}) - \cancel{\nabla p} \quad \because kT_e = 0$$

$$\Rightarrow \; m\left[\frac{\partial \vec{V}_e}{\partial t}+(\vec{V}_e\cdot\nabla)\vec{V}_e\right] = -e(\vec{E}+\vec{v}\times\vec{B})$$

$$\frac{\partial n_e}{\partial t}+\nabla\cdot(n_e\vec{V}_e)=0$$

$$\epsilon_0\nabla\cdot\vec{E} = +e(n_i-n_e)$$

$$m\left[\frac{\partial(\cancel{\vec{V}_{e0}}+\vec{V}_{e1})}{\partial t} + \underbrace{\cancel{(\vec{V}_{e0}+\vec{V}_{e1})\cdot\nabla(\vec{V}_{e0}+\vec{V}_{e1})}}_{\text{2nd order}}\right]$$

$$= -e\left[(\cancel{\vec{E}_0}+\vec{E}_1)+(\cancel{\vec{V}_{e0}}+\vec{V}_{e1})\times(\vec{B}_0+\underbrace{\cancel{\vec{B}_1}}_{\text{2nd order}})\right]$$

$$\Rightarrow \; m\frac{\partial \vec{V}_{e1}}{\partial t} = -e(\vec{E}_1+\vec{V}_{e1}\times\vec{B}_0)$$

$$\frac{\partial(\cancel{n_{e0}}+n_{e1})}{\partial t}+\nabla\cdot\left[(n_{e0}+\underbrace{\cancel{n_{e1}}}_{\text{2nd order}})(\cancel{\vec{V}_{e0}}+\vec{V}_{e1})\right]=0$$

$$\frac{\partial n_{e1}}{\partial t}+n_0\nabla\cdot\vec{V}_{e1}=0$$

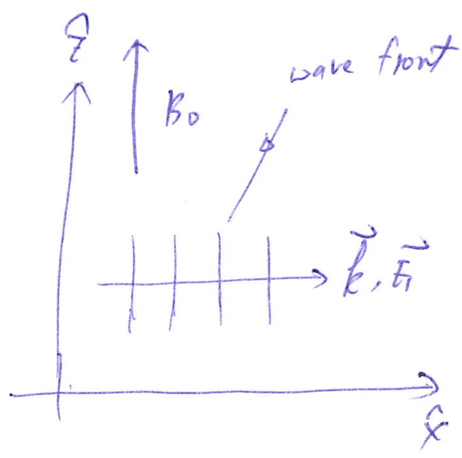
$$\epsilon_0\nabla\cdot(\cancel{\vec{E}_0}+\vec{E}_1)=+e(\cancel{n_{i0}-n_{e0}}-n_{e1})$$

$$\epsilon_0\nabla\cdot\vec{E}_1=-e\,n_{e1}$$

Longitudinal waves: $\vec{k}\parallel\vec{E}_1$

Choose $\hat{x}\parallel\vec{k}\parallel\vec{E}_1$.

$\hat{z}\parallel\vec{B}_0$



Neglect thermal motion: $KT_e = 0$

$$n m_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \nabla)\vec{v}_e\right] = -ne\left[\vec{E} + \vec{v}\times\vec{B}\right] - \cancel{\nabla p}$$

$\to$ linearized $\quad m_e \dfrac{\partial \vec{v}_1}{\partial t} = -e\vec{E}_1 - e\vec{v}_1 \times \vec{B}_0$

Continuity: $\dfrac{\partial n_{e1}}{\partial t} + n_0 \nabla\cdot\vec{v}_{e1} = 0$

Gaussian: $\epsilon_0 \nabla\cdot\vec{E}_1 = -e n_{e1}$

$$\Rightarrow \quad \begin{aligned} -i\omega m_e v_x &= -eE_1 - e v_y B_0 \\ -i\omega m_e v_y &= e v_x B_0 \quad \to \quad v_y = i\frac{eB_0}{\omega m_e} v_x \\ -i\omega m v_z &= 0 \end{aligned}$$

$$i\omega m_e v_x = +eE_1 + i\frac{e^2 B_0^2}{\omega m_e} v_x$$

$$v_x = \frac{eE/im\omega}{1 - \omega_c^2/\omega^2} \longrightarrow \infty \quad @\ \omega = \omega_c$$

$\because$ The electric field changes w/ v_x and continuously accelerates the electrons.

$$n_1 = \frac{k}{\omega} n_0 v_x$$

$$ik\epsilon_0 E_1 = -e\cdot\frac{k}{\omega} n_0 v_x = -e\frac{k}{\omega} n_0 \frac{eE_1/im\omega}{1-\omega_c^2/\omega^2}$$

$$\left(1 - \frac{\omega_c^2}{\omega^2}\right)\cancel{ik}\epsilon_0 \cancel{E_1} = \cancel{i}\frac{e^2 \cancel{E_1}}{m\omega}\frac{\cancel{k}}{\omega} n_0$$

$$\epsilon_0\left(1 - \frac{\omega_c^2}{\omega^2}\right) = \frac{e^2 n_0}{m}\cdot\frac{1}{\omega^2} = \frac{\omega_p^2}{\omega^2}$$

$$\omega^2 - \omega_c^2 = \omega_p^2 \Rightarrow \underline{\omega^2 = \omega_p^2 + \omega_c^2 = \omega_h^2} \quad \#$$

upper hybrid freq.

Electrostatic electron waves across $\vec{B}$.

The group velocity is zero as long as thermal motions are neglected.

$\vec{k} = k\hat{x}, \quad \vec{E} = E\hat{x}, \quad \vec{B}_0 = B_0\hat{z}$

$k_y = k_z = E_y = E_z = 0$

~~drop e1~~ drop the subscript e1.

$\frac{\partial}{\partial t} \to -i\omega, \quad \nabla \to i\vec{k}\cdot\hat{x}$

$$m\frac{d\vec{V}_{e1}}{dt} = -e(\vec{E}_1 + \vec{V}_{e1}\times\vec{B}) \Rightarrow -i\omega m\vec{V} = -e(\vec{E} + \vec{v}\times\vec{B}_0)$$

$$\Rightarrow \begin{cases} -i\omega m V_x = -eE - eV_yB_0 \\ -i\omega m V_y = eV_xB_0 \\ -i\omega m V_z = 0 \end{cases} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & V_z \\ 0 & 0 & B_0 \end{vmatrix}$$

$$V_y = \frac{eB_0}{-i\omega m}V_x$$

$$+i\omega m V_x = -eE + eB_0\frac{eB_0}{+i\omega m}V_x$$

$$= +eE + i\frac{e^2B_0^2}{\omega m}V_x$$

$$\Rightarrow iV_x\left(\omega m - \frac{e^2B_0^2}{\omega m}\right) = eE \qquad \frac{e^2B_0^2}{m^2} = \omega_c^2$$

$$i\omega m V_x\left(1 - \frac{e^2B_0^2}{m^2\omega^2}\right) = eE$$

$$V_x = \frac{eE/i\omega m}{1 - \omega_c^2/\omega^2} \longrightarrow \infty$$

for $\omega \to \omega_p$, ∵ the electric field changes sign w/ V_x and continuously accelerates the electrons

$$\frac{\partial n_{e1}}{\partial t} + n_0\nabla\cdot\vec{V}_{e1} = 0 \Rightarrow -i\omega n_1 + n_0 i\vec{k}\cdot\vec{V} = 0$$

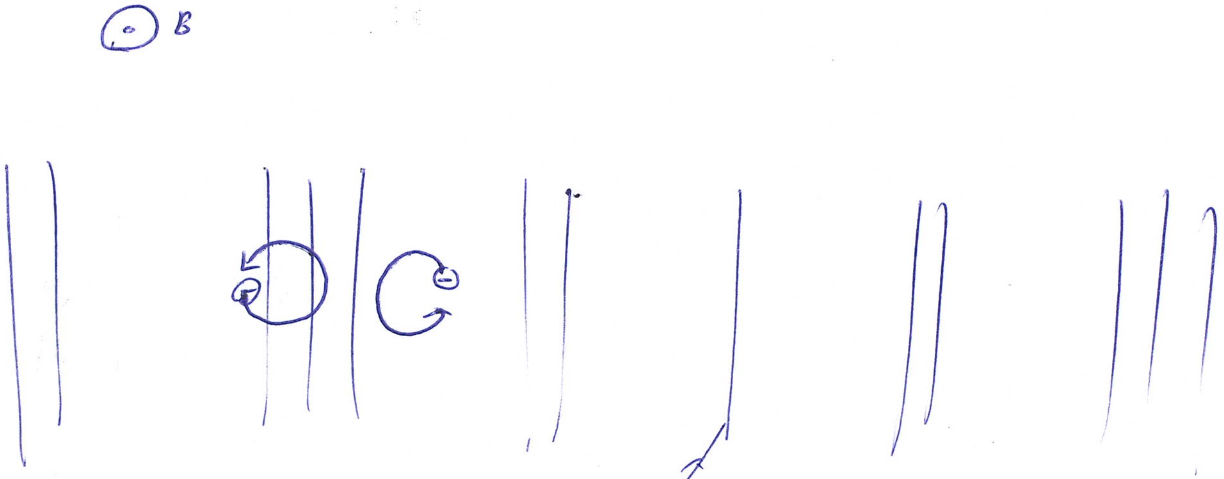
$$n_1 = \frac{k}{\omega}n_0V_x$$

$$\epsilon_0\nabla\cdot\vec{E}_1 = -en_{e1} \Rightarrow \epsilon_0 i\vec{k}\cdot\vec{E} = -en_1$$

$$\Rightarrow ik\epsilon_0 E = -en_1 = -e\frac{k}{\omega}n_0V_x$$

$$= -e\frac{k}{\omega}n_0\frac{eE/i\omega m}{1-\omega_c^2/\omega^2}$$

$$\Rightarrow \cancel{ik}\,\epsilon_0\left(1 - \frac{\omega_c^2}{\omega^2}\right)\cancel{E} = \cancel{i}\,\frac{n_0e^2\cancel{k}}{\omega^2 m}\cancel{E}$$

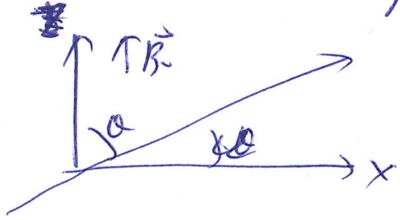


Two restoring forces:

- Electrostatic force
- Lorentz force

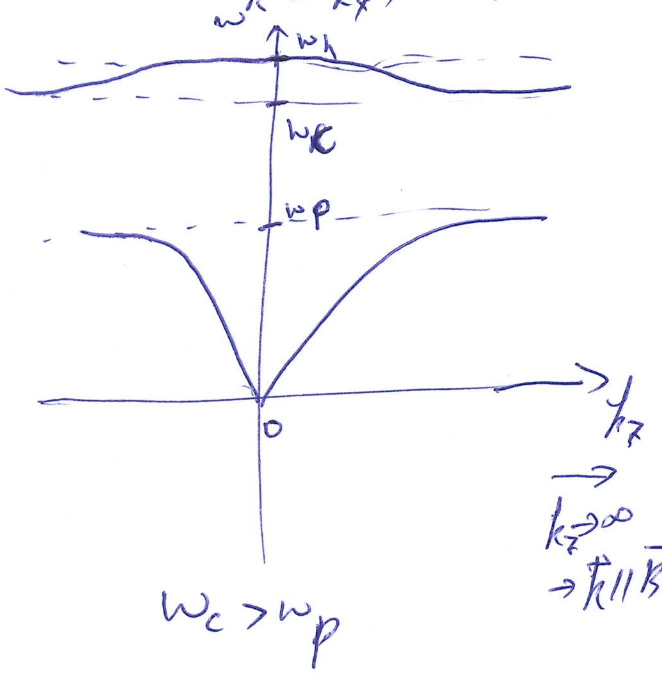
increased resorting force → $\omega \uparrow$. regards to No field case (ω_p)

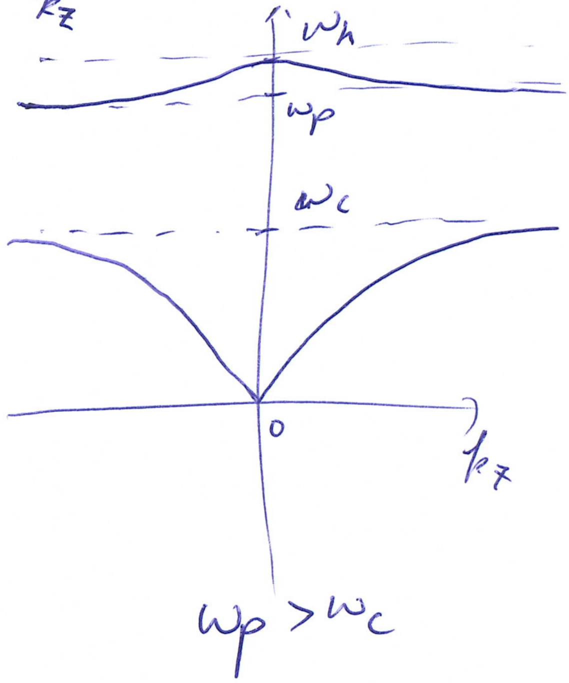
If:



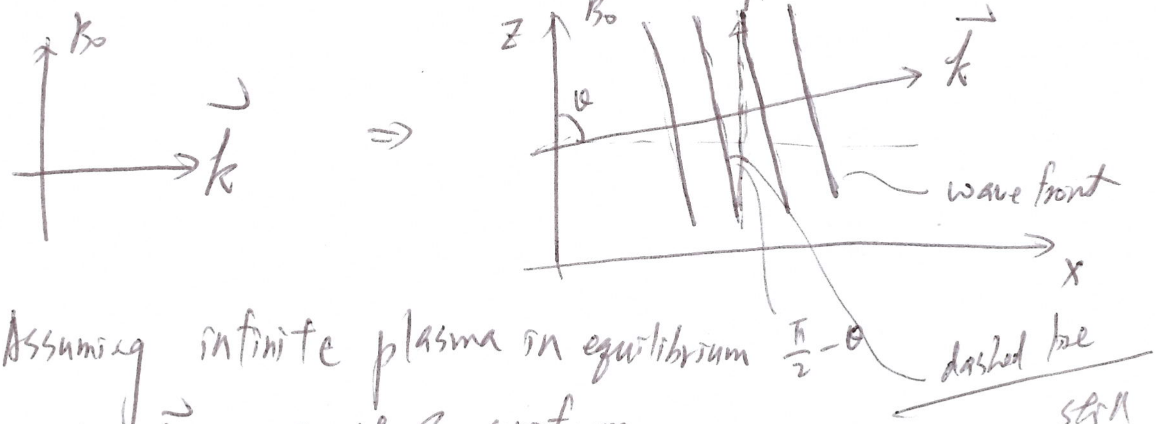
→ two possible waves — ① plasma oscillation
② modified upper hybrid oscillation.

