§2. Single-Particle Motions

* The first step is to understand how single particles behave in <u>electric</u> and <u>magnetic</u> fields.
* In this chapter, $\vec{E}$ & $\vec{B}$ are assumed to be prescribed and not affected by the charged particles

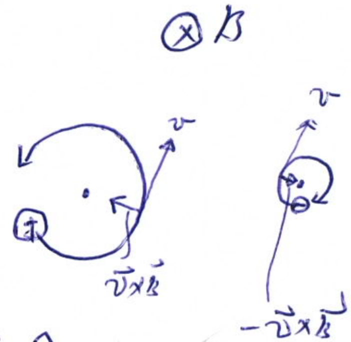
* Uniform $\vec{E}$ & $\vec{B}$
  - $\vec{E} = 0$, $\vec{B} = \text{const}$. - gyromotion
  - $\vec{E} = \text{const}$, $\vec{B} = \text{const}$. - $\vec{E} \times \vec{B}$ drift
  - $\vec{F}$ - gravitational field.
* Nonuniform $\vec{B}$ $(\vec{E} = 0)$
  - $\nabla B \perp \vec{B}$ - Grad-B drift.
  - curved B - curvature drift 
* Nonuniform $\vec{E}(x)$ $(\vec{B} = \text{const})$ (in space)
* Time-varying $\vec{E}$ ⟵ $B = \text{const}$ ($\vec{E}, \vec{B}$ uniform in space ~~$\vec{B}(x) = \text{const}(x)$~~)
* Time-varying both B, $\vec{E}$
* Adiabatic invariants
$$\begin{cases} \mu = \dfrac{m v_\perp^2}{2B} \\ J = \int_a^b v_\parallel \, ds \\ \Phi = \int \vec{B} \cdot d\vec{a} \end{cases}$$

§2.1. Uniform $\vec{E}$ and $\vec{B}$.

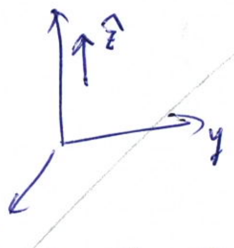
§2.1.1 $\vec{E}=0$

– cyclotron gyration.

$$m\frac{d\vec{v}}{dt}=q(\vec{v}\times\vec{B})$$



Let $\vec{B}=B\hat{z}$



p30(b)~
p30(c)

$$\begin{cases} m\dot{v}_x = qBv_y \\ m\dot{v}_y = -qBv_x \\ m\dot{v}_z = 0 \end{cases}$$

$$m\ddot{v}_x = qB\dot{v}_y = -\frac{q^2B^2}{m}v_x \Rightarrow \ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x$$

$$m\ddot{v}_y = -qB\dot{v}_x = -\frac{q^2B^2}{m}v_y \Rightarrow \ddot{v}_y = -\left(\frac{qB}{m}\right)^2 v_y$$

$$\omega_c \equiv \frac{|q|B}{m}$$ – cyclotron freq.

$$\Rightarrow v_{x,y} = v_\perp \exp(\pm i\omega_c t + i\delta_{x,y})$$ ← only the real part

Choose the phase δ so that

$$\Rightarrow \begin{cases} v_x = v_\perp \exp(i\omega_c t) = \dot{x} \\ v_y = \frac{m}{qB}\dot{v}_x = \pm\frac{1}{\omega_c}\dot{v}_x = \pm i v_\perp e^{i\omega_c t} = \dot{y} \end{cases}$$

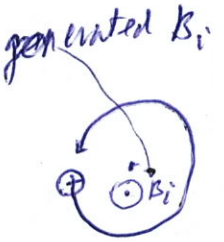
$$\Rightarrow \begin{cases} x = x_0 - i\frac{v_\perp}{\omega_c}e^{i\omega_c t} \\ y = y_0 \pm \frac{v_\perp}{\omega_c}e^{i\omega_c t} \end{cases} \Rightarrow \underline{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}}$$

$$\underset{\text{real part}}{\Longrightarrow} \begin{cases} x = x_0 + r_L\sin(\omega_c t) \\ y = y_0 \pm r_L\cos(\omega_c t) \end{cases}$$

$m\dot{v}_z = 0 \Rightarrow v_z = \text{const}, \quad z = v_z t.$ pro a

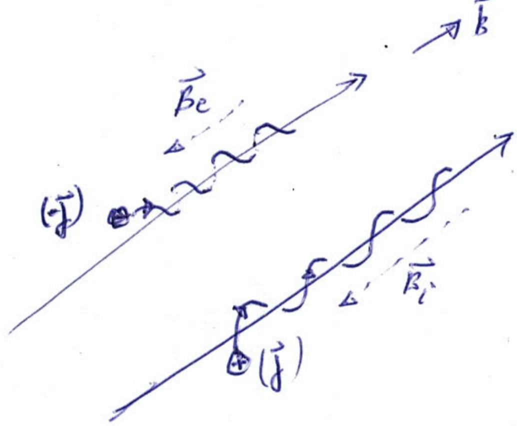
$$\begin{cases} x = x_0 + r_L \sin(\omega_c t) \\ y = y_0 \pm r_L \cos(\omega_c t) \\ z = v_z t \end{cases} \qquad \begin{cases} \omega_c = \dfrac{|q|B}{m} \\ r_L = \dfrac{v_\perp}{\omega_c} = \dfrac{m v_\perp}{|q|B} \end{cases}$$

$\otimes B$



$v_z = 0$

B_e, generated B_e.

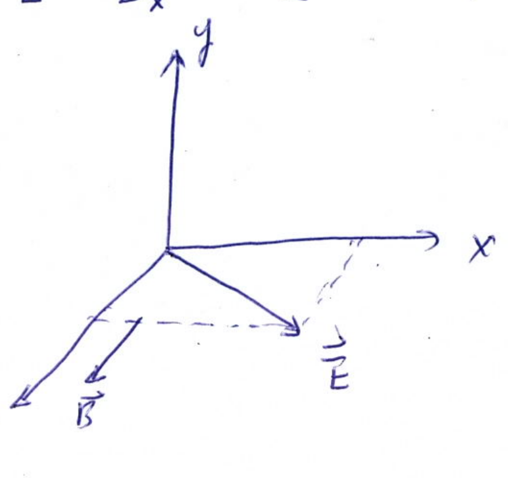


$\Rightarrow$ helix

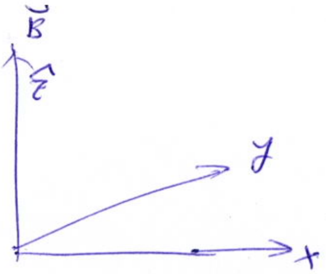
* The direction of the gyration is alway such that the magnetic field generated by the charged particle is opposite to the externally imposed field. Therefore, plasma particles tend to reduce the magnetic field $\Rightarrow$ diamagnetic.

§ 2.1.2 Finite E

$\vec{E} = \vec{E_x} + \vec{E_z}, \quad \vec{B} = \vec{B_z}$



To derive the gyromotion again.



pro

Let $\vec{B} = B\hat{z}$

$$m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})$$ q can be either positive or negative.

$$\begin{cases} m\dot{v}_x = qBv_y & \Rightarrow \quad \dot{v}_x = \frac{qB}{m}v_y \\ m\dot{v}_y = -qBv_x & \Rightarrow \quad \dot{v}_y = -\frac{qB}{m}v_x \\ m\dot{v}_z = 0 & \Rightarrow \quad \underline{\dot{v}_z = 0, \quad z = v_z t + z_0} \end{cases}$$

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = \frac{qB}{m}\left(-\frac{qB}{m}\right)v_x = -\left(\frac{qB}{m}\right)^2 v_x, \qquad \tilde{\omega} = \frac{qB}{m}$$

$\tilde{\omega}$ can be positive or negative.

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\frac{qB}{m}\left(\frac{qB}{m}\right)v_y = -\left(\frac{qB}{m}\right)^2 v_y$$

$$\Rightarrow \begin{cases} \ddot{v}_x = -\tilde{\omega}^2 v_x \\ \ddot{v}_y = -\tilde{\omega}^2 v_y \end{cases} \Rightarrow \quad \tilde{v}_x = v_\perp e^{\pm i\tilde{\omega}t}, \quad v_x = Re\{\tilde{v}_x\},\ v_y = Re\{\tilde{v}_y\}$$

$$\tilde{v}_y = \frac{m}{qB}\dot{\tilde{v}}_x = \frac{1}{\tilde{\omega}}v_\perp(\pm i\tilde{\omega})e^{\pm i\tilde{\omega}t} = \underline{\pm i v_\perp e^{\pm i\tilde{\omega}t}}$$

$$\Rightarrow \quad v_x = Re\{\tilde{v}_x\} = Re\{v_\perp e^{\pm i\tilde{\omega}t}\} = Re\{v_\perp(\cos\tilde{\omega}t \pm i\sin\tilde{\omega}t)\}$$

$$= v_\perp\cos(\tilde{\omega}t)$$

$$v_y = Re\{\tilde{v}_y\} = Re\{\pm i v_\perp e^{\pm i\tilde{\omega}t}\} = Re\{v_\perp(\pm i)(\cos\tilde{\omega}t \pm i\sin\tilde{\omega}t)\}$$

$$= Re\{v_\perp(-\sin\tilde{\omega}t \pm i\cos\tilde{\omega}t)\}$$

$$= -v_\perp\sin(\tilde{\omega}t)$$

~~for ion~~. define $\omega = \frac{|q|B}{m}$. for ion, $\omega = \tilde{\omega}$

for electron, $\omega = \frac{-|q|B}{m} = -\tilde{\omega}$

ω always positive.

$$\Rightarrow \text{For ion: } \begin{cases} v_x = v_\perp\cos\omega t \\ v_y = -v_\perp\sin\omega t \end{cases} \qquad \Rightarrow \quad \begin{cases} v_x = v_\perp\cos(\omega t) \\ v_y = \mp v_\perp\sin(\omega t) \end{cases}$$

$$\text{For electron: } \begin{cases} v_x = v_\perp\cos(-\omega t) = v_\perp\cos\omega t \\ v_y = -v_\perp\sin(-\omega t) = v_\perp\sin\omega t \end{cases}$$

p 7c

$$x = \int v_x\, dt = \int v_\perp \cos \omega t\, dt = \frac{v_\perp}{\omega} \sin \omega t + x_0$$

$$\equiv r_\perp \sin \omega t + x_0, \qquad r_\perp \equiv \frac{v_\perp}{\omega}.$$

$$y = \int v_y\, dt = \int \mp v_\perp \sin \omega t\, dt = \pm \frac{v_\perp}{\omega} \cos \omega t + y_0$$

$$= \pm r_\perp \cos \omega t + y_0$$

$$\Rightarrow \begin{cases} x = r_\perp \sin(\omega t) + x_0 \\ y = \pm r_\perp \cos(\omega t) + y_0 \end{cases} \qquad \begin{cases} v_x = v_\perp \cos(\omega t) \\ v_y = \mp v_\perp \sin(\omega t) \end{cases}$$

