3 plasma as Fluids

* In previous chapter: $\vec{E}$ & $\vec{B}$ are not prescribed but are determined by the positions & the motions of the charge themselves

* A self-consistent problem needs to be considered
  - → particles generates field by their charges ($\vec{E}$) and motions ($\vec{v} \to \vec{J}$)
  - → a time-varying situation.

* Typical plasma density $\sim 10^{12}/cm^3$ ion-electron pai
  - → impossible to predict trajectories of all partic
  - → Fluid model — identity of the individual particle is neglected, only the motion of fluid elements is taken into account.
  - → In an ordinary fluid, frequent collisions between particles keep the particle in a fluid element moving together
  - → It is surprising that the model works for plasma, which generally have infrequent collisions.

§. 3.1. Relation of Plasma physics to ordinary electromagnetics.

§ 3.1.1 Maxwell's Eq.

In vacuum:

$$\epsilon_0 \nabla\cdot\vec{E} = \sigma$$ ← free charge

$$\nabla\times\vec{E} = -\dot{\vec{B}}$$

current density

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{B} = \mu_0(\vec{J} + \epsilon_0\dot{\vec{E}})$$

In a medium:

$$\nabla\cdot\vec{D} = \sigma$$

$$\nabla\times\vec{E} = -\dot{\vec{B}}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{J} + \dot{\vec{D}}$$

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

"bound" charge & current density arising from polarization and magnetization of the medium.

In plasma, ions & electrons comprising the plasma are the equivalent of the "bound" charge & current. However, they move in a complicated way, It's hard to lump their effect into ϵ, μ.
Consequently, in plasma physics, one generally work w/ the vacuum eq. where σ, and $\vec{J}$ include All the charges and currents, both external and internal.

§ 3.1.2 Classical Treatment of Magnetic Materials.

- Each gyrating particle has magnetic moment.
  → plasma is a magnetic material w/ a permeability μ_m (use μ later)

- Ferromagnetic domains:

$$\vec{M} = \frac{1}{V}\sum_i \vec{\mu}_i$$ ← magnetization per unit volume.

bound current density: $\vec{j}_b = \nabla \times \vec{M}$

$$\frac{1}{\mu_0}\nabla \times \vec{B} = \vec{j}_f + \vec{j}_b + \epsilon_0 \dot{\vec{E}}$$

Make it into a simple form:

$$\nabla \times \vec{H} = \vec{j}_f + \epsilon_0 \dot{\vec{E}}$$ where

$$\vec{H} = \mu_0^{-1}\vec{B} - \vec{M}$$

Assuming $\vec{M} \propto \vec{B}$ or $\vec{H}$

$$\vec{M} \equiv \chi_m \vec{H}$$ ← important!!

$$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H} \equiv \mu_m \vec{H}$$

In plasma, each particle has a magnetic moment μ_α

$$\vec{M} = \frac{1}{V}\sum_i \vec{\mu}_\alpha^i$$

$$\because \mu_\alpha = \frac{m v_{\perp\alpha}^2}{2B} \propto \frac{1}{B} \Rightarrow M \propto \frac{1}{B}$$ Not linear!!

⇒ Not useful to consider a plasma as a magnetic medium.

η 3.1.3 Classical Treatment of Dielectrics p56

polarization P per volume:

$$\vec{P} = \frac{1}{V} \sum \vec{P_i}$$

bound charge density:

$$\sigma_b = -\nabla \cdot \vec{P}$$

$$\epsilon_0 \nabla \cdot \vec{E} = (\sigma_f + \sigma_b) \Rightarrow \nabla \cdot \vec{D} = \sigma_f$$

$$D = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$$

Assuming $\vec{P} = \epsilon_0 \chi_e \vec{E}$ where

$$\epsilon = (1 + \chi_e)\epsilon_0$$

→ We may try to get an expression for ϵ in plasma.

η 3.1.4 The dielectric Constant of a Plasma

polarization current, from fluctuating $\vec{E}(t)$

Continuity: $$\frac{\partial \sigma_p}{\partial t} + \nabla \cdot \vec{J}_p = 0$$

$\frac{\partial \sigma_p}{\partial t}$ → J_p

* A polarization effect does not arise in plasma unless the electric field is time varying.

* Note that $$\vec{J}_p = ne(\vec{V}_{ip} - \vec{V}_{ep}) = \frac{ne}{eB^2}(M+m)\frac{d\vec{E}}{dt}$$

$$= \frac{\rho}{B^2}\frac{d\vec{E}}{dt}$$

$V_p = \pm \frac{1}{\omega_c B}\frac{d\vec{E}}{dt}$, polarization drift

~~from~~ Also

polarization current

$$\nabla \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_p + \epsilon_0 \dot{\vec{E}})$$

$$= \mu_0(\vec{J}_f + \epsilon \dot{\vec{E}})$$

$$\epsilon = \epsilon_0 + \frac{J_p}{\dot{E}} = \epsilon_0 + \frac{\rho}{B^2}$$ ← polarizat drift

or $\epsilon_R \equiv \frac{\epsilon}{\epsilon_0} = 1 + \frac{\mu_0 \rho c^2}{B^2}$ where $c^2 = \frac{1}{\mu_0 \epsilon_0}$

* ↑ Low-frequency plasma dielectric constant for transverse motions.

Note that $\vec{J}_p = \frac{\rho}{B^2}\dot{\vec{E}}$ only valid for

$\omega^2 \ll \omega_c^2$ & $\vec{E} \perp \vec{B}$ (polarization drift)

* For $\rho \to 0$, $\epsilon_R \to 1$, like vacuum.

$B \to \infty$, $\epsilon_R \to 1$. ∵ polarization drift (rotates crazily) v_p vanishes and the particles do not move in response to the transverse electric field.

* In usual laboratory, e.g. $n = 10^{16}\ m^{-3}$, $B = 0.1\ T$

$$\frac{\mu_0 \rho c^2}{B^2} = \frac{4\pi \times 10^{-7} \times 10^{16} \times 1.67 \times 10^{-27} \times (3 \times 10^8)^2}{0.1^2} = 189$$

($\uparrow$ H)

($\epsilon_r \gg 1$)

→ E due to the particles in the plasma greatly alter the fields applied externally.

– A plasma with large ϵ shields out altering (AC) fields, just as a plasma w/ small λ_D shields out dc fields.

(w/ $\omega \ll \omega_c$)

§ 3.2 The fluid equation of motion.