$\vec{k} = k_x \hat{x} + k_z \hat{z}$ $\tan\theta = \frac{k_x}{k_z}$





§ 4.10 Electrostatic ion waves perpendicular to B p94 ②



* Assuming infinite plasma in equilibrium

n_0, $\vec{B}_0$: const & uniform

$\vec{v}_0 = \vec{E}_0 = 0$, $T_i = 0$ → ∵ acoustic wave still exists if $T_i = 0$

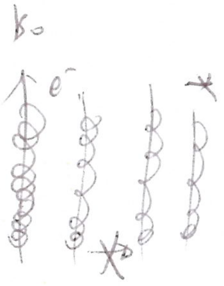
→ Electrostatic wave : $\vec{k} \times \vec{E} = 0$

$\vec{E} = -\nabla\phi$

∵ $\frac{1}{2}\pi - \theta$ is very small

∴ set $\vec{E} = \vec{E}_1 \hat{x}$,

$\nabla \to ik\hat{x}$ as far as the ion motion is concerned



* Larmor radii of electrons are very small

→ they cannot move in the x-direction to preserve charge neutrality

→ what $\vec{E}$ field does is to make electrons drift back and forth in $\hat{y}$ ($\vec{E} \times \vec{B}$ drift).

* If $\theta \neq \pi/2$, the electron can move along the dashed line (along $\vec{B}_0$) to carry charge from negative to positive region in the wave and carry out Debye length.

* ~~$\frac{\pi}{2}$ ... θ~~

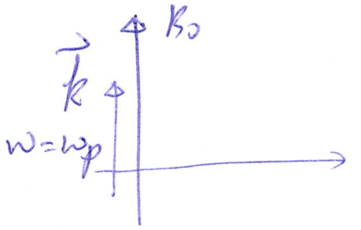
∵ $M_i >> m_e$, the path of the dashed line is too long for ions

∴ $k_z \to 0$ neglected.

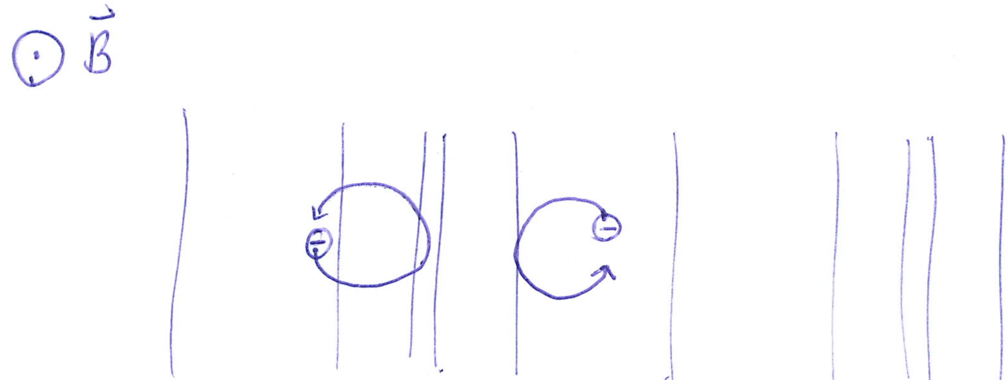
$$1-\frac{\omega_c^2}{\omega^2} = \underbrace{\frac{n_0 e^2}{\epsilon_0 m}} \cdot \frac{1}{\omega^2} \qquad \omega_p^2 \equiv \frac{n_0 e^2}{\epsilon_0 m}$$

$$1-\frac{\omega_c^2}{\omega^2} = \frac{\omega_p^2}{\omega^2}$$

$\Rightarrow$ $\underline{\omega^2 = \omega_p^2 + \omega_c^2 \equiv \omega_h^2}$ $(>\omega_p^2)$ upper hybrid freq.



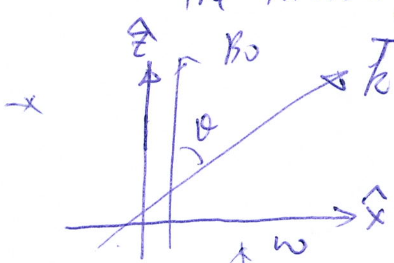
$\vec{k}$, $\omega=\omega_h$, $V_g \to 0$ w/ thermal motions



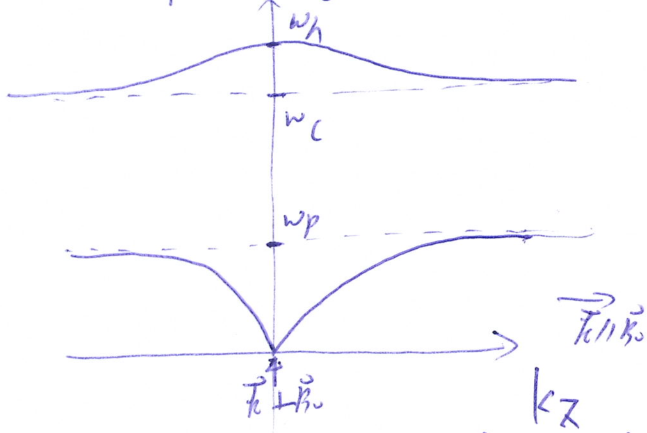
planes of const. density

Restoring force : electrostatic field + Lorentz force

The increased restoring force makes the freq. $\omega_h > \omega_p$

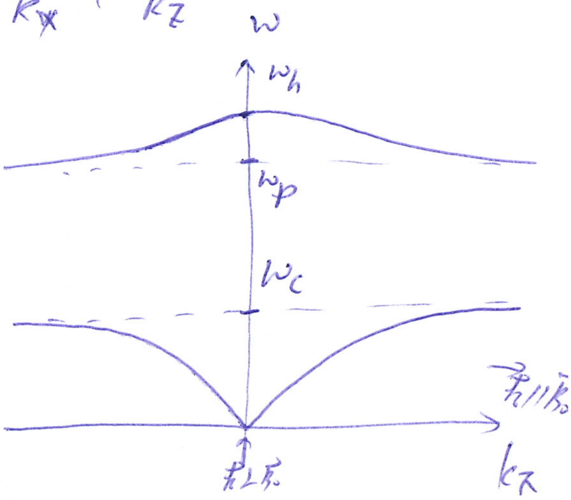


$\Rightarrow$ two possible waves
$\vec{k} = \vec{k}_x + \vec{k}_z$



$\omega_c > \omega_p$

$k_z \uparrow$: parallel to $\vec{B}_0$



$$\chi = \frac{\pi}{2} - \theta, \propto \frac{|V_i|}{|V_e|}$$

$$\simeq \sqrt{\frac{m}{M}} \quad \begin{array}{l} \leftarrow \text{for } e^- \\ \leftarrow \text{for ion} \end{array}$$

For $\chi > \sqrt{\frac{m}{M}}$: ($\chi < \sqrt{\frac{m}{M}}$ in next section)

ION eq. of motion:

$$\cancel{M \frac{\partial \vec{V}_{i1}}{\partial t}} \quad Mn\left[\frac{\partial \vec{V}}{\partial t} + \underbrace{(\vec{V}\cdot\nabla)\vec{V}}_{2^{nd}\text{ order}}\right] = nq(\vec{E} + \vec{v}\times\vec{B}) - \underbrace{\nabla P}_{T_i=0}$$

$$\Rightarrow M \frac{\partial \vec{V}_{i1}}{\partial t} = -e\nabla\phi_1 + e\vec{V}_{i1}\times\vec{B}_0$$

$$; \frac{\partial}{\partial t} \to -i\omega$$

$$\Rightarrow \begin{cases} -i\omega M V_{ix} = -eik\phi_1 + eV_{iy}B_0 \\ +i\omega M V_{iy} = +eV_{ix}B_0 \end{cases}$$

$$V_{ix} = \frac{1}{-i\omega M}\left[-iek\phi_1 + eV_{iy}B_0\right]$$

$$= \frac{ek}{M\omega}\left[\phi_1 + i\frac{B_0}{k}V_{iy}\right]$$

$$= \frac{ek}{M\omega}\left[\phi_1 + \cancel{i}\frac{B_0}{k}\frac{eB_0}{\cancel{i}\omega M}V_{ix}\right]$$

$$= \frac{ek}{M\omega}\phi_1 + \frac{e^2B_0^2}{M^2\omega^2}V_{ix}$$

$$\Rightarrow V_{ix} = \frac{ek}{M\omega}\phi_1 \frac{1}{1 - \frac{e^2B_0^2}{M^2\omega^2}}$$

$$\Omega_i \equiv \frac{eB_0}{M}$$

ion cyclotron freq.

$$= \frac{ek}{M\omega}\phi_1\left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1}$$

$$(\underbrace{n_{i0}}_{n_0} + \cancel{n_{i1}})(\underbrace{\cancel{\vec{V}_{i0}}}_{0} + \vec{V}_{i1}) \quad 2^{nd}$$

Continuity: $\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{V}_i) = 0$

$$\frac{\partial n_{i1}}{\partial t} + i\vec{k}\cdot n_0\vec{V}_{ix} = 0 \Rightarrow -i\omega n_{i1} + ik\, n_0 V_{ix} = 0$$

$$\Rightarrow n_{i1} = n_0\frac{k}{\omega}V_{ix}$$

Assuming e^- can move along $\vec{B}_0$ because of the finiteness of χ

$\Rightarrow$ Boltzmann relation for e^-:

$$n_e = n_0 \exp\left(\frac{e\phi_1}{kT_e}\right) \simeq n_0\left(1 + \frac{e\phi_1}{kT_e} + \cdots\right)$$
$$= n_0 + n_{e1}$$

$$\Rightarrow \frac{n_{e1}}{n_0} = \frac{e\phi_1}{kT_e} \quad \Rightarrow e\phi_1 = \frac{n_{e1}}{n_0}\cdot kT_e$$

* Plasma approximation (low freq.)

$n_i = n_e$

$$\Rightarrow \cancel{v}_{ix} = \frac{ek}{M\omega}\phi_1\left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1}$$
$$= \frac{k}{M\omega}\cdot\frac{n_{e1}}{n_0}kT_e\left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1}$$
$$= \frac{k}{M\omega}\frac{n_{i1}}{n_0}kT_e\left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1}$$
$$= \frac{k}{M\omega}\cdot\frac{k}{\omega}\cancel{v}_{ix}\,kT_e\left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1}$$

$$\Rightarrow \left(1 - \frac{\Omega_i^2}{\omega^2}\right) = \frac{k^2}{\omega^2}\frac{kT_e}{M}$$

$$\frac{\omega}{k} = \left(\frac{kT_e + \gamma_i kT_i}{M}\right)^{1/2} = v_s \text{ for ion wave.}$$

$$\cancel{\omega^2}\left(\omega^2 - \Omega_i^2\right) = k^2\frac{kT_e}{M}$$

$$\Rightarrow \underline{\omega^2 = \Omega_i^2 + k^2 v_s^2} \quad (\because T_i = 0)$$

dispersion relation for electrostatic ion cyclotron waves

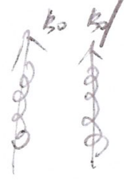
* The ion undergo an acoustic-type oscillation, but the Lorentz force constitutes a new restoring force giving rise to the Ω_i^2 term.

* $\omega^2 = k^2 v_s^2$ is valid if e^- provide Debye shielding, they do so by following long distances along $\vec{B}_0$.

Ⅶ 4.11 The lower hybrid freq.

$\theta = \pi/2$.

$\Rightarrow$ e^- are not allowed to preserve charge neutrality by following along the lines of force



$\Rightarrow$ e^- obey the full eq. of motion

NOT Boltzmann's relation.

$$m \cancel{n_e}\left[\frac{\partial \vec{v}_e}{\partial t} + \underbrace{\cancel{(\vec{v}_e\cdot\nabla)\vec{v}_e}}_{2^{nd}\text{ order}}\right] = q\cancel{n_e}\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \cancel{\nabla p}$$

$T_e = 0$ for simplicity

previously, for ion: $v_{ix} = \frac{ek}{M\omega}\phi_1\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$

$\Rightarrow$ $M \to m$, $e \to -e$, $\Omega_c \to \omega_c$.

$\Rightarrow$ $v_{ex} = -\frac{ek}{m\omega}\phi_1\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$

Continuity: $n_{i1} = n_0 \frac{k}{\omega} v_{i1}$

$\Rightarrow$ $n_{e1} = n_0 \frac{k}{\omega} v_{e1}$

plasma approximation (low freq.)

$n_0 \frac{k}{\omega} v_{i1} = n_{i1} = n_{e1} = n_0 \frac{k}{\omega} v_{e1}$

$\Rightarrow$ $\underline{v_{i1} = v_{e1}}$

$$\cancel{\frac{ek}{M\omega}}\cancel{\phi_1}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} = -\frac{\cancel{ek}}{m\cancel{\omega}}\cancel{\phi_1}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$-m\left(1 - \frac{\omega_c^2}{\omega^2}\right) = M\left(1 - \frac{\Omega_c^2}{\omega^2}\right)$$

$\because$ $\omega_c^2 = \frac{e^2B^2}{m^2}$, $\Omega_c^2 = \frac{e^2B^2}{M^2}$

$$-m\omega^2 + m\omega_c^2 = M\omega^2 - M\Omega_c^2$$

$$\omega^2(M+m) = m\omega_c^2 + M\Omega_c^2 = e^2B^2\left(\frac{1}{m} + \frac{1}{M}\right)$$

$$\omega^2 = \frac{e^2B^2}{\cancel{M+m}}\cdot\frac{\cancel{M+m}}{M\cdot m} = \frac{e^2B^2}{M\cdot m} = \Omega_c\cdot\omega_c$$

p10

$$\omega = \sqrt{\Omega_c \cdot \omega_c} \equiv \omega_\ell$$

Lower hybrid freq. *

* If poisson's eq is used, NOT plasma approximat.

$$\frac{1}{\omega_\ell^2} = \frac{1}{\omega_c \Omega_c} + \frac{1}{\Omega_p^2}$$

*

* In low-density plasma, $\frac{1}{\Omega_p^2}$ dominates

The plasma approximation is not valid at such high freq.

* Lower hybrid oscillation can be observed only if $\theta \approx \pi/2$

*

§ 4.12 Electromagnetic waves w/ $B_0 = 0$

- Transverse electromagnetic waves traveling through a plasma
- Brief review of light waves in a vacuum
- Relevant Maxwell's eqs:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \nabla \times \vec{E}_1 = -\dot{\vec{B}}_1$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \longrightarrow c^2 \nabla \times \vec{B}_1 = \dot{\vec{E}}_1$$

source free, $\vec{J} = 0$ (vacuum)

$$c^2 = \frac{1}{\epsilon_0 \mu_0}$$

$$c^2 \nabla \times (\nabla \times \vec{B}_1) = \nabla \times \dot{\vec{E}}_1 = \frac{\partial}{\partial t}(\nabla \times \vec{E}_1) = -\ddot{\vec{B}}_1$$

$\partial_t \to -i\omega$

$\nabla \to ik$

$$-(-i\omega)^2 \vec{B}_1 = c^2 i\vec{k} \times (i\vec{k} \times \vec{B}_1)$$

$$\omega^2 \vec{B}_1 = -c^2 \vec{k} \times (\vec{k} \times \vec{B}_1) = -c^2[\vec{k}(\cancel{\vec{k} \cdot \vec{B}_1}) - k^2 \vec{B}_1]$$

$$\because \nabla \cdot \vec{B} = 0 \rightarrow \nabla \cdot \vec{B}_1 = 0$$

$$\Rightarrow i\vec{k} \cdot \vec{B}_1 = 0$$

$$\Rightarrow \omega^2 \vec{B}_1 = k^2 c^2 \vec{B}_1$$

$$\Rightarrow \underline{\omega^2 = k^2 c^2} \Rightarrow \frac{\omega}{k} = c \equiv v_\phi \leftarrow \text{phase velocity}$$

- In a plasma where $\vec{B}_0 = 0$.