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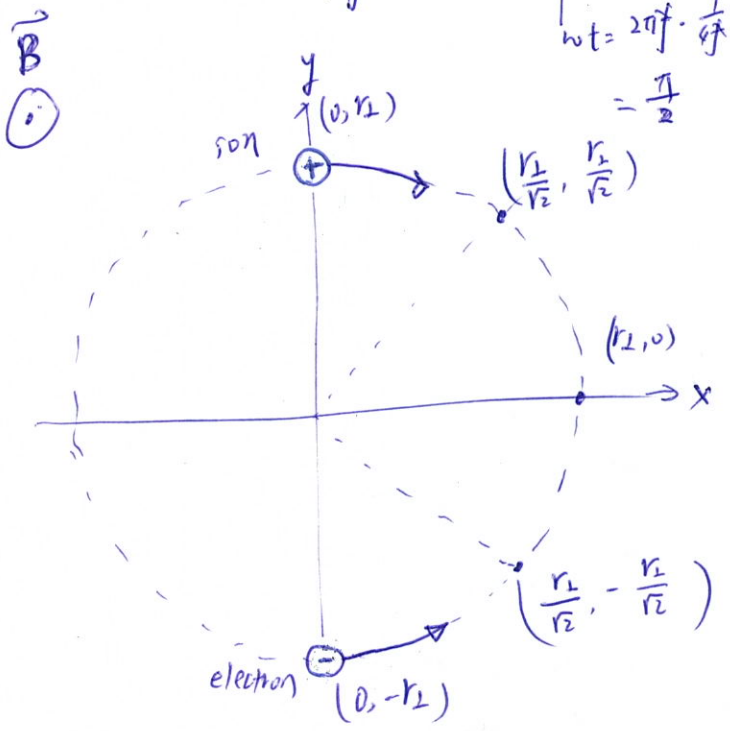
the upper sign is for ion.

Let $x_0 = y_0 = 0$ $\qquad \begin{cases} x = r_\perp \sin(\omega t) \\ y = \pm r_\perp \cos(\omega t) \end{cases}$

for $t = 0$, $\begin{cases} x = 0, \\ y = \pm r_\perp \end{cases}$ $\qquad \begin{cases} t = \frac{T}{4} = \frac{1}{4f} \\ \omega t = 2\pi f \cdot \frac{1}{4f} = \frac{\pi}{2} \end{cases}$ $\quad x = r_\perp \quad y = 0$ $\qquad t = \frac{T}{8} \quad \omega t = \frac{\pi}{4}$ $\qquad x = \frac{r_\perp}{\sqrt{2}} \quad y = \pm \frac{r_\perp}{\sqrt{2}}$



$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = \hat{x}(B\cdot v_y) + \hat{y}(-B\cdot v_x)$$

$$\begin{cases} m\frac{dv_x}{dt} = q(E_x + Bv_y) \\ m\frac{dv_y}{dt} = q(\quad -Bv_x) \\ m\frac{dv_z}{dt} = qE_z \quad \Rightarrow \quad v_z = \frac{qE_z}{m}\cdot t + v_{z0} \end{cases}$$

$$\frac{dv_x}{dt} = \frac{q}{m}E_x + \frac{qB}{m}v_y = \frac{q}{m}E_x \pm \omega_c v_y$$

$$\frac{dv_y}{dt} = -\frac{qB}{m}v_x = \mp\omega_c v_x \qquad \omega_c = \frac{|q|B}{m}$$

$$\ddot{v}_x = \pm\omega_c\dot{v}_y = -\omega_c^2 v_x$$

$$\ddot{v}_y = \mp\omega_c\dot{v}_x = \mp\omega_c\left(\frac{q}{m}\frac{B}{B}E_x \pm \omega_c v_y\right) = -\omega_c^2\left(v_y + \frac{E_x}{B}\right)$$

let $v_y' = v_y + \frac{E_x}{B} \Rightarrow \dot{v}_y' = \dot{v}_y \;;\; \ddot{v}_y' = \ddot{v}_y$

$$\Rightarrow \begin{cases} \ddot{v}_x = -\omega_c^2 v_x \Rightarrow v_x = v_\perp e^{i\omega_c t} \\ \ddot{v}_y' = -\omega_c^2 v_y' \Rightarrow v_y' = \pm i v_\perp e^{i\omega_c t} \end{cases}$$

$$\Rightarrow v_y = \pm i v_\perp e^{i\omega_c t} \underbrace{- \frac{E_x}{B}}_{\text{a drift } V_{gc} \text{ of the guiding center}}$$

$$\begin{cases} V_x = V_\perp e^{i\omega_c t} \\ V_y = \pm i \underbrace{V_\perp e^{i\omega_c t}}_{\vec{V}_{gy}\text{ - gyromotion}} - \underbrace{\frac{E_x}{B}}_{\vec{V}_{gc}\text{ - guiding center 產 drift of}} \end{cases}$$

Alternative way to derive:

$\frac{d\langle \vec{V}_{gc}\rangle}{dt} = 0 \quad \therefore \vec{V}_{gc} = \text{const}$

$$\vec{V} \equiv \vec{V}_{gc} + \vec{V}_{gy}$$

$$m\frac{d\vec{v}}{dt} = m\frac{d(\vec{V}_{gc} + \vec{V}_{gy})}{dt} = m\frac{d\vec{V}_{gy}}{dt} = q\left[\vec{E} + (\vec{V}_{gc} + \vec{V}_{gy})\times\vec{B}\right]$$

Take the time average of one cycle. $\langle m\frac{d\vec{V}_{gy}}{dt}\rangle = 0$

$\langle \vec{V}_{gy}\rangle = 0$

$$\Rightarrow \quad \vec{E} + \vec{V}_{gc}\times\vec{B} = 0 \qquad \times\vec{B}$$

$$\vec{E}\times\vec{B} + (\vec{V}_{gc}\times\vec{B})\times\vec{B} = 0$$

$$\Rightarrow \quad \vec{E}\times\vec{B} = \vec{B}\times(\vec{V}_{gc}\times\vec{B}) = (\vec{B}\cdot\vec{B})\vec{V}_{gc} - (\vec{B}\cdot\vec{V}_{gc})\vec{B}$$

$$= \vec{V}_{gc}B^2 - \vec{B}(\vec{V}_{gc}\cdot\vec{B})$$

Note that $\vec{B} = B\hat{z}$, i.e. $\vec{E}\times\vec{B} \perp \hat{z}$

let $\vec{V}_{gc} = V_{gcz}\hat{z} + \vec{V}_{gc\perp}$

$$\Rightarrow \quad \underbrace{\vec{E}\times\vec{B}}_{\perp\hat{z}} = \cancel{V_{gcz}B^2\hat{z}} + \vec{V}_{gc\perp}B^2 - \cancel{B^2 V_{gcz}\hat{z}}$$

$$\Rightarrow \quad \vec{V}_{gc\perp} = \frac{\vec{E}\times\vec{B}}{B^2} \equiv \vec{V}_E \; ; \quad V_E = \frac{E(V/m)}{B(T)}\ \frac{m}{sec}$$

P22
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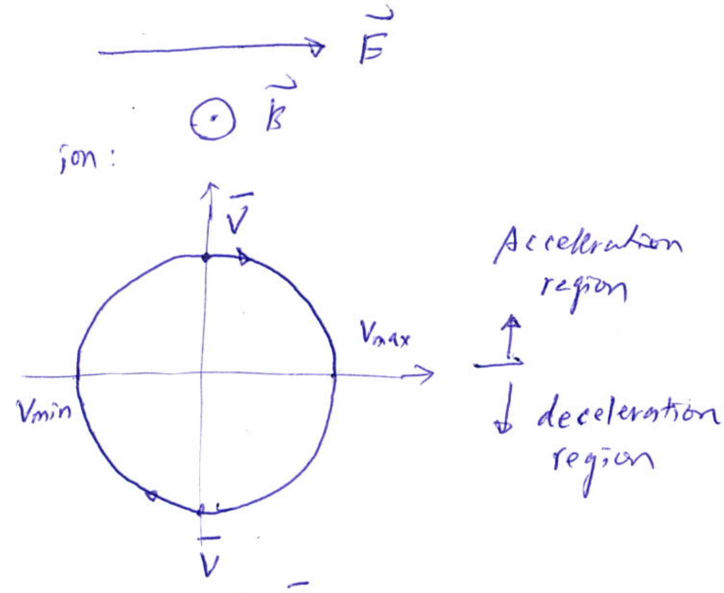
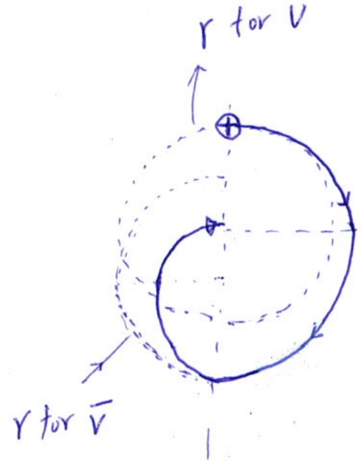
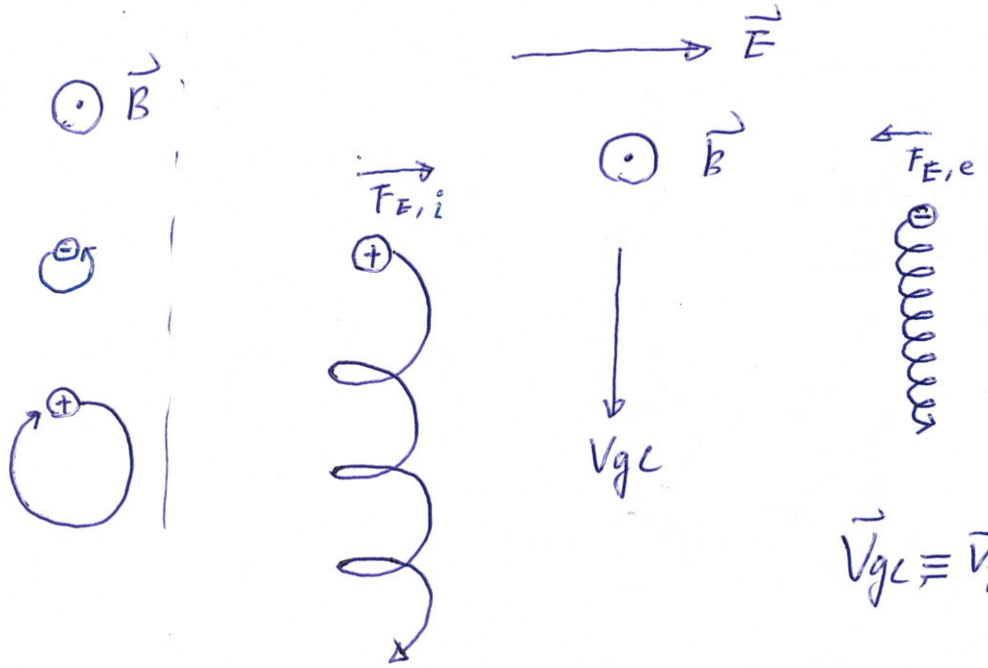


E
B
ion :
V
Vmax
Vmin
V
Acceleration region
deceleration region



r for V
r for V

V_E is independent of $q, m, v_\perp$!!