$\vec{E}, \vec{B} \rightarrow$ using Maxwell's Eqs.

plasma's response — composed of two or more interpenetrating fluids, one for each species.

— simplest case: ion fluid, electron fluid

Interact using $\vec{E}, \vec{B}$ or collision

(neutral atom fluid for partially ionized gas)

interact only through collision.

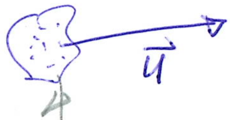
§ 3.2.1 The Convective Derivative.

Single particle: $m \dfrac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$

single particle

Assume: no collisions, no thermal motions

$\rightarrow$ all particles in a fluid element move together w/ an (average) speed $\vec{u}$, i.e. $\vec{v} = \vec{u}$



move w/ fluid

$\times n \rightarrow$

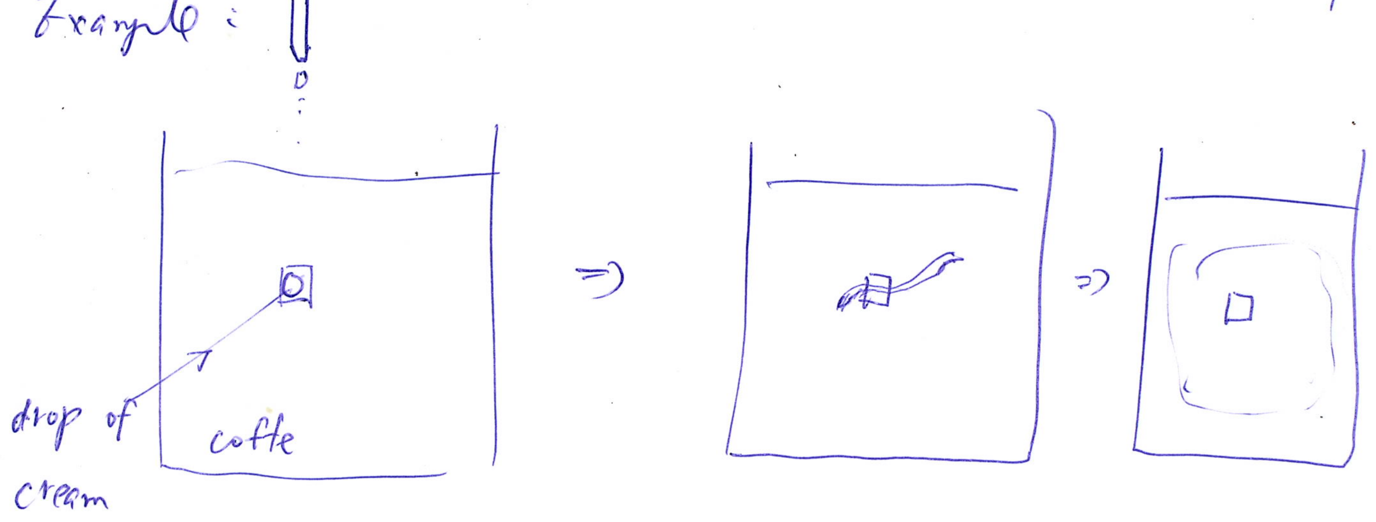
$$mn \frac{d\vec{u}}{dt} = qn(\vec{E} + \vec{u} \times \vec{B})$$

taken at the position of the particles

$\rightarrow$ impractical.

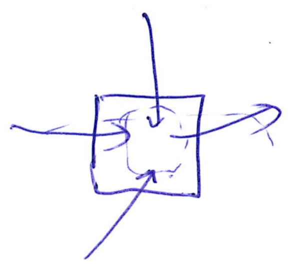
$\rightarrow$ We wish to have an equation for fluid elements ~~fix~~ fixed in space.

Example :



It's very hard to follow the drop (fluid element) since it will disperse all over the cup

Instead: the block as shown in the plot.



A fluid element at a fixed spot in the cup, retains its identity although particles continually go in and out of it.

$G(\vec{x},t)$ - any property of a fluid :

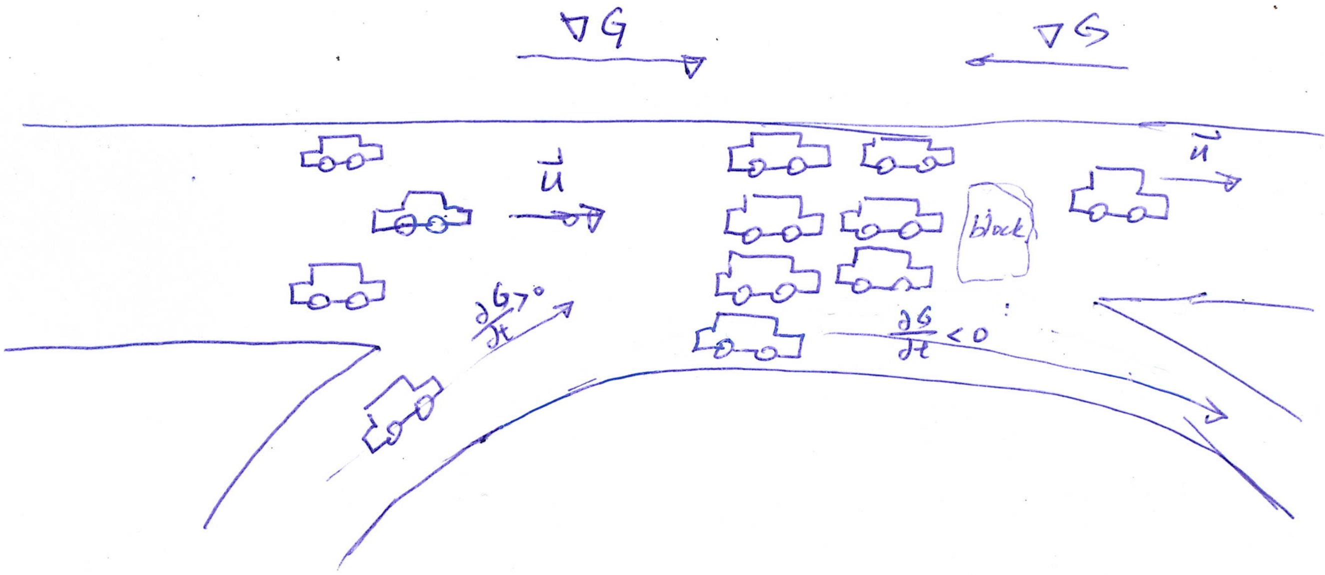
In 1D: The change of G with time:

$$\frac{dG(x,t)}{dt} = \underbrace{\frac{\partial G}{\partial t}}_{\text{change of } G \text{ @ fixed point in space}} + \underbrace{\frac{\partial G}{\partial x} \cdot \frac{dx}{dt}}_{\text{change of } G \text{ as the observer moves w/ the fluid into a region in which } G \text{ is different}} = \frac{\partial G}{\partial t} + u_x \frac{\partial G}{\partial x}$$

$$\Rightarrow \frac{dG}{dt} = \frac{\partial G}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)}_{\text{scalar differential operator}} G . \quad \text{— convective derivative.}$$

$$= \frac{DG}{Dt}.$$

Example: G : the density of cars on freeway p60



$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)\, G}_{\text{entering a trafic jam.}}$$

$\frac{\partial G}{\partial t}$: enter / exit freeway

time derivative in a fixed frame.