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \nabla \times \vec{E}_1 = -\dot{\vec{B}}_1$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \rightarrow c^2 \nabla \times \vec{B}_1 = \frac{\vec{J}_1}{\epsilon_0} + \dot{\vec{E}}_1$$

∂_t

$$c^2 \nabla \times \dot{\vec{B}}_1 = \frac{\dot{\vec{J}}_1}{\epsilon_0} + \ddot{\vec{E}}_1$$

$\nabla \times$

$$\nabla \times (\nabla \times \vec{E}_1) = -\nabla \times (\dot{\vec{B}}_1)$$

$$\parallel$$

$$\nabla(\nabla \cdot \vec{E}_1) - \nabla^2 \vec{E}_1$$

$$\nabla(\nabla\cdot\vec{E}_1) - \nabla^2\vec{E}_1 = -\nabla\times\dot{\vec{B}}_1$$
$$= -\frac{1}{c^2}\left(\frac{\dot{\vec{J}}_1}{\epsilon_0} + \ddot{\vec{E}}_1\right)$$

$\nabla \to ik$, $\partial_t \to -i\omega$

$$i\vec{k}(i\vec{k}\cdot\vec{E}_1) - (ik)^2\vec{E}_1 = -\frac{-i\omega}{\epsilon_0 c^2} J_1 - \frac{1}{c^2}(-i\omega)^2\vec{E}_1$$

$$\Rightarrow -\vec{k}(\vec{k}\cdot\vec{E}_1) + k^2\vec{E}_1 = \frac{i\omega}{\epsilon_0 c^2}\vec{J}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

Transverse waves: $\vec{k}\cdot\vec{E}_1 = 0$

$$\Rightarrow \underline{(\omega^2 - k^2c^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}\vec{J}_1}$$

For high freq. $\to$ ions are considered fixed.

$$\vec{J}_1 = -e\, n_0 \vec{v}_{e1}$$

Eq. of momentum w/ $kT_e = 0$

$$nm\left[\frac{\partial\vec{v}_e}{\partial t} + \underbrace{(\vec{v}_e\cdot\nabla)\vec{v}_e}_{2^{nd}\text{ order}}\right] = n q(\vec{E} + \overbrace{\vec{v}\times\vec{B}}^{2^{nd}\text{ order}})$$

$$\Rightarrow \frac{\partial\vec{v}_{e1}}{\partial t} = q\vec{E}_1 \Rightarrow +i\omega m\vec{v}_{e1} = +e\vec{E}_1$$
$$\underline{\vec{v}_{e1} = \frac{e\vec{E}_1}{i\omega m}}$$

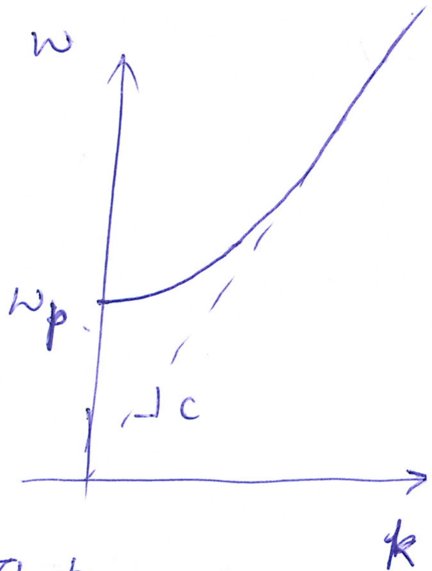
$$(\omega^2 - k^2c^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}(-e n_0 \vec{v}_{e1})$$
$$= \frac{\cancel{i\omega}}{\epsilon_0} e n_0 \frac{e\vec{E}_1}{\cancel{i\omega} m} = \underbrace{\frac{n_0 e^2}{\epsilon_0 m}}_{\omega_p^2}\vec{E}_1$$

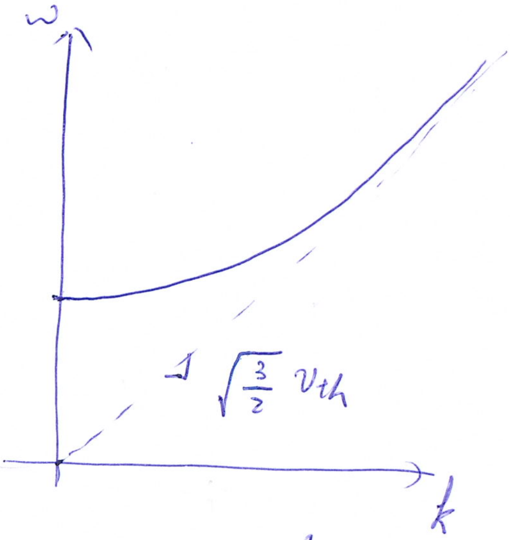
$$\Rightarrow \underline{\omega^2 = \omega_p^2 + k^2c^2}$$ dispersion relation for electromagnetic waves. w/ $B_0 = 0$

$$v_\phi^2 = \frac{\omega^2}{k^2} = c^2 + \frac{\omega_p^2}{k^2} > c^2$$

$$v_g = \frac{d\omega}{dk}: \quad \cancel{2}\omega \cdot \frac{d\omega}{dk} = \cancel{2} c^2 k$$

$$\Rightarrow \frac{d\omega}{dk} = c^2 \frac{k}{\omega} = \frac{c^2}{v_\phi} < c$$





Electromagnetic waves.
→ ordinary light waves @ large kc and are not damped by the plasma.

electron plasma waves
(from pressure gradient, electrostatic wave)
- highly damped.

Cutoff: For densities high enough such that k is no longer a real number, the wave cannot propagate.
The cutoff condition occurs at a critical density n_c such that $\omega = \omega_p$

$$n_c = \frac{m\epsilon_0\omega^2}{e^2}$$

For $n > n_c$

$$kc = [\omega^2 - \omega_p^2]^{1/2} = i[\omega_p^2 - \omega^2]^{1/2}$$

$\because$ wave: $\exp(ikx - i\omega t)$

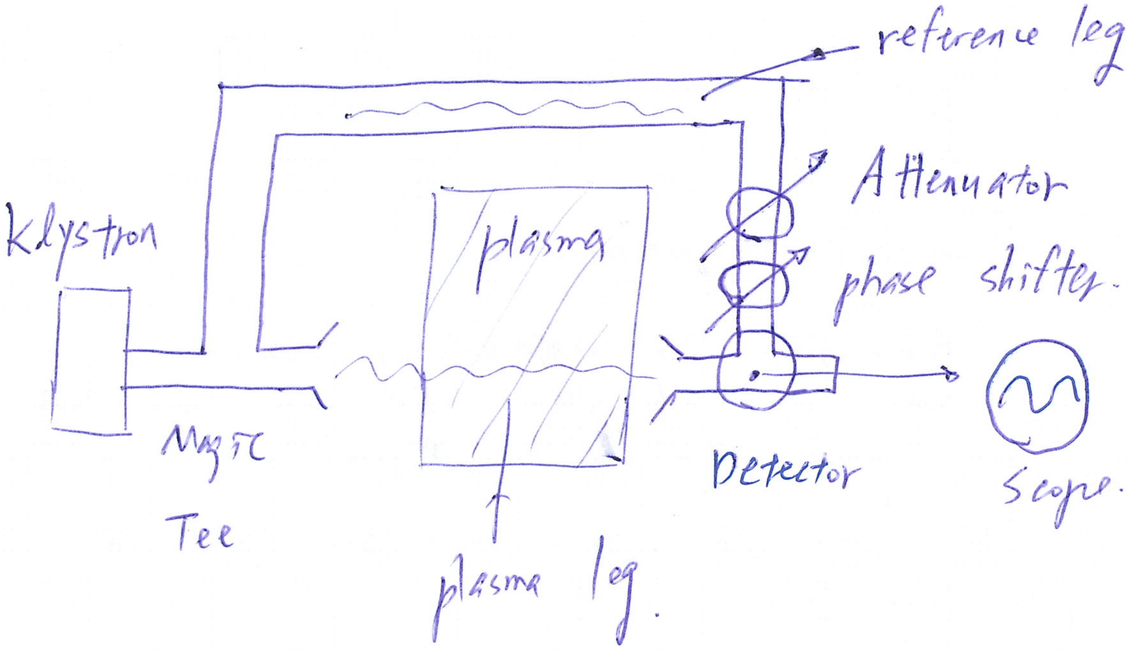
$$e^{ikx} = e^{-|k|x} = e^{-\frac{x}{\delta}}.$$

where $\delta \equiv |k|^{-1} = \dfrac{c}{(\omega_p^2 - \omega^2)^{1/2}}$ → skin depth.

For most laboratory plasma, the cutoff freq. lies in the range of microwave range.

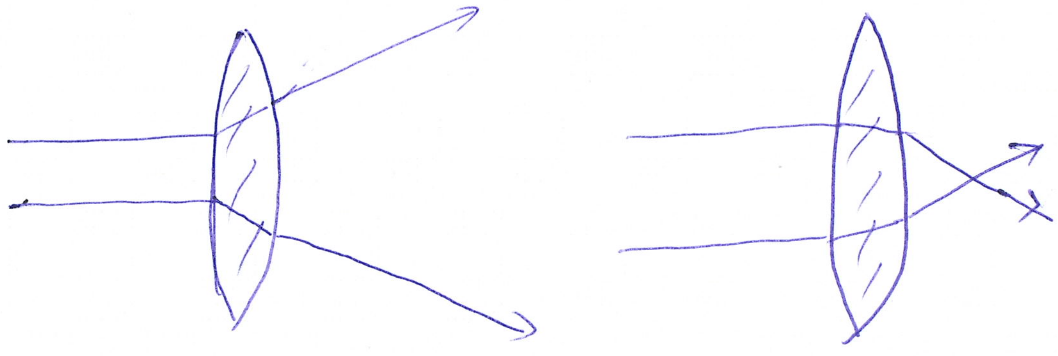
* Application: density measurement relies on the dispersion relation, or variation of index of refraction.

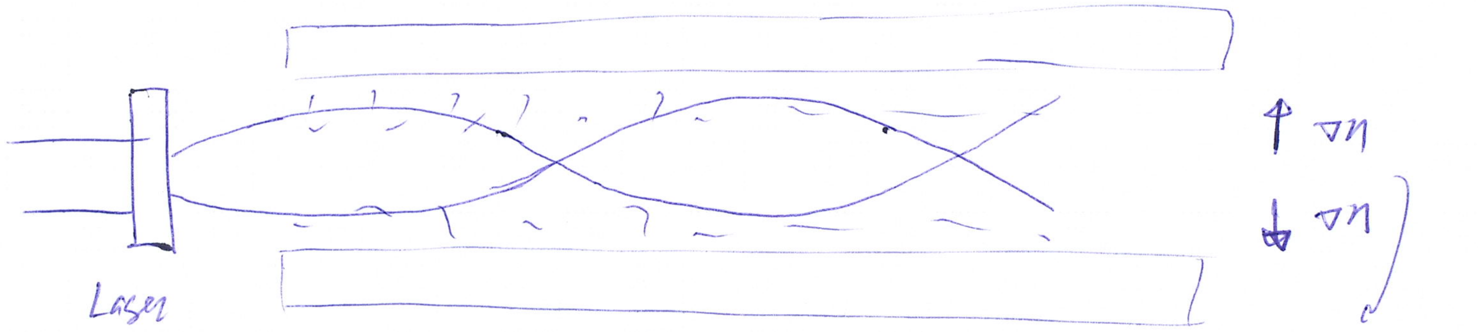
$$\hat{n} = \frac{c}{v_\phi} = \frac{ck}{\omega} < 1$$



$$\tilde{n} = \frac{c}{v_\phi} < 1$$

*

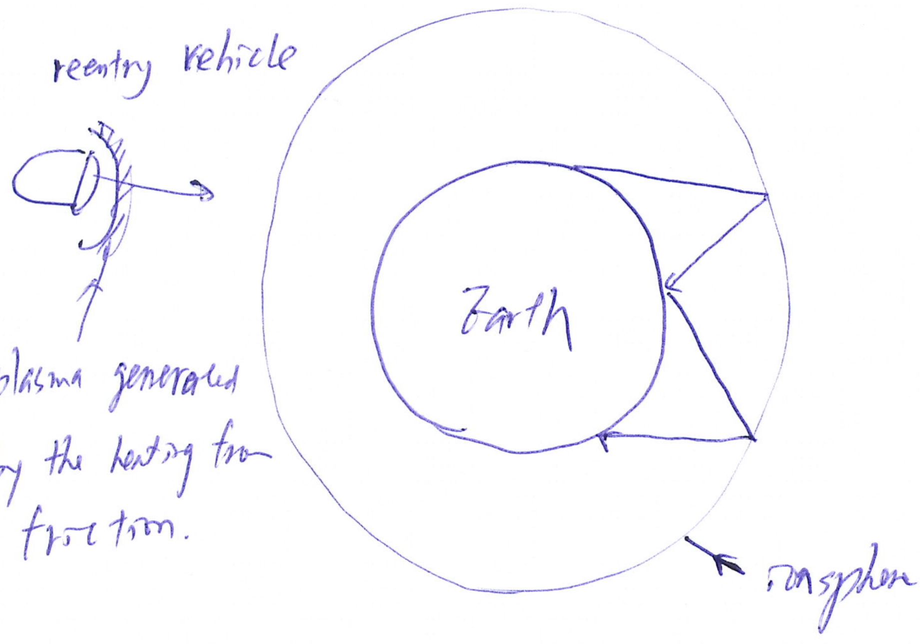




① squeezing the plasma with a pulsed coil surrounding it

② laser beam $\rightarrow$ ponderomotive force

*

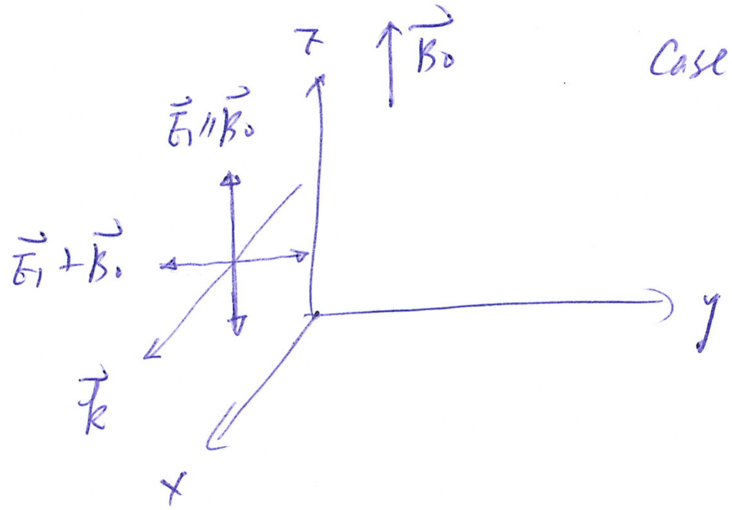


§ 4.14 Electromagnetic waves perpendicular to $\vec{B_0}$ P106

$\vec{B_0} \neq 0 \rightarrow \vec{k} \perp \vec{B_0}$. transverse wave: $\vec{k} \perp \vec{E_1}$

Case 1: $\vec{E_1} \parallel \vec{B_0}$

Case 2: $\vec{E_1} \perp \vec{B_0}$



§ 4.14.1 Ordinary wave (O-wave) $\vec{E_1} \parallel \vec{B_0}$

$\vec{B_0} = B_0 \hat{z}$, $\vec{E_1} = E_1 \hat{z}$, $\vec{k} = k\hat{x}$.