$\vec{V}_{gc} \equiv \vec{V}_E$ is called the $\vec{E}\times\vec{B}$ drift

$$\omega_c = \frac{|q|B}{m}$$

$$r_L = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$$

$$\left.\begin{array}{l} \omega_i < \omega_e \\ r_i > r_e \end{array}\right\} \rightarrow \text{canceled out}$$

Two effects

p39-II →

§ 2.1.3 Gavitational Field

$$\vec{F}_{em} = q(\vec{E} + \vec{v}\times\vec{B}) = q\vec{E} + q\,\vec{v}\times\vec{B} \equiv \underset{\uparrow\ \text{electrical force}}{\vec{F}_E} + q\,\vec{v}\times\vec{B}$$

$\vec{F}_E$ can be any kind of force

i.e. $m\frac{d\vec{v}}{dt} = \vec{F} + q\,\vec{v}\times\vec{B} \longleftrightarrow m\frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$

$$\Rightarrow \vec{V}_f = \frac{(\vec{F}/q)\times\vec{B}}{B^2} \longleftarrow \vec{V}_E = \frac{\vec{E}\times\vec{B}}{B^2}$$

$$= \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2}$$

1. all particles drift with the same velocity ~~P235~~b irrespective of their charge, mass, or energy. P234-II

2. $\vec{E}\times\vec{B}$ drift produces NO net current density

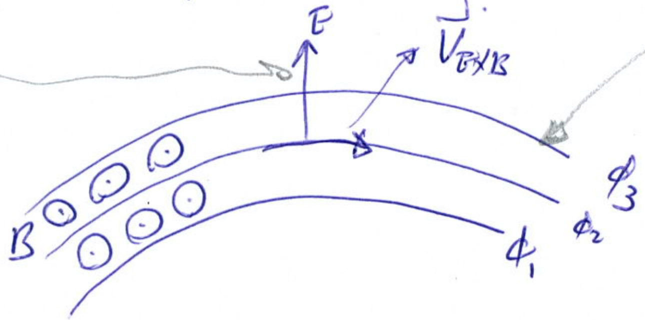
3. $\vec{E}\times\vec{B}$ drift velocity is always along a contour of const. electrostatic potential,

∵ contours of constant electrostatic potential are perpendicular to $\vec{E}$.

$$v_E = \frac{\vec{E}\times\vec{B}}{B^2}$$

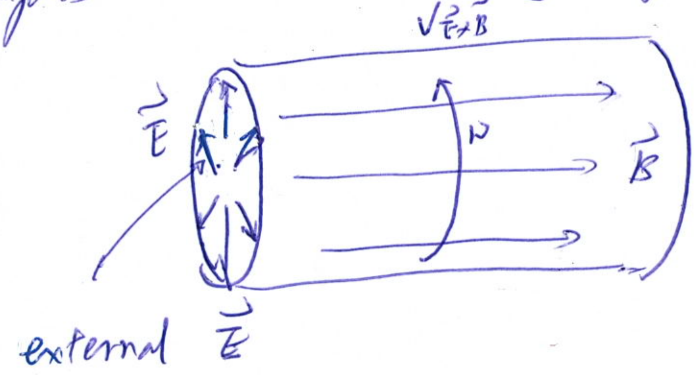
$$\vec{E} = -\nabla\phi$$

$$= -\frac{\nabla\phi\times\vec{B}}{B^2}$$



Examples :

generated by imposing a radial electric field.



If $\vec{F}$ is the force of gravity, i.e.,

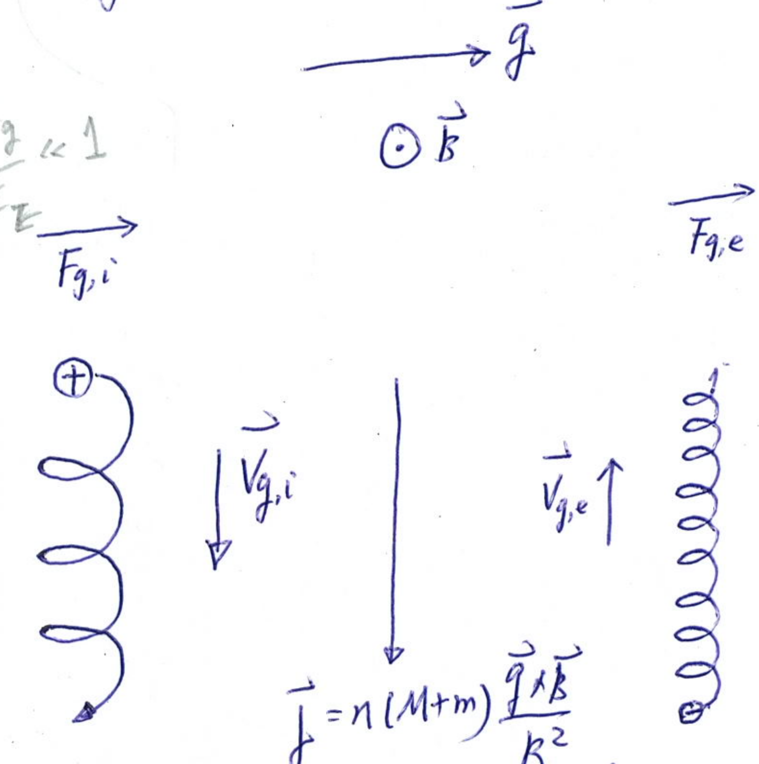
$$\vec{F} = m\vec{g}$$

$$\Rightarrow \quad \underline{\vec{V}_g = \frac{m}{q}\,\frac{\vec{g}\times\vec{B}}{B^2}} \quad \text{gravitational drift}$$

$\vec{V}_{E\times B} = \frac{\vec{E}\times\vec{B}}{B^2}$

$\dfrac{|V_g|}{|V_{E\times B}|} = \dfrac{\frac{m}{q}\frac{\vec{g}B}{B^2}}{\frac{EB}{B^2}} = \dfrac{m\vec{g}}{qE} = \dfrac{F_g}{F_E} \ll 1$

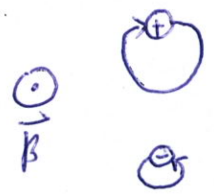
* drift $\vec{V}_g$ changes sign with the particles charge.



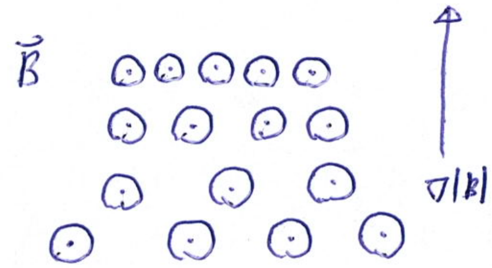
* Under a gravitational force, ions and electrons drift in opposite directions, so there is a net current density in the plasma.

* $\vec{V}_g$ is usually negligible. (problem 2-6)

§ 2.2 Nonuniform $\vec{B}$ ($\vec{E}=0$)

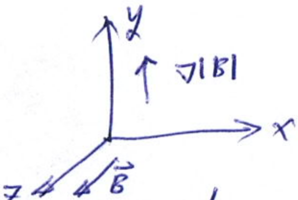


$\vec{E}=0$



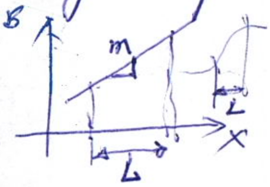
$\hat{B}=\hat{z}$

$\nabla|B| \parallel \hat{y}$



* When we introduce inhomogeneity, the problem becomes too complicated to solve exactly. "Orbit Theory", which is customary to expand in the small ratio r_L/L, is used. L is the scale length of the inhomogeneity.

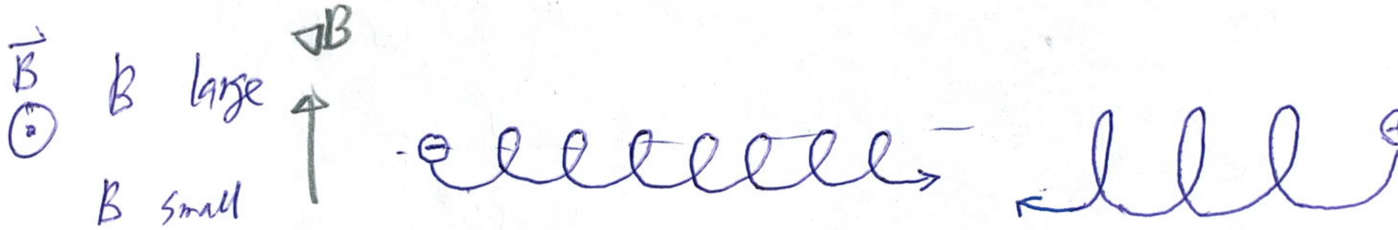
↳ $L \equiv \left(\frac{1}{B}\nabla B\right)^{-1} =$



§ 2.2.1 $\nabla B \perp \vec{B}$: Grad-B drift

* Lines of force are straight, but their density increases.

$$r_L = \frac{m v_\perp}{|q| B} \propto \frac{1}{B}$$

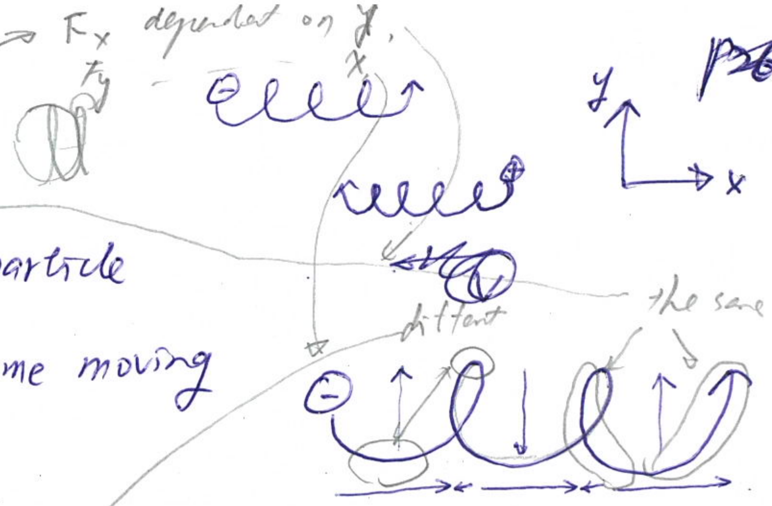


$\Rightarrow V_{drift} \propto v_\perp, \frac{r_L}{L}$

% in time ↑ % in space ↑

$\uparrow \nabla B$ B

$$\vec{F} = q\,\vec{v}\times\vec{B}$$



average

* $\bar{F}_x = 0$ since the particle spends as much time moving up as down.