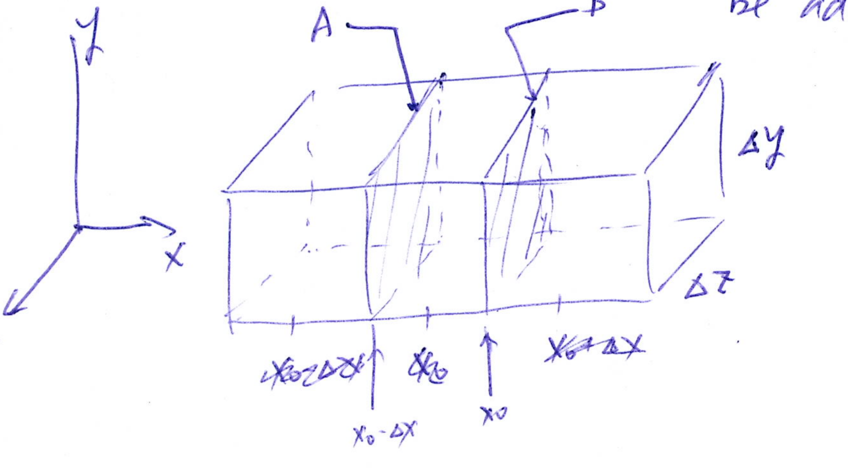
$G \rightarrow$ fluid velocity.

$$mn\frac{d\vec{u}}{dt} = qn(\vec{E} + \vec{v}\times\vec{B}) \Rightarrow mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = qn(\vec{E} + \vec{v}\times\vec{B})$$

? 3.2.5 first @ p66

? 3.2.2 The stress Tensor.

- Thermal motions $\rightarrow$ a pressure force needs to be added to



* Force arises from the random motion of particles in & out of a fluid element.

* Consider x component through the faces A & B

- #/ of particles per second through A w/ V_x.

$$\frac{\#}{\Delta t} = \cancel{\frac{n_v \cdot \Delta x \cdot \Delta y \cdot \Delta z}{\Delta t}} = n_v \cdot V_x \cdot \Delta y \cdot \Delta z$$

↑ #/ of particle per m^3 w/ V_x

$$n_v = V_x \iint f(v_x, v_y, v_z)\, dv_y \cdot dv_z$$

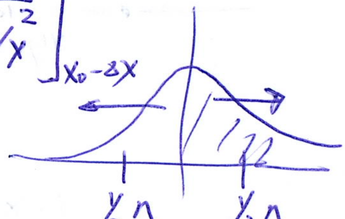
· Momentum of each particle:

$$P_{each} = m V_x$$

$$\Rightarrow P_{A+} = \sum m V_x = \sum_v \left[(n_v \cdot V_x \cdot \Delta y \cdot \Delta z) \cdot m V_x \right]$$

$$= \sum_v \left[n_v \cdot m V_x^2 \Delta y \, \Delta z \right]$$

$$= \Delta y \cdot \Delta z \cdot \left[\frac{1}{2} n \; m \overline{V_x^2} \right]_{x_0 - \Delta x}$$



$$P_{B+} = \Delta y \cdot \Delta z \cdot \left[\frac{1}{2} n \; m \overline{V_x^2} \right]_{x_0}$$

$$P_{A+} - P_{B+} = \Delta y \cdot \Delta z \cdot \frac{1}{2} m \left(\left[n \overline{V_x^2} \right]_{x_0 - \Delta x} - \left[n \overline{V_x^2} \right]_{x_0} \right)$$

$$= \Delta y \cdot \Delta z \cdot \frac{1}{2} m \, (-\Delta x) \, \frac{\partial}{\partial x} \left(n \overline{V_x^2} \right)$$

$$\left[n \overline{V_y^2} \right]_{x_0 - \Delta y} = \left[n \overline{V_x^2} \right]_{x_0} - \Delta x \frac{\partial}{\partial x} \left(n \overline{V_x^2} \right) + \dots$$

$$P_{A-} = \sum_v \left(n_v \cdot V_x \cdot \Delta y \cdot \Delta z \right)(-m V_x) = -\Delta y \Delta z \left[\frac{1}{2} n \cdot m \overline{V_x^2} \right]_{x_0 - \Delta x}$$

$$P_{B-} = -\Delta y \Delta z \left[\frac{1}{2} n \cdot m \overline{V_x^2} \right]_{x_0}$$

$$\cancel{P_B - P_A = -\Delta y \Delta z \frac{1}{2} m \left(\left[n \overline{V_y^2} \right]_{x_0} - \left[n \overline{V_x^2} \right]_{x_0 - \Delta x} \right)}$$

$$\cancel{= -\Delta y \cdot \Delta z \, \frac{1}{2} m}$$

$$P_{B-} - P_{A-} = -\Delta y \,\Delta z \left(\tfrac{1}{2} m\right)\left([n\overline{V_x^2}]_{x_0} - [n\overline{V_x^2}]_{x_0-\Delta x}\right)$$

$$= -\Delta y \,\Delta z \,\tfrac{1}{2} m \,(\Delta x)\, \frac{\partial}{\partial x}(n\overline{V_x^2})$$

$$\Rightarrow \Delta P_{Total} = (P_{A+} - P_{B+}) + (P_{B-} - P_{A-})$$

$$= -m \frac{\partial}{\partial x}(n\overline{V_x^2})\,\Delta x\,\Delta y\,\Delta z$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(n\cdot m\cdot u_x)\,\Delta x\cdot\Delta y\cdot\Delta z}_{\text{total } P} = -m\frac{\partial}{\partial x}(n\overline{V_x^2})\,\cancel{\Delta x\,\Delta y\,\Delta z}$$