$(\omega^2 - c^2k^2)\vec{E_1} = -\frac{i\omega}{\epsilon_0} \vec{J_1} = \frac{i\omega}{\epsilon_0} n_0 e \vec{V_{e1}}$

$\because \vec{E_1} = E_1 \hat{z}$ $\therefore \vec{V_{e1}} = V_{ez} \hat{z}$

$m \frac{\partial \vec{V_{e1}}}{\partial t} = -e\vec{E_1}$

$\because \vec{V} \times \vec{B} = \vec{V_z} \times \vec{B_z} = 0$

The result is the same for $\underline{B_0 = 0}$

$$\underline{\omega^2 = \omega_p^2 + c^2k^2}$$

Ordinary wave. → the one that is not affected by the magnetic field.

§ 4.14.2 Extraordinary wave, $\vec{E_1} \perp \vec{B_0}$

* The electron motion will be affected by $\vec{B_0}$

$\because \vec{V} \times \vec{B} = (\vec{V_x} + \vec{V_y}) \times \vec{B_z} \neq 0$

$\vec{k} = k\hat{x} \Rightarrow$ ~~$E_1 = E_1 \hat{y}$~~

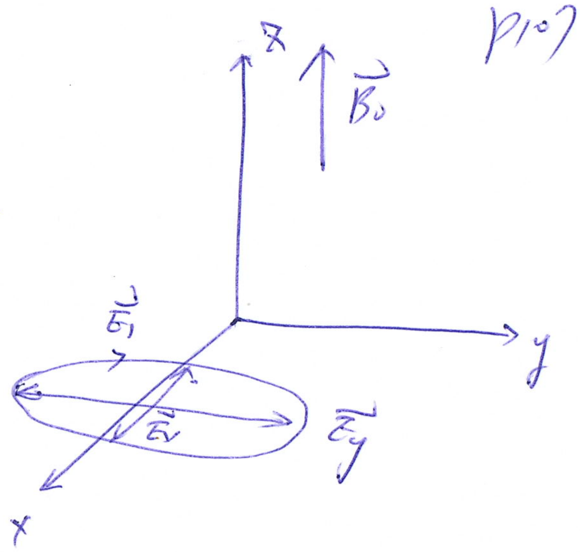
$\vec{E_1} \perp \vec{B_0} \to$ elliptically polarized,

NOT plane polarized.

a component E_x along $\vec{k}$

$\to$ partially longitudinal &

— — — — transverse.



$$\vec{E_1} = E_x \hat{x} + E_y \hat{y}$$

Momentum eq. w/ $kT_e \to 0$:

$$\cancel{n} \cdot m \left[\frac{\partial \vec{V_e}}{\partial t} + \cancel{(\vec{V_e}\cdot\nabla)\vec{V_e}}\right] = -e\cancel{n}\left(\vec{E} + \vec{V}\times\vec{B}\right)$$

2nd order

$$\to m\frac{\partial \vec{V_{e1}}}{\partial t} = -e\left(\vec{E_1} + \vec{V_{e1}}\times\vec{B_0}\right)$$

$\partial_t \to -i\omega$, $\Rightarrow$ $+i\omega m \vec{V_{e1}} = +e\left(\vec{E_1} + \vec{V_{e1}}\times\vec{B_0}\right)$

$$\vec{V_{e1}} = -i\frac{e}{\omega m}\left(\vec{E_1} + \vec{V_{e1}}\times\vec{B_0}\right)$$

$$\Rightarrow \begin{cases} V_x = -\frac{ie}{m\omega}\left(E_x + v_y B_0\right) \\ V_y = -\frac{ie}{m\omega}\left(E_y - v_x B_0\right) \end{cases} \qquad \vec{V_{e1}} = v_x\hat{x} + v_y\hat{y}$$

$$V_x = -\frac{ie}{m\omega}\left[E_x + B_0 \cdot \frac{-ie}{m\omega}\left(E_y - v_x B_0\right)\right]$$

$$= -\frac{ie}{m\omega}\left[E_x - i\frac{eB_0}{m\omega}E_y + i\frac{eB_0^2}{m\omega}V_x\right] \qquad \omega_c \equiv \frac{eB_0}{m}$$

$$= -\frac{ie}{m\omega}\left[E_x - i\frac{eB_0}{m\omega}E_y\right] + \frac{e^2B_0^2}{m^2\omega^2}V_x$$

$$\left(1 - \frac{\omega_c^2}{\omega^2}\right)V_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right) \cancel{\Rightarrow V_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)}$$

$$V_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$V_y = -\frac{ie}{m\omega}\left[E_y + B_0\frac{+ie}{m\omega}\left(E_x + V_y B_0\right)\right]$$

$$= -\frac{ie}{m\omega}\left[E_y + i\frac{eB_0}{m\omega}E_x + i\frac{eB_0^2}{m\omega}V_y\right]$$

$$= -\frac{ie}{m\omega}\left[E_y + i\frac{eB_0}{m\omega}E_x\right] + \frac{e^2B_0^2}{m^2\omega^2}V_y$$

$$\left(1-\frac{\omega_c^2}{\omega^2}\right)V_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)$$

$$\Rightarrow V_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$\cancel{(\omega^2-c^2k^2)\vec{E}_1 \quad \omega^2\vec{B}_1 = -c^2\left[\vec{k}(\vec{k}\cdot\vec{B}_1) - k^2\vec{B}_1\right]}$$

$$-\vec{k}(\vec{k}\cdot\vec{E}_1) + k^2\vec{E}_1 = \frac{i\omega}{\epsilon_0 c^2}\vec{J}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

$$\Rightarrow (\omega^2 - c^2k^2)\vec{E}_1 + c^2(\vec{k}\cdot\vec{E}_1)\vec{k} = -\frac{i\omega}{\epsilon_0}\vec{J}_1 = \frac{in_0\omega e}{\epsilon_0}\vec{V}_{e1}$$

$$\vec{J}_1 = -n_0 e\vec{V}_{e1}$$

$\vec{E}_1 = E_x\hat{x} + E_y\hat{y}$. $\quad \vec{k} = k\hat{x}$

$$(\omega^2 - \cancel{c^2k^2})E_x + \cancel{c^2kE_xk} = \frac{in_0\omega e}{\epsilon_0}V_x$$

$$= \frac{in_0\omega e}{\epsilon_0}\frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$\Rightarrow \omega^2 E_x = -\frac{i\cancel{\omega}n_0e}{\epsilon_0}\frac{e}{m\cancel{\omega}}\left(iE_x + \frac{\omega_c}{\omega}E_y\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$(\omega^2 - c^2k^2)E_y = \frac{in_0\omega e}{\epsilon_0}V_y = \frac{in_0\omega e}{\epsilon_0}\frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$(\omega^2 - c^2k^2)E_y = -\frac{i\cancel{\omega}n_0e}{\epsilon_0}\frac{e}{m\cancel{\omega}}\left(iE_y - \frac{\omega_c}{\omega}E_x\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$\omega_p^2 \equiv \frac{n_0 e^2}{\epsilon_0 m}$$

$$\omega^2\left(1-\frac{\omega_c^2}{\omega^2}\right)E_x = -i\overbrace{\frac{n_0 e^2}{\epsilon_0 m}}^{\omega_p^2}\left(iE_x + \frac{\omega_c}{\omega}E_y\right)$$

$$\cancel{(\omega^2-\omega_c^2)E_x} = \omega_p^2 E_x - i\frac{\omega_p^2\omega_c}{\omega}E_y$$

$$\left[\omega^2\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_p^2\right]E_x + i\frac{\omega_p^2\omega_c}{\omega}E_y = 0$$

$$(\omega^2-c^2k^2)\left(1-\frac{\omega_c^2}{\omega^2}\right)E_y = -i\omega_p^2\left(iE_y - \frac{\omega_c}{\omega}E_x\right)$$

$$= \omega_p^2 E_y + i\frac{\omega_p^2\omega_c}{\omega}E_x$$

$$\Rightarrow \left[(\omega^2-c^2k^2)\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_p^2\right]E_y - i\frac{\omega_p^2\omega_c}{\omega}E_x = 0$$

$$\Rightarrow \begin{bmatrix} \underbrace{\omega^2\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_p^2}_{=\ \omega^2-\omega_c^2-\omega_p^2\ =\ \omega^2-\omega_h^2} & i\frac{\omega_p^2\omega_c}{\omega} \\ -i\frac{\omega_p^2\omega_c}{\omega} & (\omega^2-c^2k^2)\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_p^2 \end{bmatrix}\begin{bmatrix}E_x\\E_y\end{bmatrix} = 0$$

ω_h ← upper hybrid freq.

$\omega^2-\omega_h^2$

the determinant needs to be zero to have non zero solution.

$$(\omega^2-\omega_h^2)\left[\omega^2-\omega_h^2-c^2k^2\left(1-\frac{\omega_c^2}{\omega^2}\right)\right] = \left(\frac{\omega_p^2\omega_c}{\omega}\right)^2$$

$$\omega^2-\omega_h^2-\frac{c^2k^2}{\omega^2}(\omega^2-\omega_c^2) = \left(\frac{\omega_p^2\omega_c}{\omega}\right)^2\frac{1}{\omega^2-\omega_h^2}$$

$$\frac{c^2k^2}{\omega^2} = \frac{\omega^2-\omega_h^2-\left(\frac{\omega_p^2\omega_c}{\omega}\right)^2/(\omega^2-\omega_h^2)}{\omega^2-\omega_c^2}$$

$\omega_h^2 = \omega_c^2 + \omega_p^2$,

$$\frac{c^2k^2}{\omega^2} = \frac{(\omega^2-\omega_h^2)^2 - \omega_p^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$= \frac{(\omega^2-\omega_h^2)(\omega^2-\omega_c^2-\omega_p^2) - \omega_p^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$= 1 - \frac{\omega_p^2(\omega^2-\omega_h^2) + \omega_p^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2(\omega^2-\omega_h^2) + \omega_p^2\omega_c^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$\omega^2\omega_p^2 - \omega_p^2\omega_c^2$

$$= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2(\omega^2-\omega_c^2-\omega_p^2) + \omega_p^2\omega_c^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2(\cancel{\omega^2-\omega_c^2}) - \omega_p^2(\cancel{\omega^2-\omega_c^2})}{(\cancel{\omega^2-\omega_c^2})(\omega^2-\omega_h^2)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2}$$

$$\Rightarrow \frac{c^2k^2}{\omega^2} = \frac{c^2}{v_\phi^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2}$$

---

dispersion relation for the extraordinary wave

$\left\{\begin{array}{l}\text{partially transverse} \\ \text{— — — longitudinal}\end{array}\right.$ w/ $\vec{k} \perp \vec{B_0}$, $\vec{E_1} \parallel \vec{B_0}$

§ 4.15 Cutoff and Resonances.

Cutoff: occurs when the index of refraction goes to zero.

$\updownarrow$ reflected

i.e. $\lambda \to \infty$ $\because$ $\hat{n} = \dfrac{ck}{\omega} \to 0$ $\quad k = \dfrac{2\pi}{\lambda}$

$v_\phi \to \pm\infty$

Resonance: occurs when the index of refraction goes to infinity

$\updownarrow$ absorbed

i.e. $\lambda \to 0$ $\because$ $\hat{n} = \dfrac{ck}{\omega} \to \infty$ $\quad k = \dfrac{2\pi}{\lambda}$

$v_\phi \to 0$

* A wave is reflected at a cutoff &

absorbed at a resonance

* Resonance of X-wave. $\Rightarrow k \to \infty$ $\qquad v_\phi = \dfrac{\omega}{k}$

$\hat{n} = \dfrac{c}{\omega/k} = \dfrac{c}{v_\phi} \to 0$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\,\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2} \longrightarrow \infty$$

$$\Rightarrow \omega \to \omega_h \Rightarrow \underbrace{\omega_h^2 = \omega_p^2 + \omega_c^2 = \omega^2}_{\text{resonance.}}$$

As $\omega \to \omega_h \Rightarrow v_\phi$ & $v_g \to 0 \Rightarrow$ wave energy is converted into upper hybrid oscillations

The wave loses its electromagnetic character & becomes an electrostatic oscillation.

* Cutoff of X-wave $\Rightarrow k \to 0$

$$\Rightarrow \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\,\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2} \to 0 \qquad \omega_h^2 = \omega_p^2+\omega_c^2$$

$$1 = \frac{\omega_p^2}{\omega^2}\,\frac{1}{(\omega^2-\omega_h^2)/(\omega^2-\omega_p^2)}$$

$$= \frac{\omega_p^2}{\omega^2}\,\frac{1}{1-\omega_c^2/(\omega^2-\omega_p^2)}$$

$$1 - \frac{\omega_c^2}{\omega^2 - \omega_p^2} = \frac{\omega_p^2}{\omega^2}$$

$$1 - \frac{\omega_p^2}{\omega^2} = \frac{\omega_c^2/\omega^2}{1 - \omega_p^2/\omega^2}$$

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 = \frac{\omega_c^2}{\omega^2} \Rightarrow 1 - \frac{\omega_p^2}{\omega^2} = \pm \frac{\omega_c}{\omega}$$

$$\underline{\omega^2 \mp \omega_c \omega - \omega_p^2 = 0}$$

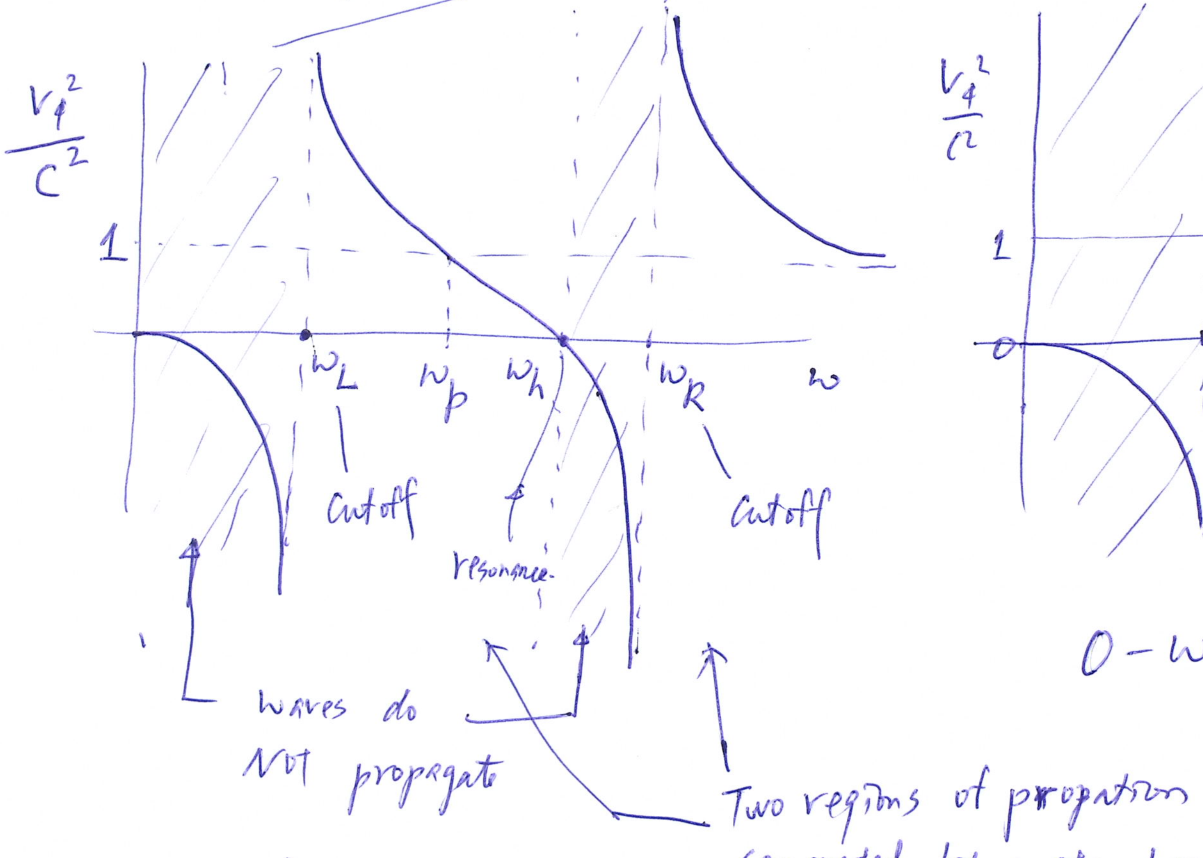
$$\omega = \frac{1}{2}\left[\pm \omega_c \pm \sqrt{\omega_c^2 + 4\omega_p^2}\right]$$

$$\omega = \frac{1}{2}\left[\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}\right] \equiv \omega_R \text{ - right-hand cutoff}$$