* To calculate $\bar{F}_y$, use the "undisturbed orbit" of the particle to find the average.

i.e. pure gyromotion.

pure gyromotion
little diff time

$$\begin{cases} V_x = V_\perp e^{i\omega_c t} \\ V_y = \pm i V_\perp e^{i\omega_c t} \end{cases} \underset{\text{real part}}{\Rightarrow} \begin{aligned} V_x &= V_\perp \cos(\omega_c t) \\ V_y &= \mp V_\perp \sin(\omega_c t) \end{aligned}$$

$$\vec{F} = q\,\vec{v}\times\vec{B} = q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & 0 \\ 0 & 0 & B \end{vmatrix} = \hat{x}(q V_y B) + \hat{y}(-q V_x B)$$

Consider $\bar{F}_y$ only. $\to F_x = \mp q V_\perp \sin(\omega_c t)\left[B_0 \pm r_\perp \cos(\omega_c t)\frac{\partial B}{\partial y}\right]$

$\Rightarrow \frac{1}{T}\int_0^T \sin(\omega_c t)\cos(\omega_c t)\,dt = 0$ $\because \nabla B \parallel \hat{y}$.

$F_x = q V_y B = \mp q V_\perp \sin(\omega_c t)\, B(y)$

$$F_y = -q V_x B = -q\, V_\perp \cos(\omega_c t)\cdot B(y)$$

$$\vec{B} = \vec{B}_0 + (\vec{r}\cdot\nabla)\vec{B} + \cdots$$

$$B_z(y) = B_0 + y\cdot\frac{\partial B}{\partial y} + \cdots \qquad y = y_0 \pm r_\perp \cos(\omega_c t) \equiv \pm r_\perp \cos(\omega_c t)$$

$y_0 = 0$

Next page

$$\Rightarrow F_y = -q V_\perp \cos(\omega_c t)\left[B_0 \pm \underbrace{r_\perp \cos(\omega_c t)}_{y}\frac{\partial B}{\partial y}\right]$$

Note that $\frac{r_L}{L} \ll 1$, L is the scale length

i.e. $L \equiv \left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^{-1} \gg r_L$

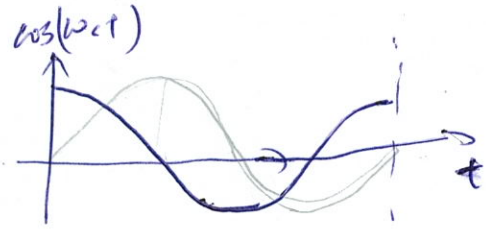
$$B(y) = B_0 + y \cdot \frac{\partial B}{\partial y} + \cdots$$

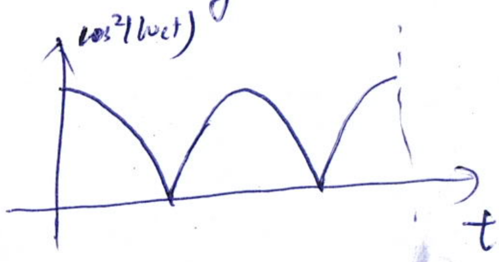
$$= B_0 + y \cdot \frac{B_0}{L} + \cdots$$

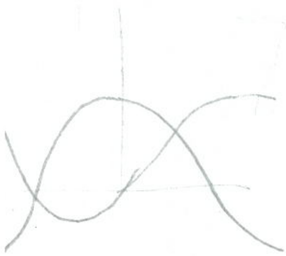
$$= B_0 \left[1 + \frac{y}{L} + \cdots\right]$$

$y \sim r_L$, $\therefore \frac{y}{L} \sim \frac{r_L}{L} \ll 1 \rightarrow$ keep the 1st term

$$F_y = -q V_\perp B_0 \cos(\omega_c t) \mp q V_\perp r_\perp \frac{\partial B}{\partial y} \cos^2(\omega_c t)$$







$$\overline{\cos(\omega_c t)} = \frac{1}{T}\int_0^T \cos(\omega_c t)\, dt = 0.$$

$$\overline{\cos^2(\omega_c t)} = \frac{1}{T}\int_0^T \cos^2(\omega_c t)\, dt = \frac{1}{2}$$

$$\overline{F_y} = \mp \frac{1}{2} q V_\perp r_\perp \left(\frac{\partial B}{\partial y}\right)$$

$\hat{y}$ $\hat{z}$ $\underline{\hat{y} \times \hat{z} = \hat{x}}$

"Gravitational Drift" $\vec{V}_{gc} = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$

$$\vec{V}_{gc} = \frac{1}{q}\frac{\overline{F_y} \cdot B}{B^2}\hat{x} = \frac{1}{q}\frac{\overline{F_y}}{B}\hat{x} = \mp \frac{1}{2}\frac{V_\perp r_\perp}{B}\left(\frac{\partial B}{\partial y}\right)\hat{x}$$

$\frac{\partial B}{\partial y} \rightarrow \nabla B \qquad \hat{x} = \hat{y} \times \hat{z} \rightarrow \hat{B} \times \widehat{\nabla B}$

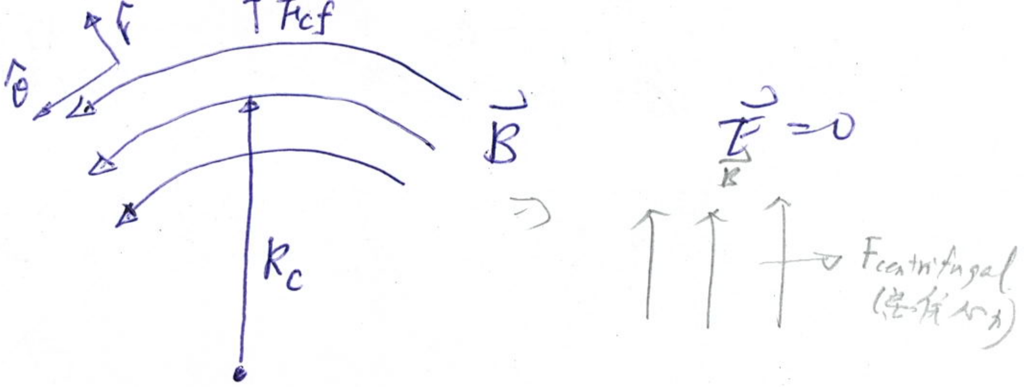
unit vector.

$\Rightarrow \quad \vec{V}_{\nabla B} = \pm \frac{1}{2} v_\perp r_\perp \frac{\vec{B} \times \nabla B}{B^2}$ grad-B drift

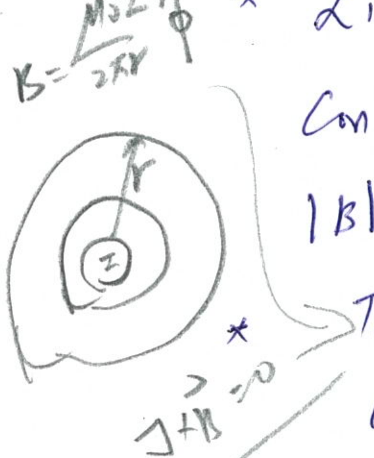
---

* It is in opposite directions for ions and electrons and causes a current transverse to $\vec{B}$.

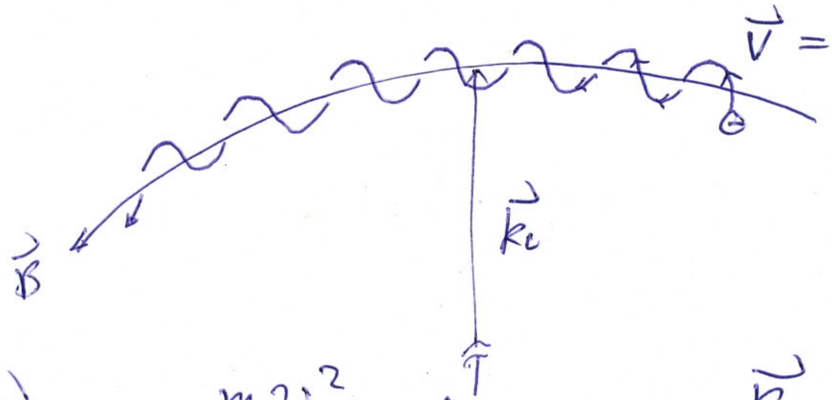
§ 2.2.2 Curved B : Curvature Drift. ($\vec{E} =$



Assume



* Lines of force to be curved with a constant radius of curvature R_c, and $|B|$ to be constant.
* The field does not obey Maxwell's eq. in a vacuum.
* A guiding center drift arises from the "centrifugal force" felt by the particles as they move along the field lines in their thermal motion.



$\vec{V} = \vec{V_\parallel} + \vec{V_\perp}$

$\vec{V_\parallel} \parallel \hat{B}$, $\vec{V_\perp} \perp \hat{B}$

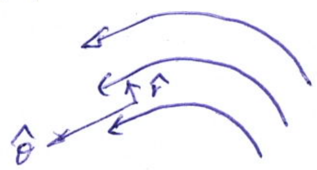
$$\vec{F}_{cf} = \frac{m v_\parallel^2}{R_c}\hat{r} = m v_\parallel^2 \frac{\vec{R_c}}{R_c^2}$$

$$\vec{V}_{gc} = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2} \Rightarrow \underline{\vec{V}_R = \frac{1}{q}\frac{\vec{F_{cf}}\times\vec{B}}{B^2} = \frac{m V_\parallel^2}{qB^2}\frac{\vec{R_c}\times\vec{B}}{R_c^2}}$$

Curvature drift.

* Consider the decrease of $|B|$ w/ radius.