~~Any~~ $V_x = u_x + V_{xr}$

$u_x = \overline{V_x}$ ← fluid velocity

V_{xr} ← random thermal velocity

In 1D: $\frac{1}{2} m \overline{V_{xr}^2} = \frac{1}{2} kT$

$$\Rightarrow \overline{V_x^2} = \overline{(u_x + V_{xr})^2} = \overline{u_x^2 + 2u_x V_{xr} + V_{xr}^2}$$

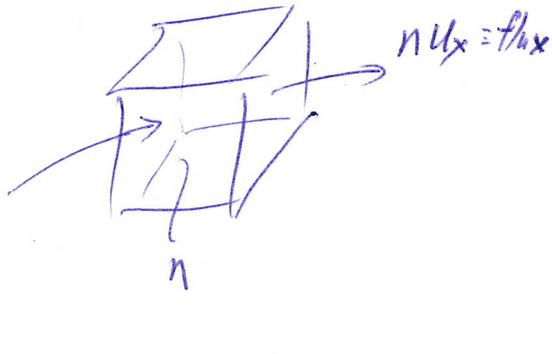
$$= u_x^2 + 2u_x \underbrace{\overline{V_{xr}}}_{=0} + \overline{V_{xr}^2} = u_x^2 + \frac{kT}{m}$$

$$\Rightarrow \frac{\partial}{\partial t}(n\, m\, u_x) = -m\frac{\partial}{\partial x}\left[n\left(u_x^2 + \frac{kT}{m}\right)\right]$$

$$mn\frac{\partial u_x}{\partial t} + \cancel{m u_x\frac{\partial n}{\partial t}} = \cancel{-m u_x\frac{\partial (n u_x)}{\partial x}} - m n u_x \frac{\partial u_x}{\partial x} - \frac{\partial}{\partial x}(nkT)$$

~~Con~~ mass conservation (continuity)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n u_x) = 0$$



$\Rightarrow$ Also, $p = nkT$

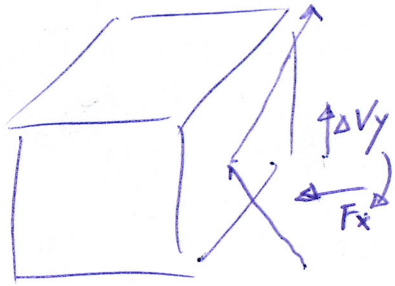
$$mn\frac{\partial u_x}{\partial t} + mnu_x\frac{\partial u_x}{\partial x} = \underbrace{-\frac{\partial P}{\partial x}}_{\text{pressure gradient force.}}$$

$$\Rightarrow mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = qn(\vec{E} + \vec{u}\times\vec{B}) - \nabla P$$

- It is a special case where ① transfer of x momentum by motion in the x direction.
  ② the fluid is isotropic, i.e., same results in y & z.

* Shear stress — can't be represented by a scalar p. but by a tensor $\overleftrightarrow{P}$, the stress tensor. whose component

$$P_{ij} = mn\overline{v_i v_j}$$



in this case $-\nabla p \rightarrow -\vec{\nabla}\cdot\overleftrightarrow{P}$

* Two simple cases:

- isotropic Maxwellian:

$$\overleftrightarrow{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \quad -\vec{\nabla}\cdot\overleftrightarrow{P} = (\partial_x, \partial_y, \partial_z)\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$
$$= (\partial_x p, \partial_y p, \partial_z p) = -\nabla p$$

- Two temperature $T_\perp, T_\parallel$ w/ $\vec{B}$.

$$\Rightarrow P_\perp = nkT_\perp, \quad P_\parallel = nkT_\parallel$$

isotropy $\perp \vec{B}$

$$\Rightarrow \overleftrightarrow{P} = \begin{pmatrix} \boxed{\begin{matrix} P_\perp & 0 \\ 0 & P_\perp \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & P_\parallel \end{pmatrix} \qquad -\nabla\cdot\overleftrightarrow{P} = -\nabla_\perp P_\perp - \nabla_\parallel P_\parallel$$

$$\overleftrightarrow{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

Off-diagonal elements → viscosity.

⇒ The resistance to shear flow

- The longer the mean free path, the farther momentum is carried, and the larger is the viscosity.

-

§ 3.2.3 Collisions.

→ If there is a neutral gas, the charged fluid will exchange momentum w/ it through collisions.

$$\Delta p \propto \vec{u} - \vec{u}_0$$

neutral fluid

$$F = \frac{\Delta p}{\Delta t} \sim -\frac{mn(\vec{u}-\vec{u}_0)}{\tau}$$

← Δp

← mean free time between collisions

$$\Rightarrow mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = qn(\vec{E} + \vec{u}\times\vec{B}) - \boxed{\nabla\cdot\overleftrightarrow{P}} - \frac{mn(\vec{u}-\vec{u}_0)}{\tau}$$

§ 3.2.4 Comparison w/ Ordinary Hydrodynamics

the same as NO $\vec{E}$. $\vec{B}$ & no collisions between different species (single specie)

Navier-Stokes eq: $\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = -\nabla p + \rho\nu\nabla^2\vec{u}$

~~kinetic~~ Kinematic viscosity coefficient

viscosity

assuming $\rho\nu\nabla^2\vec{u} = \nabla\cdot\overleftrightarrow{P} - \nabla p$ w/o magnetic field.

For fluid eq: frequent collisions between particles p65

For Plasma eq: w/o explicit statement of the collision rate

but still a good approximation:

$\because$ ① velocity distribution is assumed to be

Maxwellian $\leftarrow$ generally from freq. collision.

used only to take $\overline{v_x^2}$

$\Rightarrow$ Any other distribution w/ the same average would give us the same results.

Not very sensitive to Maxwellian.