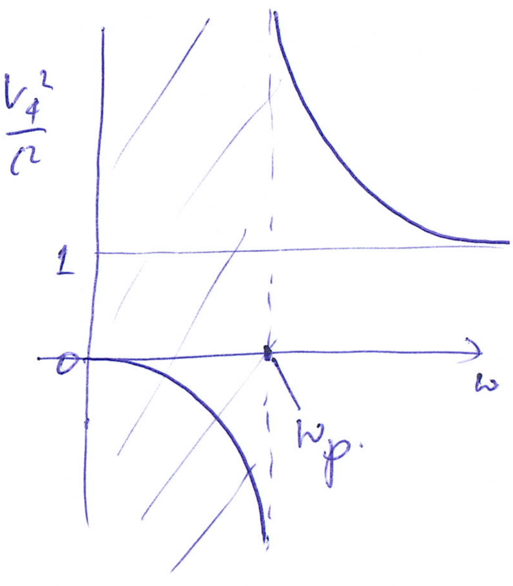
$$\cancel{\omega = \frac{1}{2}\left[\omega_c - \sqrt{\omega_c^2 + 4\omega_p^2}\right]} < 0$$

$$\omega = \frac{1}{2}\left[-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}\right] \equiv \omega_L \text{ - left-hand cutoff}$$

$$\cancel{\omega = \frac{1}{2}\left[-\omega_c - \sqrt{\omega_c^2 + 4\omega_p^2}\right]} < 0$$



X-wave



O-wave.

§4.16 Electromagnetic wave parallel to B_0

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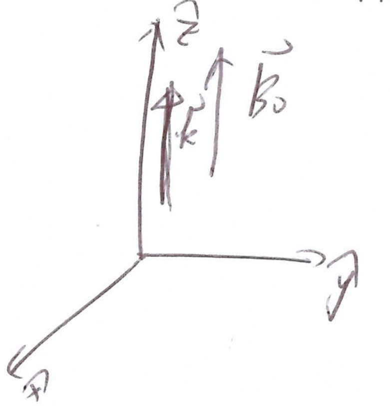
$\vec{k} = k\hat{z}$, $\quad \vec{B} = B_0\hat{z}$



Transverse $\vec{E_1}$:

$\vec{E_1} = E_x\hat{x} + E_y\hat{y}$

Assuming $KT_e = 0$.

$$m n_e\left[\frac{\partial \vec{v_e}}{\partial t} + (\vec{v_e}\cdot\nabla)\vec{v_e}\right] = -e n_e(\vec{E} + \vec{v_e}\times\vec{B}) - \cancel{\nabla p} \quad \because KT_e = 0$$

$n_e = n_0 + n_1$, $\quad \vec{v_e} = \vec{v_0} + \vec{v_1}$ $\quad \vec{E} = \vec{E_0} + \vec{E_1}$, $\quad \vec{B} = \vec{B_0} + \vec{B_1}$

$\vec{v_0} = \vec{E_0} = 0$. $\quad \partial_t n_0 = \partial_t \vec{B_0} = 0$, $\quad \nabla n_0 = 0$

$$\Rightarrow m(n_0 + \underbrace{\cancel{n_1}}_{2^{nd}\text{ order}})\left[\frac{\partial \vec{v_1}}{\partial t} + \underbrace{\cancel{(\vec{v_1}\cdot\nabla)\vec{v_1}}}_{2^{nd}\text{ order}}\right] = -e(n_0 + \underbrace{\cancel{n_1}}_{2^{nd}\text{ order}})\left[\vec{E_1} + \vec{v_1}\times(\vec{B_0} + \underbrace{\cancel{\vec{B_1}}}_{2^{nd}\text{ order}})\right]$$

$$\Rightarrow m\cancel{n_0}\frac{\partial \vec{v_1}}{\partial t} = -e\cancel{n_0}(\vec{E_1} + \vec{v_1}\times\vec{B_0})$$

$$m\frac{\partial \vec{v_1}}{\partial t} = -e(\vec{E_1} + \vec{v_1}\times\vec{B_0}). \quad \text{Eq}$$

$\partial_t \to -i\omega$: $\quad -i\omega m\vec{v_1} = -e(\vec{E_1} + \vec{v_1}\times\vec{B_0})$

$$\Rightarrow i\omega m\vec{v_1} = e(\vec{E_1} + \vec{v_1}\times\vec{B_0})$$

$i\omega m v_x = e(E_x + v_y B_0) \quad \Rightarrow v_x = -i\frac{e}{m\omega}(E_x + v_y B_0)$

$i\omega m v_y = e(E_y - v_x B_0) \quad \Rightarrow v_y = -i\frac{e}{m\omega}(E_y - v_x B_0)$

$$\Rightarrow v_x = -i\frac{e}{m\omega}\left[E_x - i\frac{eB_0}{m\omega}(E_y - v_x B_0)\right]$$

$$= -\frac{ie}{m\omega}E_x - \frac{e^2 B_0}{m^2\omega^2}(E_y - v_x B_0)$$

$$= -\frac{ie}{m\omega}E_x - \frac{e^2 B_0}{m^2\omega^2}E_y + \frac{e^2 B_0^2}{m^2\omega^2}v_x$$

$\omega_c = \frac{eB_0}{m}$

$$\left(1 - \frac{\omega_c^2}{\omega^2}\right)v_x = \frac{e}{m\omega}\left(-iE_y - \frac{eB_0}{m\omega}E_y\right) = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_p}{\omega}E_y\right)$$

$$V_x = \frac{e}{m\omega}\left(-iE_x - \frac{eB_0}{m\omega}E_y\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

ω_c/ω

P11X

$$V_y = -i\frac{e}{m\omega}\left[E_y + i\frac{eB_0}{m\omega}(E_x + V_yB_0)\right]$$

$$= -\frac{ie}{m\omega}E_y + \frac{e^2B_0}{m^2\omega^2}(E_x + V_yB_0)$$

$$\left(1-\frac{\omega_c^2}{\omega^2}\right)V_y = -\frac{ie}{m\omega}E_y + \frac{e^2B_0}{m^2\omega^2}E_x$$

$$= \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)$$

$$V_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}$$

From Maxwell's Eq: $\nabla\times\vec{E} = -\dot{\vec{B}} \Rightarrow \nabla\times\vec{E}_1 = -\dot{\vec{B}}_1$

~~$\nabla B\, i\vec{k} \rightarrow c^2\nabla\times\vec{B} = \dot{\vec{E}} \Rightarrow c^2\nabla\times\vec{B}_1 = \dot{\vec{E}}_1$~~

~~$\nabla\times(\nabla\times\vec{E}_1) = -(\nabla\times\dot{\vec{B}}_1) = -\frac{1}{c^2}\ddot{\vec{E}}_1$~~

~~$c^2\nabla\times\vec{B} = \dot{\vec{E}} + \mu_0\vec{j} \Rightarrow c^2\nabla\times\vec{B} = \dot{\vec{E}} + \mu_0\vec{j}$~~

~~$\nabla\times\vec{B} = \frac{1}{\mu_0\epsilon_0}$~~

$$\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\dot{\vec{E}} \Rightarrow c^2\nabla\times\vec{B}_1 = \frac{1}{\epsilon_0}\vec{j}_1 + \dot{\vec{E}}_1 \quad \left(c^2 = \frac{1}{\epsilon_0\mu_0}\right)$$

$$c^2\nabla\times\dot{\vec{B}}_1 = \frac{1}{\epsilon_0}\dot{\vec{j}}_1 + \ddot{\vec{E}}_1$$

$$\nabla\times(\nabla\times\vec{E}_1) = -\nabla\times\dot{\vec{B}}_1 = -\mu_0\dot{\vec{j}}_1 - \mu_0\epsilon_0\overset{(-i\omega)^2}{\ddot{\vec{E}}_1} = i\omega\mu_0\vec{j}_1 + \underbrace{\mu_0\epsilon_0}_{1/c^2}\omega^2\vec{E}_1$$

$$i\vec{k}\times(i\vec{k}\times\vec{E}_1) = -\vec{k}\times(\vec{k}\times\vec{E}) = -\vec{k}(\vec{k}\cdot\vec{E}_1) + k^2\vec{E}_1$$

$$\Rightarrow -\vec{k}(\vec{k}\cdot\vec{E}_1) + k^2\vec{E}_1 = i\omega\mu_0\vec{j}_1 + \frac{\omega^2}{c^2}\vec{E}_1 = \frac{i\omega}{\epsilon_0c^2}\vec{j}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

$\because$ Transverse wave: $\vec{k}\cdot\vec{E}_1 = 0$ (由 $\vec{E}_1 \perp \vec{k}$)

$$k^2\vec{E}_1 = \frac{i\omega}{\epsilon_0c^2} + \frac{\omega^2}{c^2}\vec{E}_1 \Rightarrow (\omega^2 - c^2k^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}\vec{j}_1$$

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$\cancel{J_1 = -n_0}$ $\qquad \vec{J} = -en\vec{v}_e$

$\rightarrow \quad \cancel{\vec{J}_0} + \vec{J}_1 = -e(n_0 + \cancel{n_1})\vec{v}_1$ 2nd order

$\rightarrow \quad \vec{J}_1 = -en_0\vec{v}_1$

$$(\omega^2 - c^2k^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}J_1$$
$$= \frac{i\omega}{\epsilon_0}en_0\vec{v}_1$$

$\cancel{\rightarrow \vec{v}_1 = -i\frac{n_0e\omega}{\epsilon_0}(\omega^2 - c^2k^2)\vec{E}_1}$

$\cancel{\begin{cases} v_x = -i\frac{n_0e\omega}{\epsilon_0}(\omega^2 - c^2k^2)E_x \\ v_y = -i\frac{n_0e\omega}{\epsilon_0}(\omega^2 - c^2k^2)E_y \end{cases}}$

$\cancel{-i\frac{n_0e\omega}{\epsilon_0}(\omega^2 - c^2k^2)E_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}}$

$\cancel{(\omega^2 - c^2k^2)E_x =}$

$$\vec{v}_1 = -i\frac{\epsilon_0}{n_0e\omega}(\omega^2 - c^2k^2)\vec{E}_1$$

$$\begin{cases} v_x = -i\frac{\epsilon_0}{n_0e\omega}(\omega^2 - c^2k^2)E_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \\ v_y = -i\frac{\epsilon_0}{n_0e\omega}(\omega^2 - c^2k^2)E_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \end{cases}$$

$$\omega_p^2 = \frac{n_0e^2}{\epsilon_0 m}$$

$$\rightarrow (\omega^2 - c^2k^2)E_x = i\frac{n_0e^2}{\epsilon_0 m}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$
$$= \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}\left(E_x - i\frac{\omega_c}{\omega}E_y\right)$$

$$\rightarrow (\omega^2 - c^2k^2)E_y = i\frac{n_0e^2}{\epsilon_0 m}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$
$$= \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}\left(E_y + i\frac{\omega_c}{\omega}E_y\right)$$

Let $\alpha \equiv \dfrac{\omega_p^2}{1-\omega_c^2/\omega^2}$

$$\begin{cases} (\omega^2 - c^2k^2 - \alpha) E_x + i\alpha \dfrac{\omega_c}{\omega} E_y = 0 \\ -i\alpha \dfrac{\omega_c}{\omega} E_x + (\omega^2 - c^2k^2 - \alpha) E_y = 0 \end{cases}$$

~~determi~~

$$\begin{pmatrix} \omega^2 - c^2k^2 - \alpha & i\alpha \dfrac{\omega_c}{\omega} \\ -i\alpha \dfrac{\omega_c}{\omega} & \omega^2 - c^2k^2 - \alpha \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

~~Non~~ Non zero solutions occur when determinant $= 0$

i.e. $(\omega^2 - c^2k^2 - \alpha)^2 - \alpha^2 \dfrac{\omega_c^2}{\omega^2} = 0$

$$\omega^2 - c^2k^2 - \alpha = \pm \alpha \frac{\omega_c}{\omega}$$

$$\omega^2 - c^2k^2 = \alpha\left(1 \pm \frac{\omega_c}{\omega}\right) = \frac{\omega_p^2}{1-\omega_c^2/\omega^2}\left(1 \pm \frac{\omega_c}{\omega}\right)$$

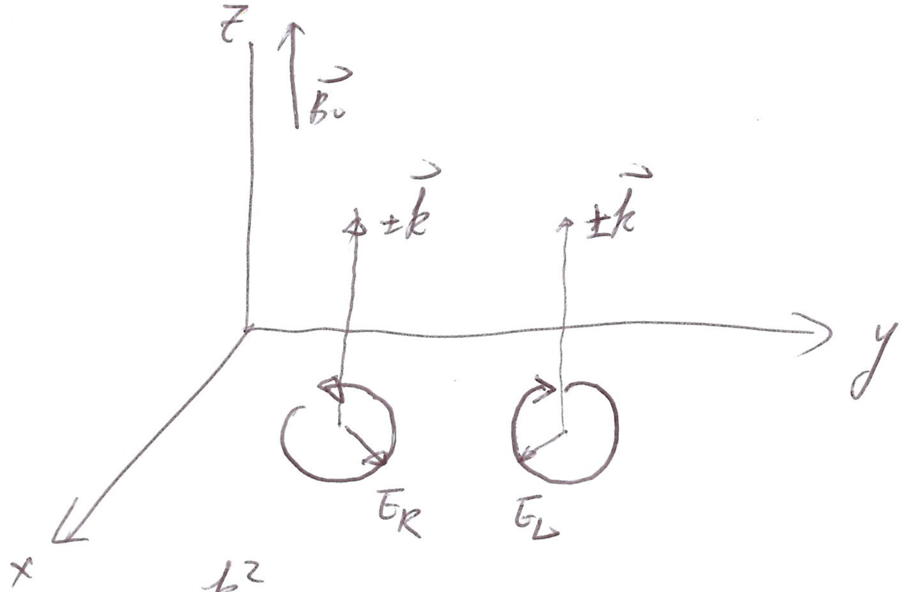
$$= \frac{\omega_p^2}{(1-\omega_c/\omega)(1+\omega_c/\omega)}\left(1 \pm \frac{\omega_c}{\omega}\right)$$

$$= \frac{\omega_p^2}{1 \mp \omega_c/\omega}$$

~~$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{1}$~~

$n^2 = \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2/\omega^2}{1-\omega_c/\omega}$ — R wave, right hand circular polarized

$n^2 = \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2/\omega^2}{1+\omega_c/\omega}$ — L wave, left hand circular polarized.



k^2

$\because \frac{c^2 k^2}{n^2} = \cdots$

$\therefore$ the direction of the $\vec{E}$ vector is independent of the sign of $\vec{k}$,
the polarization is the same for wave propagates in the opposite direction.