- Note that $\nabla\times\vec{B} = 0$ in vacuum.
- In cylindrical coordinate of



$\rightarrow \nabla\times\vec{B} \parallel \hat{z}$

$\because \vec{B} = B\hat{\theta}$, $\nabla B \parallel \hat{r}$

$$\therefore \nabla\times\vec{B} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ B_r & rB_\theta & B_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & 0 & 0 \\ 0 & rB_\theta & 0 \end{vmatrix}$$

$$= \hat{z}\,\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r}$$

$B_\theta = \frac{\alpha}{r}$

$\nabla|B| = \frac{\partial}{\partial r}\left(\frac{\alpha}{r}\right) = \alpha\frac{\partial}{\partial r}($

$\because \nabla\times\vec{B} = 0$

$$(\nabla\times B)_z = \frac{1}{r}\frac{\partial rB_\theta}{\partial r} = 0 \Rightarrow B_\theta \propto \frac{1}{r}$$

$B|_{R_c} = \frac{\alpha}{R_c}$

$$|B| \propto \frac{1}{R_c} \qquad \frac{\nabla|B|}{|B|} = \frac{1}{1/R_c}\cdot\frac{\partial\, 1/r}{\partial r}\bigg|_{R_c}\hat{r} = -R_c\frac{\hat{r}}{R_c^2} = -\frac{\vec{R_c}}{R_c^2}$$

$$V_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \nabla B}{B^2} \frac{|B|}{|B|}$$

$$-\frac{\vec{R_c}}{R_c^2}$$

$$= \pm \frac{1}{2} v_\perp r_L \frac{\cancel{|B|}}{|B^2|} \vec{B} \times \frac{\nabla B}{|B|}$$

$$= \mp \frac{1}{2} v_\perp r_\perp \frac{1}{|B|} \vec{B} \times \frac{\vec{R_c}}{R_c^2} \qquad r_\perp = \frac{v_\perp}{\omega_c}$$

$$= \pm \frac{1}{2} \frac{v_\perp^2}{\omega_c} \frac{\vec{R_c} \times \vec{B}}{R_c^2 |B|} \qquad \omega_c = \frac{|q| B}{m}$$

$$= \pm \frac{1}{2} \frac{v_\perp^2}{|q| B/m} \frac{\vec{R_c} \times \vec{B}}{R_c^2 |B|}$$

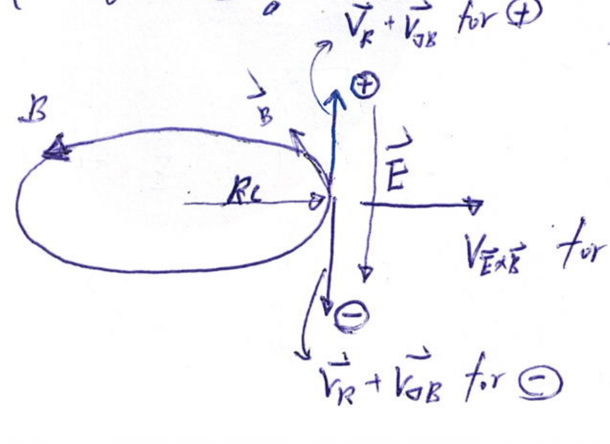
$$= \frac{1}{2} \frac{m}{q} v_\perp^2 \frac{\vec{R_c} \times \vec{B}}{R_c^2 B^2}$$

$$\vec{V_R} + \vec{V_{\nabla B}} = \frac{m}{q} \frac{\vec{R_c} \times \vec{B}}{R_c^2 B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)$$

---

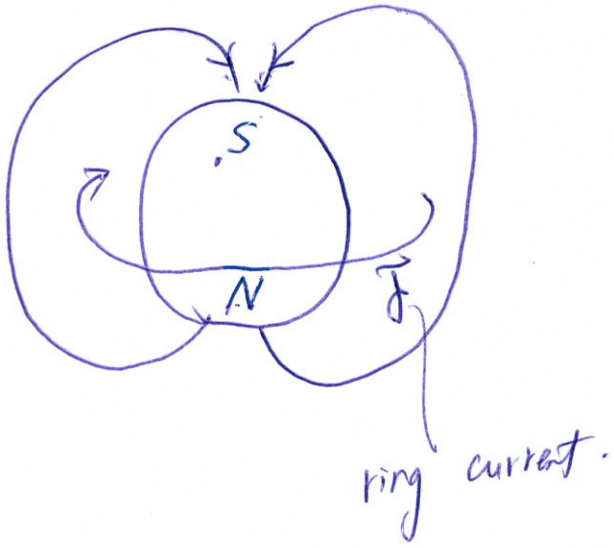
p30b first

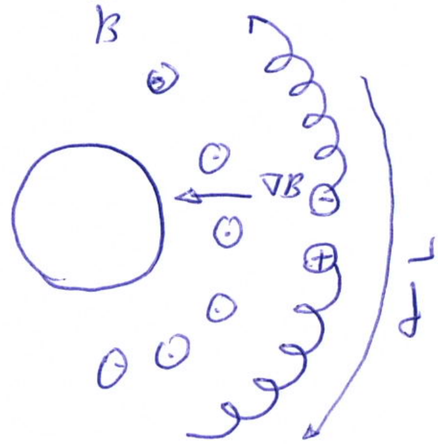
* If one bends a magnetic field into a torus for the purpose of confining a thermonuclear plasma, the particle will drift out of the torus. ~~..~~



* Curve + Grad B drifts cause charge separation in a ~~Tokamak~~ torus (Tokamak)

*Example of gradient & curvature drift





polar view

$j \propto \rho$, $\frac{1}{2}mv^2$ of particles in radiation belt.

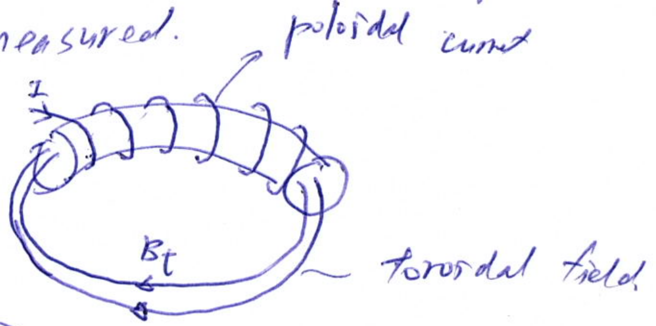
* Magnetic storms: a dense energetic plasma is injected into the magnetic field

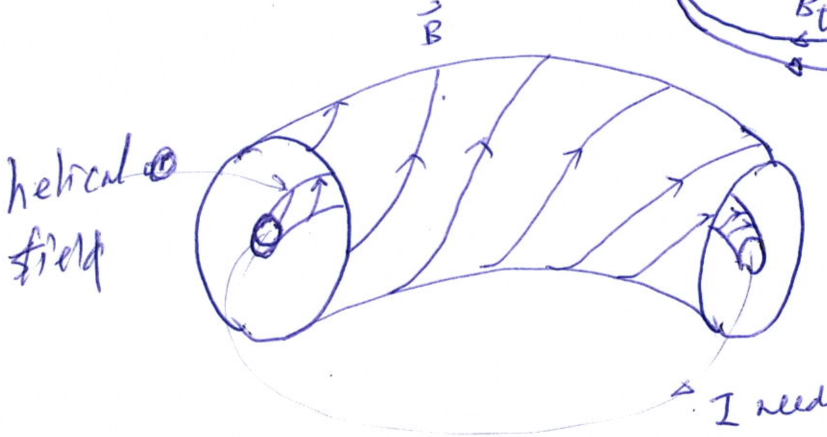
$\Rightarrow$ $\vec{j}$ ↑ $\Rightarrow$ $\vec{B}$ is disturbed

$\Rightarrow$ through measuring $\vec{B}$, total energy of the particles in $\vec{j}$ can be measured.

back to p30 →

* Toroidal magnetic field:



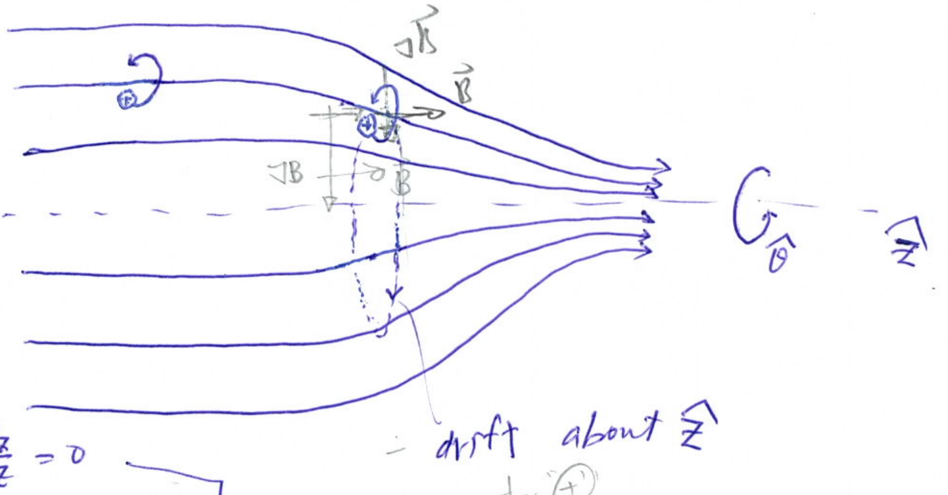


Tokamak: ~~current in~~ toroidal current needed in plasma

Stellarator: external helical current

§ 2.2.3. $\nabla B \parallel \vec{B}$: Magnetic Mirrors

$\vec{B} = B(z)\,\hat{z}$



Axisymmetric:

$B_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$

$B_r \neq 0$

$\nabla \cdot \vec{B} = 0$

$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \cancel{\frac{1}{r}\frac{\partial B_\theta}{\partial \theta}} + \frac{\partial B_z}{\partial z} = 0$

$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_r) = -\frac{\partial B_z}{\partial z} \neq 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

If @ $r=0$, $\frac{\partial B_z}{\partial z}$ is given & does not vary much w/ r (weak funct. of r)

$\Rightarrow \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z}$

$$\Rightarrow rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}\,dr \approx -\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\int_0^r r\,dr = -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

$$\Rightarrow B_r = -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0}.$$

$\Rightarrow \vec{B}$ has component both in $\hat{r}$ & $\hat{z}$

$\nabla|B|$ is in $\hat{z}$.

$\Rightarrow \nabla|B| \times \vec{B}$ is in θ direction

Alternatively, $\nabla|B|$ has $\hat{r}$ component, $\vec{B}$ is in r-z plane.

$\nabla|B| \times \vec{B}$ is in θ direction

· The variation of $|B|$ w/ r causes a <u>grad-B drift</u> of guiding centers about the axis of symmetry ($\hat{z}$). $\Leftarrow$

$\star$ $\because \frac{\partial B}{\partial \theta} = 0$

$\therefore$ NO radial grad-B drift.