~~② w/ freq. collisions, the particle comes~~

②: B field !!

when a particle is accelerated (by $\vec{E}$)

$\rightarrow$ allowed to free stream w/o collisions

$\rightarrow$ w/ freq. collisions, the particle comes to a limiting velocity $\propto \vec{E}$

i.e. $\vec{v} = \mu\vec{E}$ $\leftarrow$ mobility.

w/ B, $\rightarrow$ gyromotion, drift ($\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2}$)

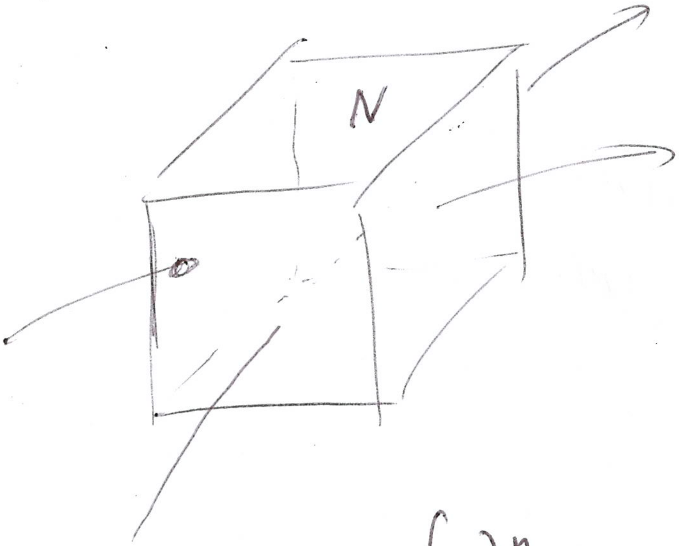
$\rightarrow$ a collisionless plasma behaves like a collisional fluid.

Nevertheless, particles do free-stream along $\vec{B}$,

$\rightarrow$ fluid picture is not suitable.

Conclusion: For motion perpendicular to $\vec{B}$, the fluid theory is a good approximation.

§ 3.2.5 Equation of Continuity



$$\frac{\partial N}{\partial t} = \underbrace{\int_V \frac{\partial n}{\partial t}\, dV}_{\#\text{ in the box}} = -\oint \underbrace{n\vec{u}}_{\text{flux}} \cdot d\vec{s} = -\int_V \nabla\cdot(n\vec{u})\, dV$$

divergence theory

⇒ it holds for any volume V.

$$\Rightarrow \quad \frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0 \quad (+\ \text{source} - \text{sink})$$

equation of continuity or
continuity equation.

§ 3.2.6 Equation of State.

- to closed the system of equations

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0$$

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = qn(\vec{E} + \vec{v}\times\vec{B}) - \nabla\cdot\overleftrightarrow{P} - \frac{mn(\vec{u}-\vec{u}_0)}{\tau}$$

① thermodynamic equation: (Adiabatic compression p67)

$\rho = \frac{M}{V} \to \rho V = M \equiv \text{const.}$

$$pV^{\gamma} = \text{const} \quad p = C_{\text{const}}\,\rho^{\gamma}, \quad \gamma \equiv \frac{C_p}{C_V}$$

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n} \qquad (\rho = n \cdot m)$$

$p = Cm^{\gamma} \cdot n^{\gamma}$

$$\frac{\nabla p}{p} = \frac{Cm^{\gamma} \cdot \gamma n^{\gamma-1} \nabla n}{Cm^{\gamma} n^{\gamma}} = \gamma \frac{\nabla n}{n}$$

② Isothermal compression:

$$\nabla p = \nabla(nkT) = kT\nabla n + n\nabla(\cancel{kT})$$

$$= \frac{kT\nabla n}{nkT}$$

nkT

treat as γ → $\gamma = 1$ $\quad \frac{\nabla p}{p} = \frac{\nabla n}{n}$

③ Adiabatic compression: kT will also change

$$\gamma = \frac{2+N}{N} \qquad \frac{\nabla p}{p} = \frac{kT\nabla n + n\nabla(kT)}{nkT}$$

* The validity of the equation of state requires that heat flow be negligible, i.e., thermal conductivity is low.

→ * more likely to be true in directions $\perp \vec{B}$ than $\parallel \vec{B}$

$$= \frac{\nabla n}{n} + \frac{\nabla(kT)}{kT}$$

$$\equiv \gamma \frac{\nabla n}{n}$$

3.2.7 The complete set of Fluid equation p68

$$\sigma = n_i q_i + n_e q_e$$

$$\vec{J} = n_i q_i \vec{V_i} + n_e q_e \vec{V_e}$$

$$\epsilon_0 \nabla\cdot\vec{E} = n_i q_i + n_e q_e \quad \leftarrow \quad \nabla\cdot\vec{E} = \frac{\sigma}{\epsilon_0}$$

$$\nabla\times\vec{E} = -\dot{\vec{B}} \qquad \nabla\cdot\underbrace{(\nabla\times\vec{E})}_{0} = -\nabla\cdot\dot{\vec{B}} = -\frac{\partial}{\partial t}(\nabla\cdot\vec{B})$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\cdot\vec{B} = 0$$

$$\mu_0^{-1}\nabla\times\vec{B} = \underbrace{n_i q_i \vec{V_i} + n_e q_e \vec{V_e}}_{\vec{J}} + \epsilon_0 \dot{\vec{E}}$$

$\nabla\cdot$

$$\nabla\cdot\frac{(\nabla\times\vec{B})}{\mu_0} = \nabla\cdot[n_i q_i \vec{V_i} + n_e q_e \vec{V_e}] + \epsilon_0 \nabla\cdot\dot{\vec{E}}$$

$$= q_i \nabla\cdot(n_i\vec{V_i}) + q_e \nabla\cdot(n_e\vec{V_e}) + \epsilon_0\nabla\cdot\dot{\vec{E}}$$

$$= -q_i\frac{\partial n_i}{\partial t} - q_e\frac{\partial n_e}{\partial t} + \epsilon_0\nabla\cdot\dot{\vec{E}}$$

$$= \frac{\partial}{\partial t}\left[-q_i n_i + q_e n_e) + \epsilon_0\nabla\cdot\vec{E}\right] = 0$$

$$m_i n_i\left[\frac{\partial \vec{V_j}}{\partial t} + (\vec{V_j}\cdot\nabla)\vec{V_j}\right] = q_j n_j(\vec{E} + \vec{V_j}\times\vec{B}) - \nabla P_j$$

(neglect collisions & viscosity)

$$\frac{\partial n_j}{\partial t} + \nabla\cdot(n_j\vec{V_j}) = 0$$

$$P_j = C_j n_j^{\gamma_j}$$

j = ion, electron.

Unknown: $n_i, n_e, P_i, P_e, \vec{V_i}, \vec{V_e}, \vec{E}, \vec{B}$ (16 unknown)

$\Rightarrow$ Eqs: 18 $\rightarrow$ 16 eqs

$\Rightarrow$ Self-consistent set of fluids.

§3.3 Fluid drifts perpendicular to B

- Fluid - composed of many individual particles
  → the fluid has drifts perpendicular to $\vec{B}$ if the individual guiding centers have such drift.
- ∇p → appears only in fluid.