Summery: for electromagnetic wave w/ $\vec{B_0}$.

* $\vec{k} \parallel \vec{B_0} \longrightarrow \begin{cases} R\text{-wave} \\ L\text{-wave} \end{cases}$ circularly-polarized wave

* $\vec{k} \perp \vec{B_0} \longrightarrow \begin{cases} O\text{-wave} \\ X\text{-wave} \end{cases}$

O-wave: plane-polarized wave, $\vec{E_1} \parallel \vec{B_0}$

X-wave: ellipitically-polarized wave, $\vec{E_1} \perp \vec{B_0}$.

→ Cutoff & Resonance:

Resonance: $k \to \infty$

R-wave: $\omega = \omega_c$, $k \to \infty$

- resonance w/ the cyclotron motion of the electrons.
  - → the direction of ~~gyration of~~ rotation of the plane of polarization is the same as the direction of the gyration of electrons.
  - ⇒ the wave loses its energy in continuously accelerating the electrons

L-wave: it does NOT have a resonance for positive ω.

- If ion motions are included, L-wave would have a resonance at $\omega = \Omega_c$ due to ion gyration.

Cutoff: $k \to 0$

R-wave. $\frac{c^2k^2}{\omega^2} = 0 = 1 - \frac{\omega_p^2/\omega^2}{1-\omega_c/\omega} \Rightarrow \frac{\omega_p^2}{\omega^2} = 1 - \frac{\omega_c}{\omega} = \frac{\omega - \omega_c}{\omega}$

$\Rightarrow \omega^2 - \omega_c\omega - \omega_p^2 = 0$

$\omega = \frac{\omega_c \pm \sqrt{\omega_c^2 + 4\omega_p^2}}{2}$

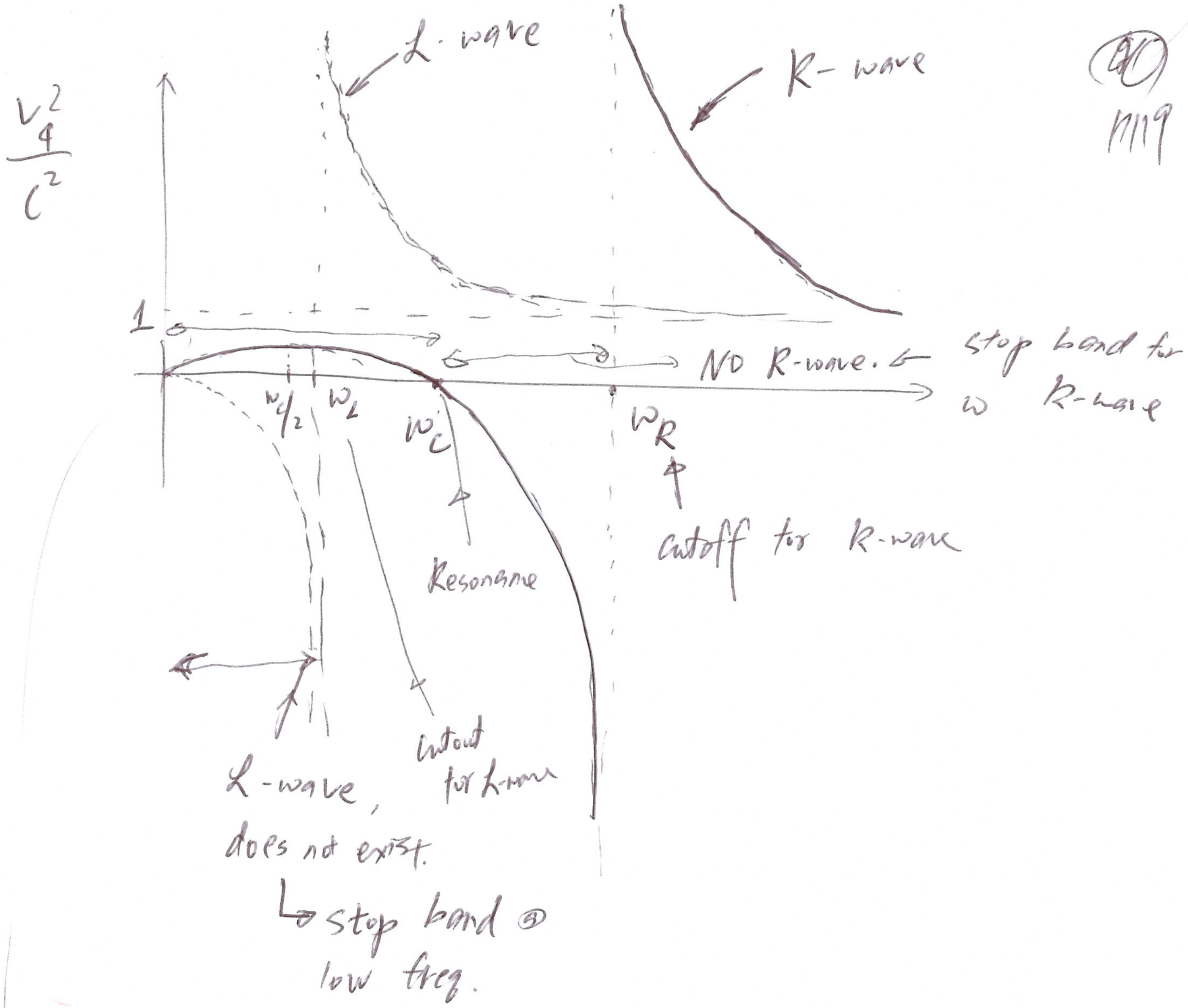
$\rightarrow \frac{\omega_c + \sqrt{\omega_c^2 + \omega_p^2}}{2}$

L-wave: $\frac{c^2k^2}{\omega^2} = 0 = 1 - \frac{\omega_p^2/\omega^2}{1+\omega_c/\omega} \Rightarrow \frac{\omega_p^2}{\omega^2} = 1 + \frac{\omega_c}{\omega} = \frac{\omega + \omega_c}{\omega}$

$\Rightarrow \omega^2 + \omega_c\omega - \omega_p^2 = 0$

$\omega = \frac{-\omega_c \pm \sqrt{\omega_c^2 + 4\omega_p^2}}{2}$

$\rightarrow \frac{-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}}{2}$



A second band of propagation w/ $v_\phi < c$ below ω_c. The wave in this low-frequency region is called the "whistler mode" and is of extreme importance in the study of ionospheric phenomena.

§4.17 Experimental Consequenses.

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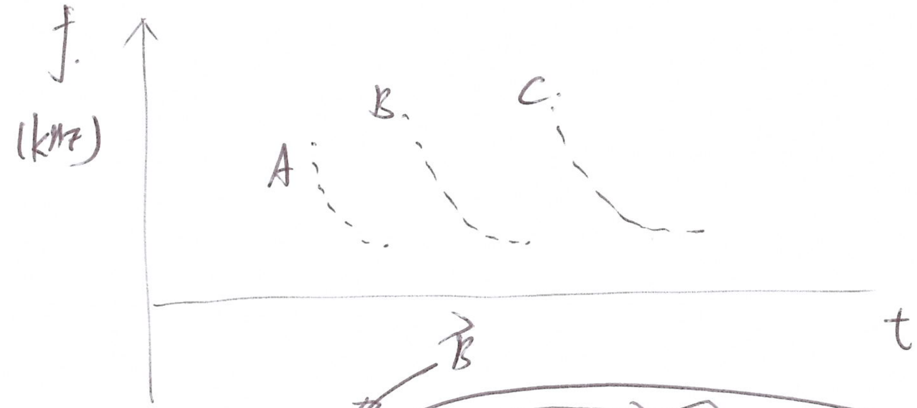
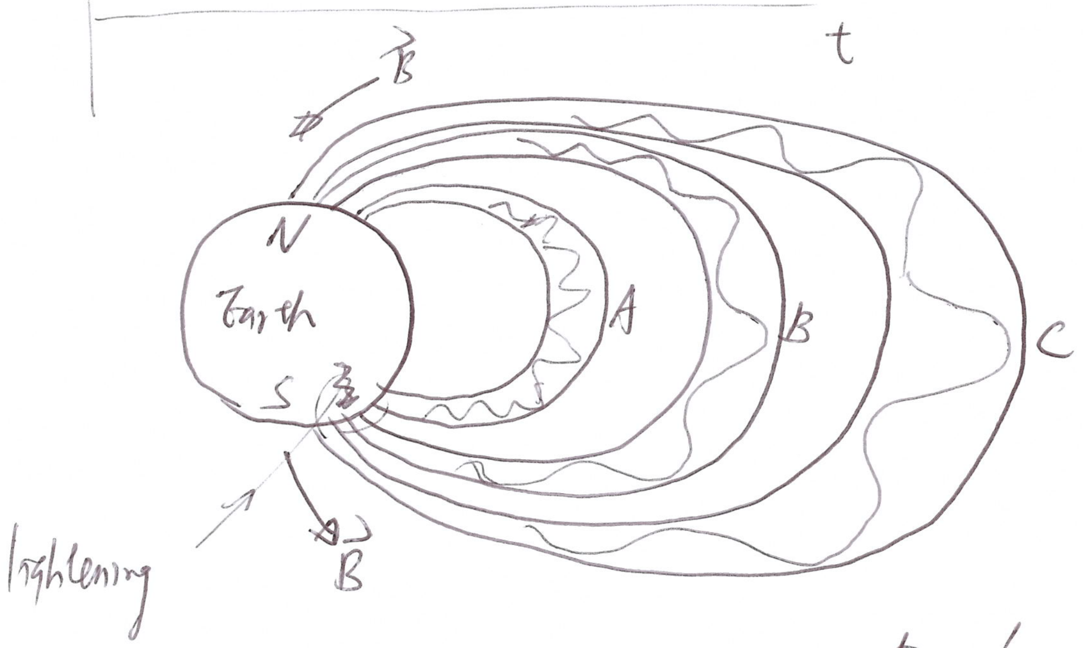
§4.17.1 The whistler mode.





→ generate waves. → R waves propagate along $\vec{B}$.

for $\omega < \omega_c/2$, V_ϕ increases w/ freq.

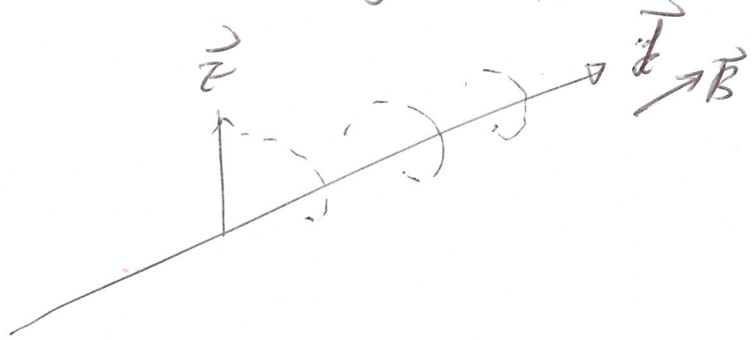
→ the low frequencies arrive later, giving rise to the descending tone.

* Several whistles can be produced by a single lightning flash because of propagating along different tubes of force A, B, C.

* ∵ $\omega < \omega_c$, ~~...~~

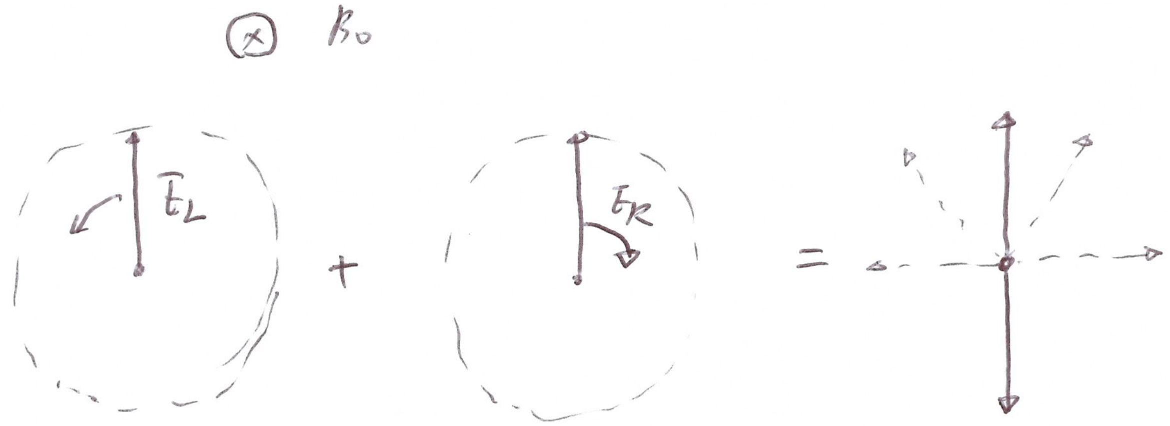
↳ $f \sim 100$kHz.

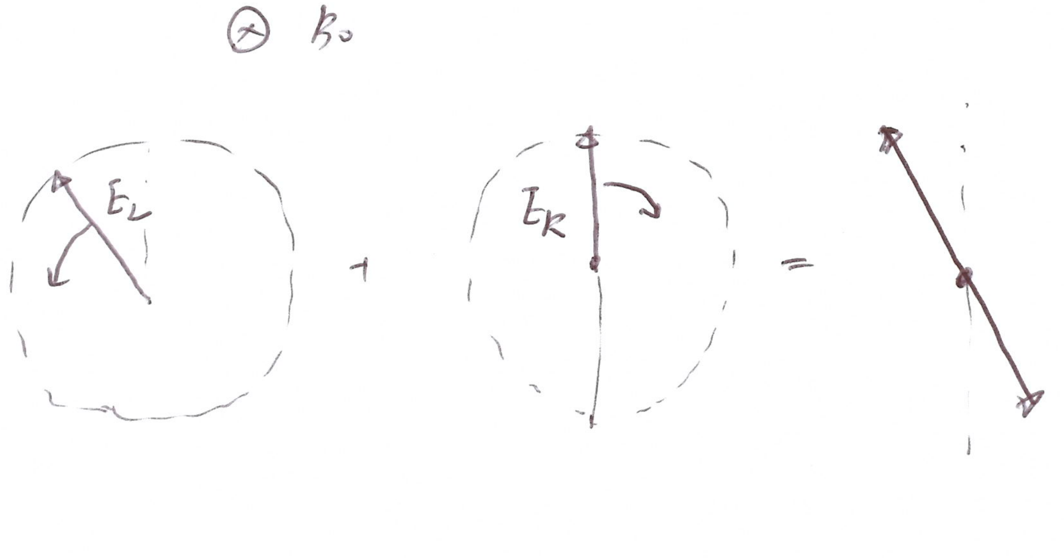
7 4.17.2. Faraday Rotation.