Lorentz force: $F = q(\vec{v}\times\vec{B}) = q\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ v_r & v_\theta & v_z \\ B_r & B_\theta & B_z \end{vmatrix}$ p32

$$\begin{cases} F_r = q(\underset{①}{v_\theta B_z} - v_z \cancel{B_\theta}^{\,0}) \\ F_\theta = q(-\underset{②}{v_r B_z} + \underset{③}{v_z B_r}) \\ F_z = q(v_r \cancel{B_\theta}^{\,0} - \underset{④}{v_\theta B_r}) \end{cases}$$

$v_E = \frac{\vec{E}\times\vec{B}}{B^2}$

$v_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2}$

① + ② : usual Larmor gyration.

③ : $(v_z B_r) \to = 0$ @ axis, ∵ $r=0$

$-\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \neq 0$: azimuthal force causes a drift in the radial direction.

④ : $F_z = -q v_\theta B_r = \frac{1}{2} \underset{\pm|q|}{q}\, \underset{\mp v_\perp}{v_\theta}\, r \left[\frac{\partial B_z}{\partial z}\right]_{r=0}$ ← interesting. return force

* - Average over one gyration.
- Consider a particle whose guiding center lies on the axis

$\frac{1}{2} \underset{\pm|q|}{q}\, \underset{\mp v_\perp}{v_\theta}\, r \left[\frac{\partial B}{\partial z}\right]$ "−"

⇒ v_θ is a const. ← gyro motion

$v_\theta = \mp v_\perp$ depend on q.

$r = r_L$ ← Larmor radius.

$r_L = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B}$

$\omega_c = \frac{|q|B}{m}$

$$\bar{F}_z = -\frac{1}{2}|q| v_\perp r_L \left[\frac{\partial B_z}{\partial z}\right]_{r=0} = -\frac{1}{2}\cancel{|q|}\frac{m v_\perp^2}{\cancel{|q|}B}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

$$= -\frac{1}{2}\frac{m v_\perp^2}{B}\left[\frac{\partial B_z}{\partial z}\right]_{r=0} = -\mu\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \text{ where}$$

$\mu \equiv \frac{1}{2}\frac{m v_\perp^2}{B}$ — magnetic moment.

In general, force on a diamagnetic particle:

$$\vec{F}_{\parallel} = -\mu \frac{\partial B}{\partial s} = -\mu \nabla_{\parallel} B$$

$d\vec{s}$: a line element along $\vec{B}$

* Note that the magnetic moment μ is the same as the magnetic moment of a current loop w/ area A & current I :

$$\mu = I \cdot A \qquad \because \omega = 2\pi f = \frac{2\pi}{T}$$

$\vec{B}$

$$I = \frac{q}{t} = q \cdot \frac{\omega_c}{2\pi}, \quad A = \pi r_L^2 = \pi \frac{v_\perp^2}{\omega_c^2}$$

$$\mu = q \cdot \frac{\cancel{\omega_c}}{2\cancel{\pi}} \cdot \cancel{\pi} \frac{v_\perp^2}{\omega_c^{\cancel{2}}} = \frac{q}{2} \frac{v_\perp^2}{\omega_c} \qquad \omega_c = \frac{|q| B}{m} \neq \frac{qB}{m}$$

$$= \frac{\cancel{q}}{2} \frac{m v_\perp^2}{\cancel{q} B} = \frac{1}{2} \frac{m v_\perp^2}{B}$$

* $B \to$ stronger/weaker $\to$ Larmor radius r_L changes

$\Rightarrow \mu$ remains invarient. will show it in the following

$$F_{\parallel} = m \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s} \quad \times v_{\parallel}$$

$$m v_{\parallel} \frac{dv_{\parallel}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right) = -\mu \frac{\partial B}{\partial s} \cdot v_{\parallel} = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}$$

$v_{\parallel} = \frac{ds}{dt}$

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{\partial B}{\partial s} \cdot \frac{ds}{dt} \leftarrow \text{variation of B seen by the particle}$$

B doesn't change w/ time $\Rightarrow \frac{\partial B}{\partial t} = 0$ $\quad \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right) + \mu \frac{dB}{dt} = 0$

Note that the particle's energy is conserved $\leftarrow$ why? (HW)

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2 + \frac{1}{2} m v_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2 + \mu B\right) = 0$$

$$\because \mu = \frac{1}{2} \frac{m v_\perp^2}{B}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2 + \mu B\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2\right) + \frac{d}{dt}(\mu B) = 0$$

$$\Rightarrow \quad \cancel{-\mu\frac{dB}{dt}} + \underbrace{\frac{d}{dt}(\mu B)}_{=\,\cancel{\mu\frac{dB}{dt}} + B\frac{d\mu}{dt}} = 0 \quad \Rightarrow B\frac{d\mu}{dt} = 0$$

$$\Rightarrow \quad \underline{\frac{d\mu}{dt} = 0}$$

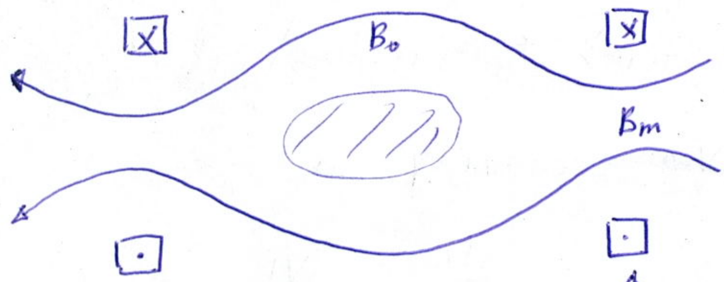
* The invariance of μ is the basis for one of the primary schemes for plasma confinement: <u>the magnetic mirror</u>

$\mu = \frac{1}{2}\frac{mV_{\perp}^2}{B}$

energy conservation.

$\Rightarrow B \uparrow \Rightarrow V_{\perp} \uparrow \Rightarrow V_{\parallel} \downarrow \longrightarrow V_{\parallel} = 0 \longrightarrow$ reflect.

$\vec{F}_{\parallel}$



throat of the mirror

* The trapping is not perfect:
  ① $V_{\perp} = 0$, no magnetic moment, $\Rightarrow$ No force along $\vec{B}$.
  ② particles w/ small $\frac{V_{\perp}}{V_{\parallel}}$ @ mid plane $(B = B_0)$ may escape if B_m is not enough.

For a particle:

@ mid plane: $V_\perp = V_{\perp 0}$, $V_\parallel = V_{\parallel 0}$, $B = B_0$

@ turning point: $V_\perp = V_\perp'$, $V_\parallel = V_\parallel'$, $B = B'$

$$\mu = \frac{1}{2}\frac{m V_{\perp 0}^2}{B_0} = \frac{1}{2}\frac{m V_\perp'^2}{B'} \Rightarrow \frac{V_{\perp 0}^2}{B_0} = \frac{V_\perp'^2}{B'}$$

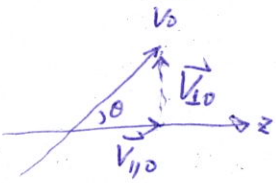
Energy conservation:

total speed @ mid plane

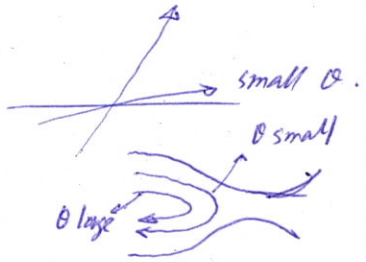
$$V_\parallel'^2 + V_\perp'^2 \equiv V_\perp'^2 = V_{\parallel 0}^2 + V_{\perp 0}^2 = V_0^2$$

$V_\parallel'^2 = 0$ @ turning point

$$\Rightarrow \frac{B_0}{B'} = \frac{V_{\perp 0}^2}{V_\perp'^2} = \frac{V_{\perp 0}^2}{V_0^2} \equiv \sin^2\theta.$$

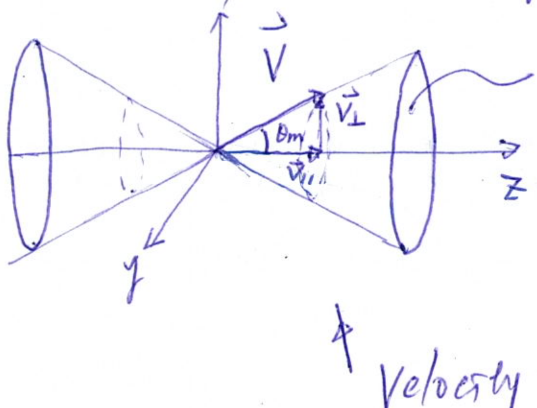


* θ is the pitch angle of the orbit in the weak-field region.
  - For particle w/ smaller θ → mirror in a region of higher B.



  - If $B' > B_m$ → the particle does not mirror. i.e., $V_\parallel' \neq 0$

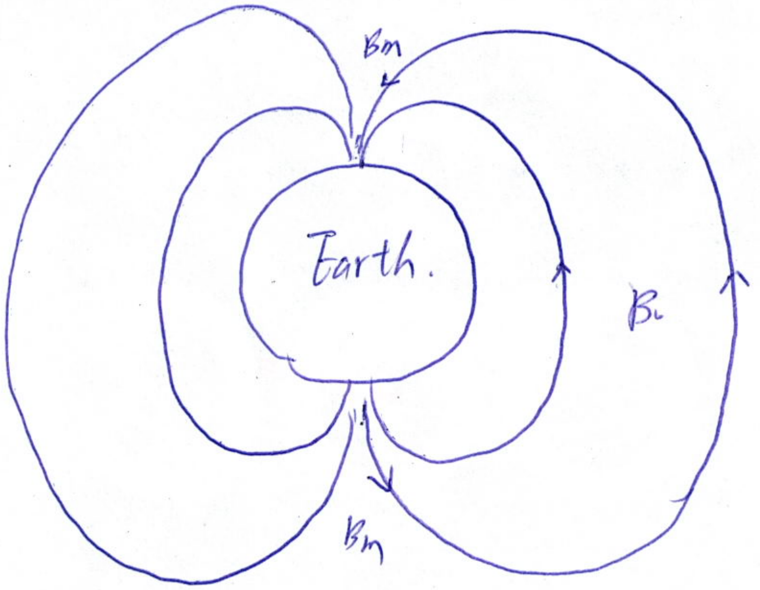
The smallest θ: $\frac{B_0}{B_m} = \sin^2\theta_m \equiv \frac{1}{R_m}$ ← mirror ratio



loss cone. → particle in this region can not be confined.

↑ velocity space.

* loss cone is independent of q, m. p30
  → electrons & ions are equally confined.
  w/o collisions
* Mirror confined plasma is NEVER isotropic.
* w/ collisions, particles are lost when they change their pitch angle in a collision and are scattered into the loss cone.
* Electrons have larger collision freq. than ions
  ⇒ electrons are lost more easily.
* 1st mirror motion was proposed by Enrico Fermi
* Another example: Van Allen belts.