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

v.s. $$mn\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = qn(\vec{E} + \vec{v}\times\vec{B}) - \nabla p$$

⇒ a drift associated w/ ∇p which does not occur for individual particles

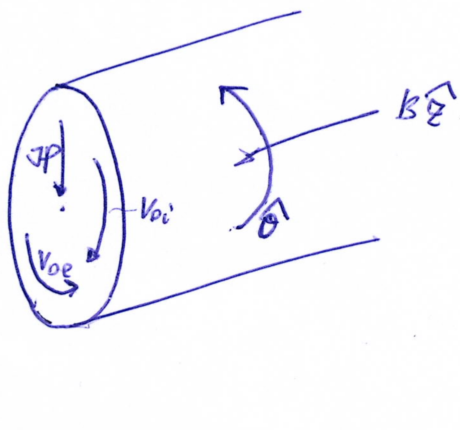
$$mn\left[\underset{①}{\frac{\partial\vec{u}}{\partial t}} + \underset{②}{(\vec{u}\cdot\nabla)\vec{u}}\right] = \underset{③}{qn(\vec{E} + \vec{v}\times\vec{B})} - \nabla p$$

$$\frac{①}{③} \approx \left|\frac{mn\, i\omega V_\perp}{qn V_\perp B}\right| \approx \frac{\omega}{\omega_c}$$

where $\frac{\partial}{\partial t} \to i\omega$. concerned only w/ $V_\perp$

$\because$ $\omega \ll \omega_c$, $\therefore$ ① is neglected.

* neglect $(\vec{u}\cdot\nabla)\vec{u}$ for now and varify later.
* let $\vec{E}$ & $\vec{B}$ uniform, but ∇n, $\nabla p \neq 0$
  this is a usual condition in MCF



$$0 = qn[\vec E + \vec v\times\vec B] - \nabla p \qquad \times\vec B$$

$$0 = qn[\vec E\times\vec B + (\vec v_\perp\times\vec B)\times\vec B] - \nabla p\times\vec B$$

$$\cancel{(\vec A\times\vec B)\times\vec C = (\vec A\cdot\vec C)\vec B - (\vec B\cdot\vec B)\vec C}$$

$$\cancel{(\vec v_\perp\times\vec B)\times\vec B = (\vec v_\perp\cdot\vec B)\vec B - (\vec v_\perp\cdot\vec B)\vec B}$$

$$\vec A\times(\vec B\times\vec C) = (\vec A\cdot\vec C)\vec B - (\vec A\cdot\vec B)\vec C$$

$$(\vec v_\perp\times\vec B)\times\vec B = -\vec B\times(\vec v_\perp\times\vec B) = -\left[(\vec B\cdot\vec B)\vec v_\perp - (\vec B\cdot\vec v_\perp)\vec B\right]$$

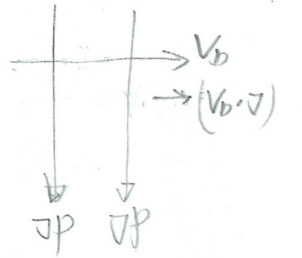
$$= -B^2\vec v_\perp + \overset{0}{(\vec B\cdot\vec v_\perp)}\vec B = -B^2\vec v_\perp$$

$$\Rightarrow 0 = qn[\vec E\times\vec B - B^2\vec v_\perp] - \nabla p\times\vec B$$

or $$\vec v_\perp = \underbrace{\frac{\vec E\times\vec B}{B^2}}_{\vec E\times\vec B\text{ drift}} - \underbrace{\frac{\nabla p\times\vec B}{qnB^2}}_{\text{Diamagnetic drift}} \equiv \vec v_E + \vec v_D$$

$\vec E\times\vec B$ drift

Diamagnetic drift.



~~∴ $\vec v_E$ & $\vec v_D$ ⊥~~

- For $\vec E = 0$, $\vec v = \vec v_D \perp \nabla p$

∇V should $\parallel \nabla p$

to verify neglecting ② term in p19 $\rightarrow$ ∴ $(\vec v\cdot\nabla)\vec v = 0$

$(\vec v_D\cdot\nabla)\vec v_D$ $\perp v_D$

Uniform $\perp\nabla p$ $\Rightarrow \vec v_D$ const $\perp \nabla p$

$\nabla p \parallel -\nabla\phi$

- For $\vec E \neq 0$, $\vec E = -\nabla\phi \neq 0$,

if $\nabla\phi$ & ∇p are in the same direction. $\Rightarrow \vec E \parallel \nabla p \Rightarrow \vec v = \vec v_E + \vec v_D \perp (\hat E, \nabla p)$

$\vec v_E = \frac{\vec E\times\vec B}{B^2}$

$\vec v_E \perp \nabla p \parallel \nabla V$

$\vec v \perp \nabla p \Rightarrow (\vec v\cdot\nabla)\vec v = 0$

i.e. $\vec v \perp \nabla V$ $\quad (\vec v_E\cdot\nabla)\vec v_E$ $\perp \vec v_E$

$p = nkT$

Note that $p = Cn^\gamma \Rightarrow \frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}$

$$\Rightarrow \vec v_D = -\frac{\nabla p\times\vec B}{qnB^2} = -\gamma\frac{p}{n}\frac{\nabla n\times\vec B}{qnB^2} = \gamma\frac{p}{n}\frac{\left(\frac{\vec B}{B}\right)\times\nabla n}{qnB} = \gamma\frac{p}{n}\frac{\hat z\times\nabla n}{qnB}$$

isothermal
$p = nkT$

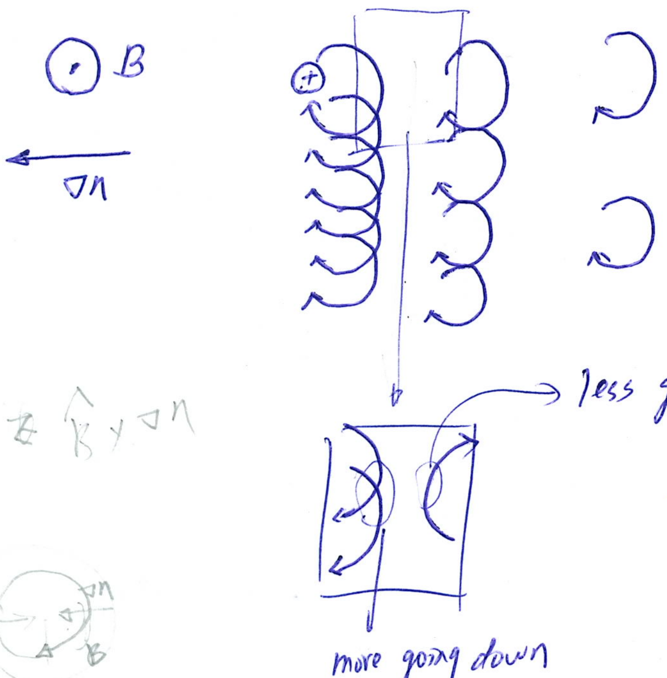
$$\longrightarrow = \gamma\frac{kT}{qB}\frac{\hat z\times\nabla n}{n} = \pm\gamma\frac{kT}{eB}\frac{\hat z\times\nabla n}{n}$$