- A plane-polarized wave sent along a magnetic field in a plasma will suffer a rotation of its plane of polarization.

→ from difference in phase velocity of the R & L waves.





* Since the L-wave travels more slowly, it will have undergone $N+\epsilon$ cycles at the position where the R wave has undergone N cycles. ~~The~~ ~~vector then have~~ The plane of polarization is seen to have rotated.

→ rotation angle → ω_p^2 → n_e w/ given distance

→ NOT useful comparing to microwave interferometer unless the density is so high that refraction becomes a problem.

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7) 4.18 hydromagnetic waves

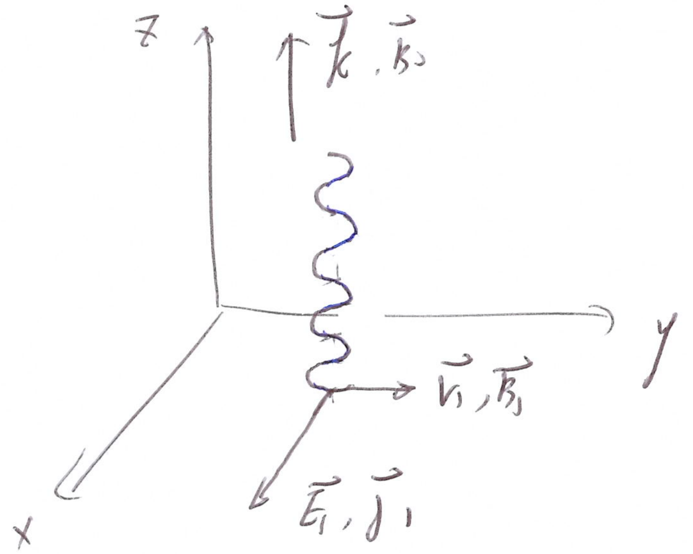
* $\vec{B}_0 \neq 0$, Low-freq. → ions ~~~~ NOT fixed.

- hydromagnetic wave along B_0
  - Alfvén wave

- Magnetosonic wave.

* Alfvén wave:

$\vec{k} \parallel \vec{B}_0$, $\vec{E}_1, \vec{j}_1 \perp \vec{B}_0$. $\vec{B}_1, \vec{v}_1 \perp \vec{B}_0, \vec{E}_1$



$$\nabla\times\vec{E} = -\dot{\vec{B}} \Rightarrow \nabla\times\vec{E}_1 = -\dot{\vec{B}}_1$$

$$\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\dot{\vec{E}} \Rightarrow \nabla\times\vec{B}_1 = \mu_0\vec{J}_1 + \mu_0\epsilon_0\dot{\vec{E}}_1$$

$\nabla\times$:

$$\nabla\times(\nabla\times\vec{E}_1) = -\nabla\times\dot{\vec{B}}_1 = -\frac{\partial}{\partial t}\nabla\times\vec{B}_1$$

$$= -\frac{\partial}{\partial t}\left(\mu_0\vec{J}_1 + \mu_0\epsilon_0\dot{\vec{E}}_1\right)$$

$$\nabla(\nabla\cdot\vec{E}_1) - \nabla^2\vec{E}_1 = -\mu_0\dot{\vec{J}}_1 - \mu_0\epsilon_0\ddot{\vec{E}}_1$$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

$\nabla\to ik,\ \partial_t\to -i\omega$

$$\Rightarrow \cancel{-\vec{k}(\vec{k}\cdot\vec{E}_1)}^{\,0} + k^2\vec{E}_1 = \mu_0\epsilon_0\omega^2\vec{E}_1 + i\omega\mu_0\vec{J}_1$$

$$= \frac{\omega^2}{c^2}\vec{E}_1 + \frac{i\omega}{\epsilon_0c^2}\vec{J}_1$$

$\because \vec{k} = k\hat{z},\ \vec{E}_1 = E_1\hat{x}$

$$\therefore\ +k^2E_1 = \frac{\omega^2}{c^2}E_1 + \frac{i\omega}{\epsilon_0c^2}j_1$$

$\because$ Low-freq. is considered

i.e. both electrons & ions are considered.

$$\therefore\ J_1 = -n_e e V_e + n_i e V_i$$

$$= n_0 e(V_{i1} - V_{e1}) \equiv n_0 e(V_{ix} - V_{ex})$$

$$\Rightarrow \underline{\epsilon_0(\omega^2 - k^2c^2)E_1 = -i\omega n_0 e(V_{ix} - V_{ex})}$$

- Neglect thermal motion, i.e., $T_i = 0$

Momentum eq. for ion:

$$\cancel{M\frac{\partial V_i}{\partial t} =}$$

$$Mn_i\left[\frac{\partial\vec{V}_i}{\partial t} + (\vec{V}_i\cdot\nabla)\vec{V}_i\right] = n_i e(\vec{E} + \vec{V}_i\times\vec{B})$$

$\xrightarrow{\text{linearized}}$

$$Mn_i\frac{\partial\vec{V}_{i1}}{\partial t} = n_i e(\vec{E}_1 + \vec{V}_{i1}\times\vec{B}_0)$$

$$M\frac{d\vec{V}_{i1}}{dt} = e\vec{E}_1 + e\vec{V}_{i1}\times B_0,$$

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$\partial t \rightarrow -i\omega$

$\nabla \rightarrow ik\hat{z}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & 0 \\ 0 & 0 & B_0 \end{vmatrix} = \hat{x}(V_yB_0) + \hat{y}(-V_xB_0)$$

$$\begin{cases} -i\omega M V_{ix} = eE_1 + eV_{iy}B_0 \\ -i\omega M V_{iy} = -eV_{ix}B_0 \end{cases}$$

$$V_{ix} = \frac{1}{-i\omega M}\left[eE_1 + eV_{iy}B_0\right]$$

$$V_{iy} = \frac{eB_0}{i\omega M}V_{ix}$$

$$V_{ix} = \frac{1}{-i\omega M}\left[eE_1 + \frac{e^2B_0^2}{i\omega M}V_{ix}\right]$$

$$= \frac{e}{-i\omega M}E_1 + \frac{e^2B_0^2}{M^2\omega^2}V_{ix}$$

$$\Omega_c \equiv \frac{eB_0}{M}$$

$$\left(1 - \frac{\Omega_c^2}{\omega^2}\right)V_{ix} = \frac{ie}{M\omega}E_1$$

$$V_{ix} = \frac{ie}{M\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1$$

$$V_{iy} = \frac{eB_0}{iM\omega}\cdot\frac{ie}{M\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1$$

$$= \frac{e}{M\omega}\frac{\Omega_c}{\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1$$

Similarly, for electron, $M \rightarrow m$, $e \rightarrow -e$, $\Omega_c \rightarrow -\omega_c$

$$V_{ex} = \frac{-ie}{m\omega}\left(1 - \frac{(-\omega_c)^2}{\omega^2}\right)^{-1}E_1 \xrightarrow{\omega_c^2 \gg \omega^2} \frac{+ie}{m\omega}\left(+\frac{\omega^2}{\omega_c^2}\right)E_1 \rightarrow 0$$

$$= \frac{-ie}{m\omega}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}E_1$$

$$V_{ey} = +\frac{e}{m\omega}\frac{(+\omega_c)}{\omega}\left[1 - \frac{(\omega_c)^2}{\omega^2}\right]^{-1}E_1$$

$$= \frac{e}{m}\frac{\omega_c}{\omega^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}E_1 \xrightarrow{\omega_c^2 \gg \omega^2} \frac{e}{m}\cdot\frac{\omega_c}{\omega^2}\cdot\frac{-\omega^2}{\omega_c^2}E_1$$

$$= -\frac{e}{m}\cdot\frac{m}{eB_0}E_1 = -\frac{E_1}{B_0}$$

$\vec{E}\times\vec{B}$ drift in $\hat{y}$ direction in the limit of $\omega_c^2 >> \omega^2$,

the Larmor ~~radius~~ gyrations of the electrons are neglected, they simply have $\vec{E}\times\vec{B}$ drift in $\hat{y}$

$$\epsilon_0(\omega^2 - c^2k^2)\cancel{E_1} = -i\omega n_0 e(V_{ix} - V_{ex})$$

$$\approx -i\omega n_0 e V_{ix}$$

$$= +i\omega n_0 e \frac{ie}{M\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1$$

$$= \frac{n_0 e^2}{M}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}\cancel{E_1}$$

$$\Omega_p^2 \equiv \frac{n_0 e^2}{\epsilon_0 M}$$

$$\Rightarrow \quad \omega^2 - c^2k^2 = \Omega_p^2\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$$

Assuming $\omega^2 << \Omega_c^2$ → hydromagnetic waves have freq. bellow ion cyclotron freq.

$$\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} = \frac{\omega^2}{\omega^2 - \Omega_c^2} \approx -\frac{\omega^2}{\Omega_c^2}$$

$$\omega^2 - c^2k^2 \approx \Omega_p^2 \cdot \frac{\omega^2}{\Omega_c^2}$$

$$= -\omega^2 \cdot \frac{n_0 \cancel{e^2}}{\epsilon_0 \cancel{M}} \cdot \frac{M^2}{\cancel{e^2} B_0^2}$$

$$= -\omega^2 \frac{n_0 M}{\epsilon_0 B_0^2}$$

$n_0 M \equiv \rho$ ← mass density

$$= -\omega^2 \frac{\rho}{\epsilon_0 B_0^2}$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \rho\epsilon_0/B_0^2} = \frac{c^2}{1 + \frac{\rho\mu_0}{B_0^2}c^2}$$

The denominator is recognized as the relative dielectric constant for low-freq. perpendicular motions.

$$\frac{\omega}{k} \equiv \frac{c}{\sqrt{\epsilon_R \mu_R}} = \frac{c}{\sqrt{\epsilon_R}} \quad \text{for } \mu_R = 1$$

$\because$ $\epsilon \gg 1$ for most Laboratory plasma

$$\frac{\omega}{k} \equiv v_\phi = \frac{c}{\sqrt{1 + \left(\frac{\rho \mu_0}{B_0^2}\right) c^2}} \approx \frac{\cancel{c}}{\sqrt{\rho \mu_0} \cdot \frac{\cancel{c}}{B_0}}$$

$$= \underline{\frac{B_0}{\sqrt{\mu_0 \rho}}} \equiv V_A$$

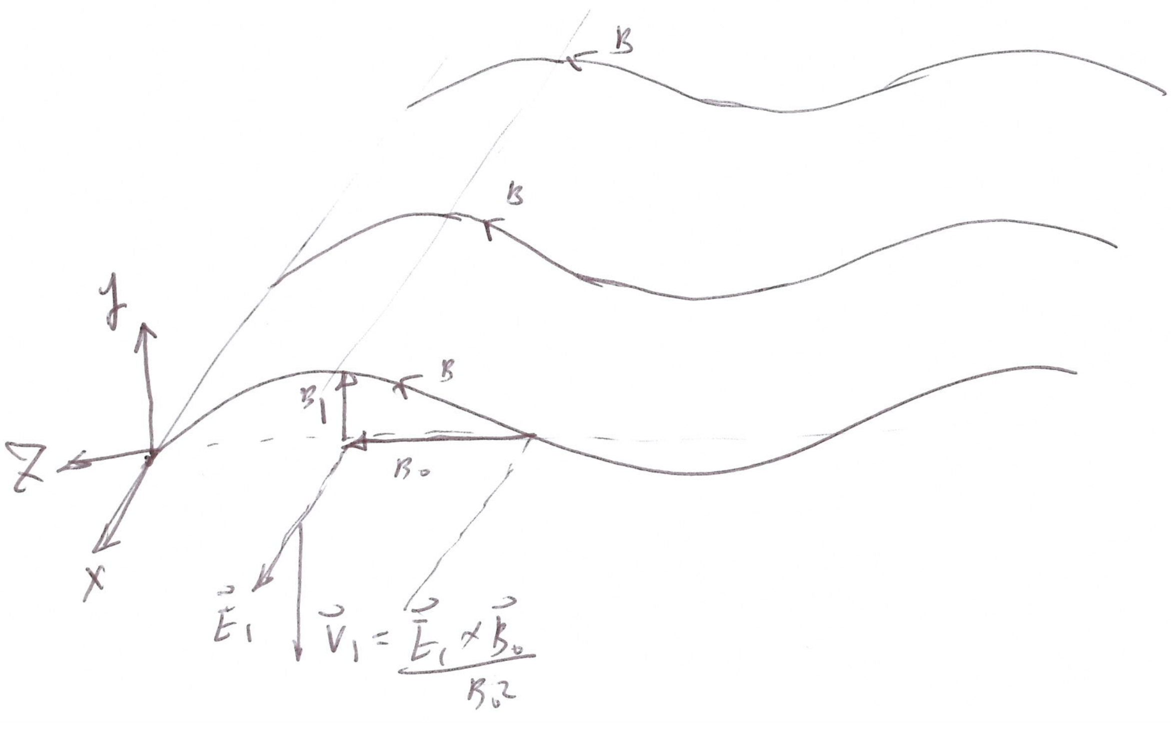
The <u>hydromagnetic waves</u> travel <u>along $\vec{B_0}$</u> at a constant velocity V_A. Alfvén velocity:

$$\underline{V_A \equiv \frac{B}{\sqrt{\mu_0 \rho}}}$$

$$\epsilon_R = \frac{\epsilon}{\epsilon_0} = 1 + \frac{c^2}{V_A^2}$$

$\because$ V_A is small

$\therefore$ <u>ϵ_R is large</u>



$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1 \Rightarrow E_{x1} = \frac{\omega}{k} B_{y1}$

$i\vec{k} \times \vec{E}_1 = +i\omega \vec{B}_1$

small perturbation $B_{y1} \longrightarrow E_{x1}$ in $\hat{x}$ for $\frac{\omega}{k}$ on $\hat{z}$

$E_{x1} \longrightarrow \underline{\vec{E}_1 \times \vec{B}_0}$ drift in $-\hat{y}$

$\omega^2 \ll \Omega_i^2 \longrightarrow$ both ions & electrons have the same drift v_y

$\longrightarrow$ fluid moves up & down in $\hat{y}$ w/ $v_y = \left|\frac{E_{x1}}{B_0}\right| = \frac{\omega}{k}\left|\frac{B_{y1}}{B_0}\right|$

∵ the ripple ~~drag~~ in the field is moving by at the phase velocity $\frac{\omega}{k}$

∴ the line of force is also moving downward.

∴ The fluid & the field lines oscillate together as if the particles were stuck to the lines

or The line of force act as if they were mass-loaded strings under tension.

∴ Alfven wave can be regarded as the propagating disturbance occuring when the strings are plucked,

→ <u>plasma is frozen to lines of force and</u> ~~moves~~ moving <u>w/ them</u> is a useful one to understand many low-freq. plasma phenomena.