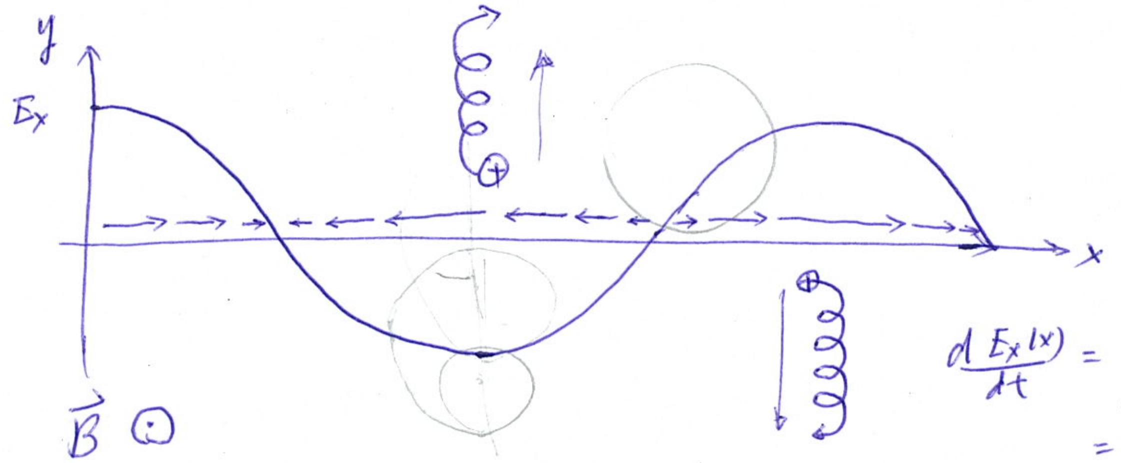


7. 2.3 Nonuniform E field.

B = uniform $= B\hat{z}$

$\vec{E} = E_0 \cos(kx)\,\hat{x}$ $\quad \lambda = \frac{2\pi}{k}$

↳ result of a sinusoidal distribution of charges.



$$\frac{d E_x(x)}{dt} = \frac{\partial E_x(x)}{\partial t} + \frac{\partial E_x}{\partial x}\frac{dx}{dt}$$
$$= \frac{\partial E_x}{\partial x}\cdot\frac{dx}{dt} \neq 0$$

$$m\frac{d\vec{v}}{dt} = q\left[\vec{E}(x) + \vec{v}\times\vec{B}\right]$$

$$\begin{cases} \dot{v}_x = \frac{qB}{m} v_y + \frac{q}{m} E_x(x) & \Rightarrow \ddot{v}_x = \frac{qB}{m}\dot{v}_y + \frac{q}{m}\dot{E}_x(x) \\ \dot{v}_y = -\frac{qB}{m} v_x & \Rightarrow \ddot{v}_y = -\frac{qB}{m}\dot{v}_x \\ \dot{v}_z = 0 & \end{cases}$$

$$\omega \equiv \frac{|q|B}{m}$$

$\frac{\partial E_x}{\partial y}$ $\cdot$ V_x ($\sin \omega t$) 1^s $\sim r_L \sin \omega t$ 1^{st} ...

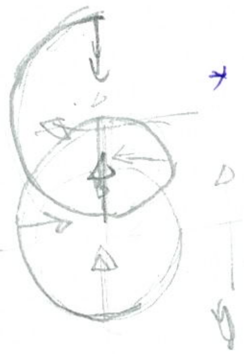
$$\Rightarrow \begin{cases} \ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x + \left(\frac{qB}{m}\right)\cdot\frac{\dot{E}_x}{B} = -\omega^2 v_x \pm \omega\frac{\dot{E}_x}{B} \\ \ddot{v}_y = -\left(\frac{qB}{m}\right)^2 v_y - \left(\frac{qB}{m}\right)\left(\frac{qB}{m}\right)\frac{E_x}{B} = -\omega^2 v_y - \omega^2\frac{E_x}{B}. \end{cases}$$

* Assuming that the electric field is weak.
"undisturbed orbit" is used to evaluate $E_x(x)$
as an approximation, i.e.,

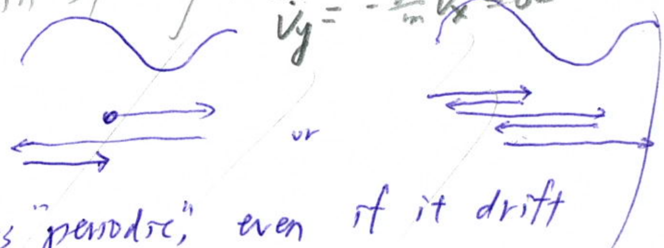
$$x = x_0 + r_L \sin(\omega t)$$

$$E_x(x) = E_0 \cos(kx)$$
$$= E_0 \cos[k(x_0 + r_L \sin(\omega t))]$$

$$\Rightarrow \ddot{V}_y = -\omega^2 V_y - \omega^2 \frac{E_0}{B} \cos[k(x_0 + r_L \sin \omega t)]$$



* We would like to find a solution which is the sum of a gyration at ω and a steady drift v_E. v_E is what we are interested in. Therefore, gyratory motion can be taken out by averaging over a cycle. $\overline{V_x} = 0$ in symmetry, accele = decelle $\overline{\dot{V}_y} = -\frac{qq}{m}\overline{V_x} = 0$



$\overline{\dot{V}_x} \neq 0$

$\overline{\dot{V}_y} = 0$

$$\ddot{V}_x = -\omega V_x \pm \omega \frac{\dot{E}_x}{B}$$

$$\overline{\ddot{V}_x} = 0 = -\omega^2 \overline{V_x} \pm \omega \frac{\overline{\dot{E}}}{B}$$

$\overline{V_x} = \overline{V_0 \sin \omega t} = 0$

$\overline{\dot{E}} = \frac{dE}{dx} \overline{V_x} = \frac{dE}{dx} \overline{V_0 \sin \omega t} = 0$

E_x is "periodic," even if it drift along $\hat{x}$, the average will be zero.

$\overline{V_y} = $ const, $\overline{\dot{V}_y} = 0$, $\overline{\ddot{V}_y} = 0$

$E_x = E_0 \cos[kx_0 + r \sin \omega t]$

$\frac{dE}{dt} = \frac{dE}{dx} \cdot \frac{dx}{dt} = \frac{dE}{dx} \cdot V_x$

$\ddot{V}_x = \left(-\omega \pm \frac{\omega}{B} \frac{dE}{dx}\right) V_x$

$\Rightarrow \ddot{V}_x = \omega' V_x$

$V_x = e^{\pm i \omega' t}$

$\overline{V_x} = 0$

symmetric

$$\overline{\ddot{V}_y} = 0 = -\omega_c^2 \overline{V_y} - \omega_c^2 \frac{\overline{E}}{B}$$

$$= -\omega_c^2 \overline{V_y} - \omega_c^2 \frac{E_0}{B} \overline{\cos[k(x_0 + r_L \sin \omega t)]}$$

$$\cos[k(x_0 + r_L \sin \omega t)] = \cos[kx_0 + kr_L \sin \omega t]$$
$$= \cos kx_0 \cos(kr_L \sin \omega t) - \sin kx_0 \sin(kr_L \sin \omega t)$$

(for small Larmor radius, i.e., $kr_L \ll 1$.

$\epsilon \equiv kr_L \qquad \cos \epsilon = 1 - \frac{1}{2}\epsilon^2 + \cdots$

$\qquad\qquad\quad \sin \epsilon = \epsilon + \cdots$)

$$\approx \cos(kx_0)\left(1 - \frac{1}{2}k^2 r_L^2 \underbrace{\sin^2 \omega t}_{\to \frac{1}{2} \text{ after averaging}}\right) - \sin(kx_0) \cdot \underbrace{k r_L \sin(\omega t)}_{\to 0 \text{ after averaging}}$$

2nd order; 1st order

$$\Rightarrow \overline{V_y} = -\frac{E_0}{B} \cos(kx_0)\left(1 - \frac{1}{4}k^2 r_L^2\right) = -\underbrace{\frac{E_x(x_0)}{B}}_{\frac{E \times B}{B^2}}\left(1 - \frac{1}{4}k^2 r_L^2\right)$$

$\vec{E}\times\vec{B}$ drift:

homogeneous $\vec{E}$: $\vec{V}_E = \dfrac{\vec{E}\times\vec{B}}{B^2}$

inhomogeneous $\vec{E}$: $\vec{V}_E' = \dfrac{\vec{E}\times\vec{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right)$

$\vec{V}_E' < \vec{V}_E$ since the particle spends more time in regions of weaker $\vec{E}$.

* The correction term depends on the 2nd derivative of $\vec{E}$.

* For an arbitrary variation of $\vec{E}$:

  $ik \to \nabla$.

  $\Rightarrow \vec{V}_E' = \left(1 + \frac{1}{4} r_L^2 \nabla^2\right) \dfrac{\vec{E}\times\vec{B}}{B^2}$

  finite-Larmor-radius effect.

$r_L = \dfrac{m v_\perp}{|q| B} = \dfrac{\sqrt{2}\sqrt{m}\cdot\sqrt{m v_\perp^2/2}}{|q| B} = \dfrac{\sqrt{2mE}}{|q| B} \propto m$

depend on r_L $\to$ depends on species

r_L for ion $>$ r_L for electron.

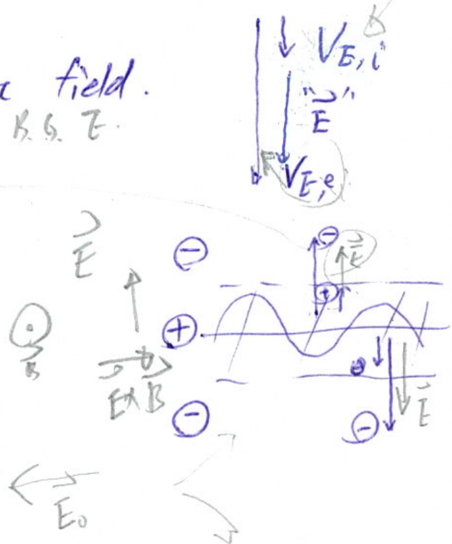
electron drift faster

$\Rightarrow$ charge separation occurs

$\Rightarrow$ generating another Electric field.

of $\vec{E}$ has feedback B.G. $\vec{E}$

$\Rightarrow$ drift instability



* Comparing to $V_{\nabla B} \propto \vec{B}\times\nabla B \propto k r_L$

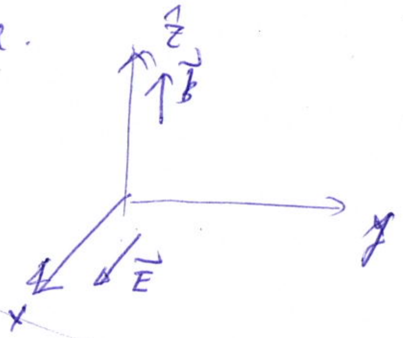
  $V_{\nabla E} \propto \nabla^2 \propto k^2 r_L^2$

  @ large k, i.e. smaller λ.

  non-uniform-E field effect is more important.