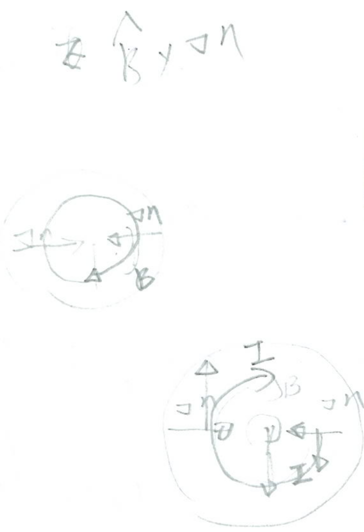
$\nabla n = n' \hat{r}$

$\Rightarrow \quad V_{Di} = \dfrac{KT_i}{eB} \dfrac{n'}{n} \hat{\theta} \qquad \left(n' \equiv \dfrac{\partial n}{\partial r} < 0 \right)$

$V_{De} = -\dfrac{KT_e}{eB} \underbrace{\dfrac{n'}{n}}_{} \hat{\theta} \longrightarrow \dfrac{1}{n}\dfrac{\partial n}{\partial r} \equiv \dfrac{1}{\Lambda}$ ↳ density scale length

$\Rightarrow \quad V_D = \dfrac{KT(eV)}{B} \cdot \dfrac{1}{\Lambda} \left(\dfrac{m}{sec} \right)$





$\Rightarrow$ More ions moving downward than upward, since the downward-moving ions come from a region of higher density.

Therefore, a fluid drift perpendicular to ∇n and B even though the guiding centers are stationary.

$\star$ V_D dependent on q

~~independent on ...~~

$\rightarrow$ diamagnetic current, for $q = Z = 1$

$J_D = ne(\vec{V}_{Di} - \vec{V}_{De}) = (KT_i + KT_e) \dfrac{\vec{B} \times \nabla n}{B^2}$

$\rightarrow$ particle picture : NO current if the guiding center do NOT drift

fluid picture : current J_D flows whenever there is a ∇p

Curvature drift , gravitational drift
∇B drift
(nonuniform E field drift) → complicated

* Curvature drift :

The curvature drift exists in the fluid ~~model~~ picture since the centrifugal force is felt by all particles in a fluid element as they move around a bend in the magnetic field.

$$F_{cf} = \frac{m v_{\parallel}^2}{R_c}\hat{r} \longrightarrow \overline{F}_{cf} = \frac{n m \overline{v_{\parallel}^2}}{R_c}\hat{r}$$

$$= \frac{n k T_{\parallel}}{R_c}\hat{r}$$

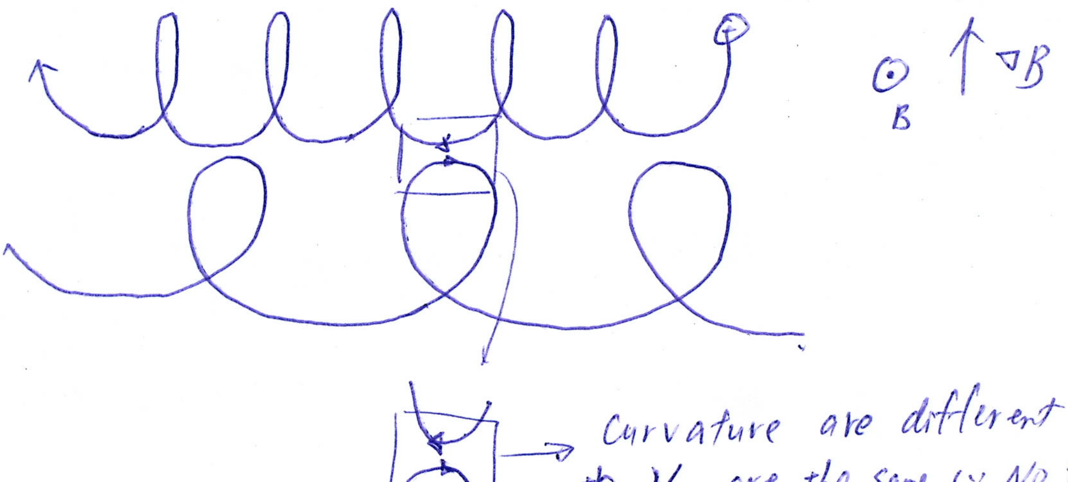
$$\Rightarrow \vec{V}_R = \frac{n k T_{\parallel}}{n q}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} = \frac{k T_{\parallel}}{q}\frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$$

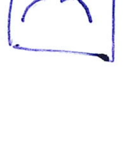
* Gravitational drift

$$\vec{F} = m\vec{g} \rightarrow \vec{F} = M n \vec{g},$$

$$V_g = \frac{\vec{F}}{nq}\frac{\times \vec{B}}{B^2} = \frac{Mn}{nq}\frac{\vec{g}\times\vec{B}}{B^2} = \frac{M}{q}\frac{\vec{g}\times\vec{B}}{B^2}$$

* ∇B drift : does NOT exist for fluids





→ Curvature are different but the $V_\perp$ are the same ($\because$ No $\vec{E}$) & # of particles are the same
⇒ perfect cancellation.