* it's accurate as long as NO $\vec{E}$ along $\vec{B}$

* As $\vec{E}_1$ fluctuates → ion lag behind e^- due to its inertia

→ polarization drift $\vec{v}_p \parallel \vec{E}_1$ ($\hat{x}$)

→ cause a current $\vec{j}_1 \parallel \vec{E}_1$ ($\hat{x}$)

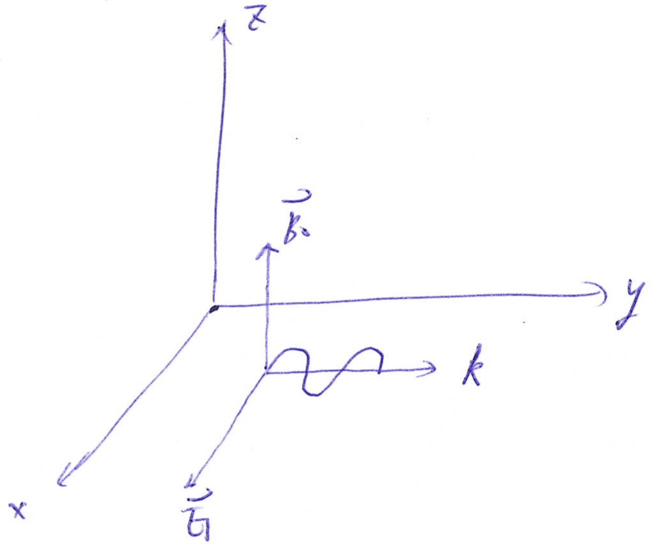
→ $\vec{j}_1 \times \vec{B}_0$ force on fluid in ($\hat{y}$) and 90° out of phase w/ $\vec{v}_1$

→ the ion inertia always causes an overshoot & a sustained oscillation

§ 4.19 Magnetosonic waves

- Low freq.
- EM waves propagate across $\vec{B_0}$

$\vec{B_0} = B_0 \hat{z}$, $\vec{E_1} = E_1 \hat{x}$. $\vec{k} = k\hat{y}$



$\vec{E_1} \times \vec{B_0}$ drifts lie along $\vec{k}$

→ the plasma will be compressed and released in the course of the oscillation.

→ ∇p is kept:

For ions:

$$M n_i \left(\frac{\partial \vec{V_i}}{\partial t} + (\vec{V_i} \cdot \nabla) \vec{V_i} \right) = e n_0 (\vec{E} + \vec{V_i} \times \vec{B}) - \gamma_i k T_i \nabla n_i$$

$n_i = n_0 + n_1$, $\vec{V_i} = \vec{V_{i0}} + \vec{V_{i1}} = \vec{V_{i1}}$, $\vec{E} = \vec{E_0} + \vec{E_1} = \vec{E_1}$, $\vec{B} = \vec{B_0} + \vec{B_1}$

Linearized →

$$M n_0 \frac{\partial \vec{V_{i1}}}{\partial t} = e n_0 (\vec{E_1} + \vec{V_{i1}} \times \vec{B_0}) - \gamma_i k T_i \nabla n_1$$

$$\Rightarrow \begin{cases} M \cancel{n_0} \dfrac{\partial V_{ix}}{\partial t} = e \cancel{n_0} (E_x + V_{iy} B_0) \\ M n_0 \dfrac{\partial V_{iy}}{\partial t} = e n_0 (-V_{ix} B_0) - \gamma_i k T_i \nabla n_1 \end{cases}$$

$\partial_t \to -i\omega$, $\nabla \to i\vec{k} = ik\hat{y}$ →

$$\begin{cases} -i\omega M V_{ix} = e(E_x + V_{iy} B_0) \Rightarrow V_{ix} = \dfrac{ie}{M\omega}(E_x + V_{iy} B_0) \\ -i\omega M n_0 V_{iy} = e n_0 (-V_{ix} B_0) - ik\gamma_i k T_i n_1 \Rightarrow V_{iy} = \dfrac{ie}{M\omega}(-V_{ix} B_0) + \dfrac{k}{\omega} \dfrac{\gamma_i k T_i}{M} \dfrac{n_1}{n_0} \end{cases}$$

Continuity: $\frac{\partial n}{\partial t} + \nabla(n\vec{v}) = 0$

Linearized $\xrightarrow{\partial_t \to -i\omega,\ \nabla \to ik}$ $-i\omega n_1 + i\vec{k}\, n_0 \cdot \vec{V}_{i1} = 0$

$$n_1 = \frac{k}{\omega} n_0 V_{iy} \Rightarrow \frac{n_1}{n_0} = \frac{k}{\omega} V_{iy}$$

$$\Rightarrow V_{iy} = -\frac{ie}{M\omega} V_{ix} B_0 + \frac{k}{\omega}\frac{\gamma_i k T_i}{M}\frac{k}{\omega} V_{iy}$$

$$= -i\frac{eB_0}{M\omega} V_{ix} + \frac{k^2}{\omega^2}\frac{\gamma_i k T_i}{M} V_{iy} \qquad \Omega_c \equiv \frac{eB_0}{M}$$

$$A \equiv \frac{k^2}{\omega^2}\frac{\gamma_i k T_i}{M}$$

$$\Rightarrow (1-A) V_{iy} = -\frac{i\Omega_c}{\omega} V_{ix}$$

$$V_{ix} = \frac{ie}{M\omega}\left(E_x + V_{iy} B_0\right)$$

$$= \frac{ie}{M\omega}\left[E_x + B_0 \frac{-i\Omega_c}{\omega(1-A)} V_{ix}\right]$$

$$= \frac{ie}{M\omega} E_x + \underbrace{\frac{eB_0}{M\omega}}_{\Omega_c} \frac{\Omega_c}{\omega}(1-A)^{-1} V_{ix}$$

$$= \frac{ie}{M\omega} E_x + \frac{\Omega_c^2}{\omega^2}(1-A)^{-1} V_{ix}$$

$$\left[1 - \frac{\Omega_c^2}{\omega^2}(1-A)^{-1}\right] V_{ix} = \frac{ie}{M\omega} E_x$$

$(-i\omega)^2$

-1

Note that $\nabla \times (\nabla \times \vec{E}_1) = -\mu_0 \dot{\vec{J}}_1 - \mu_0 \epsilon_0 \ddot{\vec{E}}_1$

$$-\vec{k}(\vec{k}\cdot\vec{E}_1) + k^2 \vec{E}_1 = -\mu_0 \dot{\vec{J}}_1 - \mu_0\epsilon_0 \ddot{\vec{E}}_1 = i\omega\mu_0 \vec{J}_1 + \mu_0\epsilon_0\omega^2 \vec{E}_1$$

$$= i\omega\mu_0 \vec{J}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

$\vec{k}\cdot\vec{E}_1$: 令

$$(\omega^2 - k^2 c^2)\vec{E}_1 = -i\omega\mu_0 c^2 \vec{J}_1 \qquad c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$\Rightarrow \epsilon_0(\omega^2 - k^2c^2) E_x = -i\omega n_0 e (V_{ix} - V_{ex})$$

for ① small electron mass.

$\longrightarrow \omega^2 \ll \omega_c^2 \qquad \omega_c^2 = \frac{eB_0}{m} \to \infty$

② $\omega^2 \ll k^2 V_{Th\,e}^2$

$\cancel{V_{ix} = \frac{ie}{M\omega} E_x}$ $\quad V_{ix}\left(1 - \frac{\Omega_i^2/\omega^2}{1-A}\right) = \frac{ie}{M\omega} E_y$

$\xrightarrow[\substack{e \to -e \\ \Omega_i^2 \to \omega_c^2}]{M \to m} \quad V_{ex}\left(1 - \frac{\omega_c^2/\omega^2}{1-A_e}\right) = \frac{-ie}{m\omega} E_x, \quad A_e = \frac{k^2}{\omega^2}\frac{\gamma_e k T_e}{m_e}.$

$\Rightarrow V_{ex} \frac{-\omega_c^2/\omega^2}{1-A_e} \simeq -\frac{ie}{m\omega} E_x$

$\cancel{V_{ex} = \frac{\omega^2}{\omega_c^2}[}$ $\quad V_{ex} \simeq \frac{ie}{m\omega}\frac{\omega^2}{\omega_c^2}\left[1 - \frac{k^2}{\omega^2}\frac{\gamma_e k T_e}{m}\right] E_x$

$\longrightarrow -\frac{ie}{m\omega}\frac{\omega^2}{\omega_c^2}\frac{k^2}{\omega^2}\frac{\gamma_e k T_e}{m} E_x$

$= -\frac{ik^2}{\omega}\frac{e}{m}\frac{m^2}{e^2 B_0^2}\frac{\gamma_e k T_e}{m} E_y$

$= -\frac{ik^2}{\omega B_0^2}\frac{\gamma_e k T_e}{e} E_x$

$\epsilon_0 (\omega^2 - k^2 c^2) E_x = -i\omega n_0 e \left[\frac{ie}{M\omega} E_x \left(1 - \frac{\Omega_i^2/\omega^2}{1-A}\right)^{-1} + \frac{ik^2}{\omega B_0^2}\frac{\gamma_e k T_e}{e} E_x\right]$

$= n_0 e \left[\frac{e}{M} E_x \left(\frac{1-A}{1-A-\Omega_i^2/\omega^2}\right) + \frac{k^2 M}{B_0^2}\frac{\gamma_e k T_e}{Me} E_x\right]$

for $\omega^2 \ll \Omega_i^2$ — low freq.

$\rightarrow$ $1-A$ is neglected relative to Ω_i^2/ω^2

$\Rightarrow (\omega^2 - k^2 c^2) = \frac{-n_0 e^2}{\epsilon_0 M}\frac{1-A}{\Omega_i^2/\omega^2} + n_0 e \frac{k^2 M}{B_0^2}\frac{\gamma_e k T_e}{Me}$

$= -\frac{\Omega_p^2}{\Omega_i^2}\omega^2(1-A) + \frac{k^2}{\epsilon_0 \mu_0 V_A^2}\frac{\gamma_e k T_e}{M}$

$= -\frac{\Omega_p^2}{\Omega_i^2}\omega^2(1-A) + \frac{k^2 c^2}{V_A^2}\frac{\gamma_e k T_e}{M}$

$V_A^2 = \frac{B^2}{\mu_0 \rho} = \frac{B^2}{\mu_0 n_0 M}$

$\frac{n_0 e^2}{\epsilon_0 M} = \Omega_p^2$

$c^2 = \frac{1}{\epsilon_0 \mu_0}$

Note that

$$\frac{\Omega_p^2}{\Omega_c^2} = \frac{n_0 \cancel{e^2}}{\epsilon_0 \cancel{M}} \cdot \frac{M^{\cancel{2}}}{\cancel{e^2} B^2} = \frac{n_0 M}{\epsilon_0 B^2} = \frac{\rho}{\epsilon_0 B^2} = \underbrace{\frac{1}{\epsilon_0 \mu_0}}_{c^2} \underbrace{\frac{\mu_0 \rho}{B^2}}_{1/V_A^2}$$

$$= \frac{c^2}{V_A^2}$$

$$\omega^2 - k^2c^2 = -\frac{\Omega_p^2}{\Omega_i^2}\omega^2(1-A) + \frac{k^2c^2}{V_A^2}\frac{\gamma_e kT_e}{M}$$

$$= -\frac{c^2}{V_A^2}\omega^2(1-A) + \frac{k^2c^2}{V_A^2}\frac{\gamma_e kT_e}{M}$$

$$= -\frac{c^2}{V_A^2}\omega^2\left[1 - \frac{k^2}{\omega^2}\frac{\gamma_i kT_i}{M}\right] + \frac{k^2c^2}{V_A^2}\frac{\gamma_e kT_e}{M}$$

$$= -\frac{c^2\omega^2}{V_A^2} + \frac{k^2c^2}{V_A^2}\frac{\gamma_i kT_i}{M} + \frac{k^2c^2}{V_A^2}\frac{\gamma_e kT_e}{M}$$

$$\omega^2\left(1 + \frac{c^2}{V_A^2}\right) = k^2c^2\left[1 + \frac{\gamma_e kT_e + \gamma_i kT_i}{M V_A^2}\right] = k^2c^2\left(1 + \frac{V_s^2}{V_A^2}\right)$$

where $V_s \equiv \frac{\gamma_e kT_e + \gamma_i kT_i}{M}$ – acoustic speed.

$$\Rightarrow \quad \frac{\omega^2}{k^2} = c^2 \frac{V_s^2 + V_A^2}{c^2 + V_A^2}$$

dispersion relation for the magnetosonic wave propagating $\perp B_0$

- an acoustic wave → compressions & rarefactions are produced not by motions along $\vec{E}$, BUT by $\vec{E}\times\vec{B}$ drift across $\vec{E}$
- for $B_0 \to 0 \Rightarrow V_A \to 0 \Rightarrow$ ordinary ion acoustic wave.
- For $KT \to 0$, $V_s \to 0 \Rightarrow \nabla p \to 0 \Rightarrow$ modified Alfvén wave.
- For magnetosonic wave, $V_\phi > V_A \Rightarrow$ "fast" hydromagnetic wave.

§ 4.20 Summary of elementary plasma waves

- Electrostatic : $\vec{E}_1 \parallel \vec{k}$
  - Electron waves (high freq.)
    - $\vec{B}_0 = 0$ or $\vec{k} \parallel \vec{B}_0$, $\omega^2 = \omega_p^2 + \frac{3}{2} k^2 V_{th}^2$ Plasma oscillation
    - $\vec{k} \perp \vec{B}_0$, $\omega^2 = \omega_p^2 + \omega_c^2 = \omega_h^2$ Upper hybrid oscillation.
  - Ion waves (Low freq.)
    - $\vec{B}_0 = 0$ or $\vec{k} \parallel \vec{B}_0$, $\omega^2 = k^2 V_s^2 = k^2 \frac{\gamma_e K T_e + \gamma_i K T_i}{M}$ Acoustic waves
    - $\vec{k} \perp \vec{B}_0$, $\omega^2 = \Omega_c^2 + k^2 V_s^2$. Electrostatic ion cyclotron waves
      or $\omega^2 = \omega_\ell^2 = \Omega_c \omega_c$. Lower hybrid oscillations.

- Electromagnetic : $\vec{E}_1 \perp \vec{k}$
  - Electron waves (high freq.)
    - $\vec{B}_0 = 0$. $\omega^2 = \omega_p^2 + k^2 c^2$ Light waves
    - $\vec{k} \parallel \vec{B}_0$, $\begin{cases} \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} & \text{R-wave, whistler mode.} \\ \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2/\omega^2}{1 + \omega_c/\omega} & \text{L-wave} \end{cases}$
    - $\vec{k} \perp \vec{B}_0$ $\begin{cases} \vec{E}_1 \parallel \vec{B}_0 : \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2}{\omega^2} & \text{O-wave} \\ \vec{E}_1 \perp \vec{B}_0 : \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2}{\omega^2} \dfrac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2} & \text{X-wave} \end{cases}$
  - Ion waves: (Low freq.)
    - $\vec{B}_0 = 0$ None
    - $\vec{k} \parallel \vec{B}_0$ $\omega^2 = k^2 V_A^2$ Alfvén wave
    - $\vec{k} \perp \vec{B}_0$ $\dfrac{\omega^2}{k^2} = c^2 \dfrac{V_s^2 + V_A^2}{c^2 + V_A^2}$ Magnetosonic wave.