3.4 Fluid drifts parallel to B

$\vec{B} = B\hat{z}$

$$mn\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = qn(\vec{E} + \vec{v}\times\vec{B}) - \nabla p$$

For $\hat{z}$ component, (v_z)

$$mn\left[\frac{\partial v_z}{\partial t} + \cancel{(\vec{v}\cdot\nabla) v_z}\right] = qn\left(E_z + \cancel{(\vec{v}\times\vec{B})_z}\right) - \frac{\partial p}{\partial z}$$

for $\ll \frac{\partial v_z}{\partial t}$, e.g., v_z is spatially uniform

$\because \vec{v}\times\vec{B} \perp \vec{B} = \hat{z}$ $\therefore (\vec{v}\times\vec{B})_z = 0$

$$\Rightarrow \frac{\partial v_z}{\partial t} = \frac{q}{m}E_z - \frac{1}{mn}\frac{\partial p}{\partial z}$$

For $\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$ & $p = nkT$

$$\frac{1}{mn}\frac{\partial p}{\partial z}\frac{p}{p} = \frac{1}{mn}\frac{\partial n}{\partial z}\gamma\frac{p}{n} = \frac{1}{mn}\frac{\partial n}{\partial z}\cdot\gamma\frac{nkT}{n}$$

$$= \frac{\gamma kT}{mn}\frac{\partial n}{\partial z}$$

$$\Rightarrow \frac{\partial v_z}{\partial t} = \frac{q}{m}E_z - \frac{\gamma kT}{mn}\frac{\partial n}{\partial z}$$

→ The fluid is accelerated along $\vec{B}$ under the combined electrostatic and pressure gradient forces.

for $m \to 0$ (m_e). $\frac{\partial v_z}{\partial t} \ll$ the other two terms $\Rightarrow qE_z \sim \frac{\gamma kT}{n}\frac{\partial n}{\partial z}$.

Let $q = -e$, $E = -\nabla\phi = -\frac{\partial \phi}{\partial z}$

$$\Rightarrow e\frac{\partial \phi}{\partial z} = \frac{\gamma kT_e}{n}\frac{\partial n}{\partial z}$$

- Electrons are so mobile that their heat conductivity is almost infinite. $\Rightarrow$ assume isothermal electron, $\gamma = 1$

$$e\int d\phi = \underset{=1}{\gamma} kT_e \int \frac{dn}{n} \Rightarrow e\phi = \underset{=1}{\gamma} kT_e \ln \frac{n}{n_0}$$

$$= kT_e \ln n + \underset{-kT_e \ln n_0}{C}$$

$$\Rightarrow \underline{n = n_0 \exp\left(\frac{e\phi}{kT_e}\right)}$$

Boltzmann relation for electrons !!

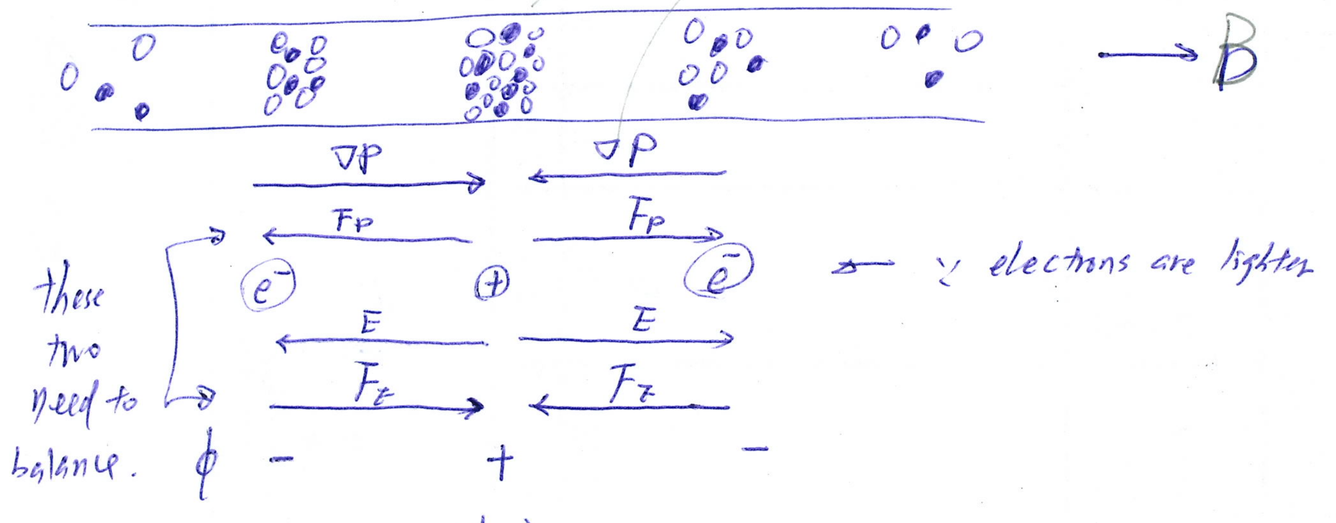
* physical picture:
  - electron → accelerated to high energy very quickly w/ force ($\frac{\partial v_z}{\partial t} \approx 0$)
  - → ions are left behind
  - → electrostatic & pressure gradient forces on (e^- must be closely in balance)

$$\frac{q}{m}E_z \longleftrightarrow \frac{1}{mn}\frac{\partial p}{\partial z}$$

balance

for both e^- & ion

KT const.



$$\Rightarrow n = n_0 \exp\left(\frac{e\phi}{kT_e}\right)$$

* There is enough charge to set up the $\vec{E}$ field required to balance the forces on the electrons.

§3.5 The plasma approximation.

~~To~~ Poisson's eq: $\nabla \cdot \vec{E} = \frac{\sigma}{\epsilon_0}$ ← from $\sigma \to \vec{E}$

In plasma: $\begin{cases} \vec{E} \text{ is found from the equations of motion} \\ \text{Poisson's eq is to find } \sigma \end{cases}$

∵ plasma has tendency to remain <u>neutral</u>.

$\vec{E}$ must adjust itself so that the orbits of the electrons and ions preserve neutrality.

$E \to \sigma$ → ~~the~~ {The charge density is of secondary importance; it will adjust itself so that Poisson's eq. is satisfied.

- True only for low-freq. motions.

- In plasma, $n_i = n_e$ and $\nabla \cdot \vec{E} \neq 0$!!
  - ↳ plasma approximation.
  - ↳ Do NOT use Poisson's eq. to obtain $\vec{E}$ unless it is unavoidable.
  - ↳ as long as motions (wave) are slow enough, that both ions and electrons have time to move, it is a good approximation to $\nabla \cdot \vec{E} = \frac{\sigma}{\epsilon_0} \rightarrow n_e = n_i$
  - ↳ If only one species can move and the other can NOT follow, e.g., high freq. electron waves, plasma approximation is not valid. ⇒ find $\vec{E}$ from Maxwell's eq. not from eq. of motion
  - ↳ For ion waves, low freq. ⇒ good for plasma approximation.