§. 2.4 Time-varying E field

$\vec{E}$ & $\vec{B}$ are uniform in space.
but ($\vec{E} \perp \vec{B}$) varying in time.



$$\vec{E} = E_0 e^{i\omega t}\,\hat{x} = E_x \hat{x}$$

$$\dot{E}_x = i\omega E_0 e^{i\omega t} = i\omega E_x$$

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$\begin{cases} \dot{v}_x = \frac{qB}{m} v_y + \frac{q}{m} E_x & \Rightarrow \ddot{v}_x = \frac{qB}{m}\dot{v}_y + \frac{q}{m}\dot{E} \\ \dot{v}_y = -\frac{qB}{m} v_x & \Rightarrow \ddot{v}_y = -\frac{qB}{m}\dot{v}_x \end{cases} \qquad \omega_c = \frac{|q|B}{m}$$

$$\Rightarrow \ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x + \frac{qB}{m}\frac{\dot{E}}{B} = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}}{B}$$

$$= -\omega_c^2 v_x \pm i\omega\cdot\omega_c \frac{E_x}{B}$$

$$= -\omega_c^2\left(v_x \mp i\frac{\omega}{\omega_c}\frac{E_x}{B}\right)$$

note that E_x is oscillating

$$\ddot{v}_y = -\left(\frac{qB}{m}\right)^2 v_y - \left(\frac{qB}{m}\right)^2 \frac{E_x}{B}$$

$$= -\omega_c^2 v_y - \omega_c^2 \frac{E_x}{B}$$

let
$$\begin{cases} \tilde{v}_p \equiv \pm\frac{i\omega}{\omega_c}\frac{E_x}{B} \\ \tilde{v}_E \equiv -\frac{E_x}{B} \end{cases} \Rightarrow \begin{cases} \ddot{v}_x = -\omega_c^2(v_x - \tilde{v}_p) \\ \ddot{v}_y = -\omega_c^2(v_y - \tilde{v}_E) \end{cases}$$

To find a solution which is the sum of a drift and a gyratory motion

$$\begin{cases} v_x = v_\perp e^{i\omega_c t} + \tilde{v}_p \\ v_y = \underbrace{\pm i v_\perp e^{i\omega_c t}}_{\text{gyro motion}} + \tilde{v}_E \end{cases}$$

$$\dot{V}_x = i\omega_c V_\perp e^{i\omega_c t} + \dot{\tilde{V}}_p = i\omega_c V_\perp e^{i\omega_c t} + \left(\frac{\pm i\omega}{\omega_c}\right)\frac{\dot{E}_x}{B}$$

$i\omega E_x$

$$= i\omega_c V_\perp e^{i\omega_c t} \pm \frac{i\omega}{\omega_c}\frac{i\omega E_x}{B}$$

$$= i\omega_c V_\perp e^{i\omega_c t} + i\omega \tilde{V}_p$$

$$\ddot{V}_x = -\omega_c^2 V_\perp e^{i\omega_c t} - \omega^2 \tilde{V}_p \quad \cancel{\mp i\omega_c^2 (V_x - \tilde{V}_p)}$$

$(V_x - \tilde{V}_p)$

Note $V_x = V_\perp e^{i\omega t} + \tilde{V}_p$

$$\ddot{V}_x = -\omega_c^2 (V_x - \tilde{V}_p) - \omega^2 \tilde{V}_p$$

$$= -\omega_c^2 V_x + (\omega_c^2 - \omega^2)\tilde{V}_p$$

$$\dot{V}_y = \pm i V_\perp (i\omega_c) e^{i\omega_c t} + \left(-\frac{\dot{E}_x}{B}\right) = \mp \omega_c V_\perp e^{i\omega_c t} - i\omega\frac{E_x}{B}$$

$\tilde{V}_E$

$$= \mp \omega_c V_\perp e^{i\omega_c t} + i\omega \tilde{V}_E$$

$$\ddot{V}_y = \mp i\omega_c^2 V_\perp e^{i\omega_c t} - \omega^2 \tilde{V}_E, \qquad \text{Note } V_y = \pm i V_\perp e^{i\omega t} + \tilde{V}_E$$

$$= -\omega_c^2 (V_y - \tilde{V}_E) - \omega^2 \tilde{V}_E$$

$$= -\omega_c^2 V_y + (\omega_c^2 - \omega^2)\tilde{V}_E$$

pure drift:

$$\ddot{V}_x = -\omega_c^2 (V_x - \tilde{V}_p)$$
$$\ddot{V}_y = -\omega_c^2 (V_y - \tilde{V}_E)$$

$$\begin{cases} \ddot{V}_x = -\omega_c^2 V_x + (\omega_c^2 - \omega^2)\tilde{V}_p \\ \ddot{V}_y = -\omega_c^2 V_y + (\omega_c^2 - \omega^2)\tilde{V}_E \end{cases} \iff \begin{cases} \ddot{V}_x = -\omega_c^2 V_x + \omega_c^2 \tilde{V}_p \\ \ddot{V}_y = -\omega_c^2 V_y + \omega_c^2 \tilde{V}_E \end{cases}$$

extra term

$\Rightarrow$ Assuming that E varies slowly. i.e. $\underline{\omega^2 \ll \omega_c^2}$

There are two drifting of the guiding center.

① $\hat{y}$: $\tilde{V}_E \perp \vec{B}, \vec{E}$, usual $\vec{E}\times\vec{B}$ drift.

the difference is that it oscillates slowly at the frequency ω.

② $\hat{x}$: Polarization drift along the direction of $\vec{E}$.

$$\tilde{V}_p = \pm \frac{i\omega}{\omega_c}\frac{E_x}{B}, \quad i\omega \to \frac{\partial}{\partial t}$$

$$V_p \equiv \pm \frac{1}{\omega_c B}\frac{d\vec{E}}{dt}$$

∵ ions & electrons drift in opposite directions:

⇒ polarization current.

for $Z = 1$

$$j_p = n\cdot e\cdot (V_{ip} - V_{ep})$$
$$= n\cdot e\cdot\left(\frac{1}{\omega_{ci}B} + \frac{1}{\omega_{ce}B}\right)\frac{d\vec{E}}{dt}$$
$$= n\cdot \cancel{e}\left(\frac{M}{\cancel{e}B^2} + \frac{m}{\cancel{e}B^2}\right)\frac{d\vec{E}}{dt}$$
$$= \frac{n(M+m)}{B^2}\frac{d\vec{E}}{dt}$$
$$= \frac{\rho}{B^2}\frac{d\vec{E}}{dt}$$

ρ: mass density.

goes to p42a

* $t=0$, ions @ rest, $\vec{B} = B\hat{z}$, $\vec{E} = 0$

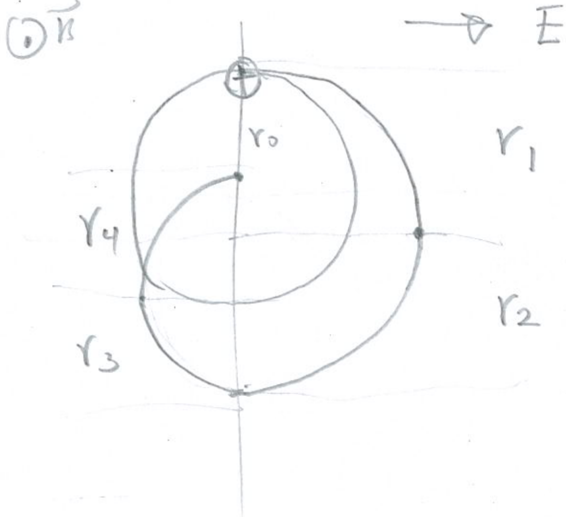
skip

⇒ $\vec{E} = E_x(t)\hat{x}$ ↑ ⇒ ion is accelerated ⇒ $V_x > 0$ drift along $\vec{E}$ (+)

⇒ $F = q(\vec{v}\times\vec{B})$ ⇒ $\vec{E}\times\vec{B}$ drift

⇒ If $\vec{E}$ is reversed ⇒ decelerate ⇒ $V_x < 0$ drift along $\vec{E}$ (−)

pY2a



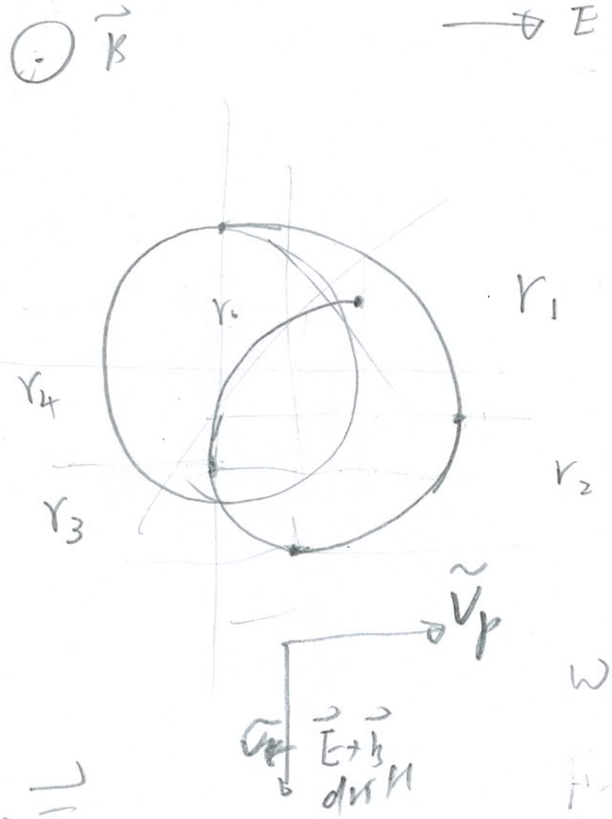
$r_1 = r_2 > r_0$

$r_3 = r_4 < r_0$

$\vec{E} \times \vec{B}$ drift

$\vec{E} = E_0 \sin(kt)$

$\sim E_0 kt$



$r_1 > r_2$

$r_3 < r_2$

$r_4 > r_3$

∴ decelerate more than acceler

$r_2 < r_0$

$r_3 < r_2 < r_0$

r_4

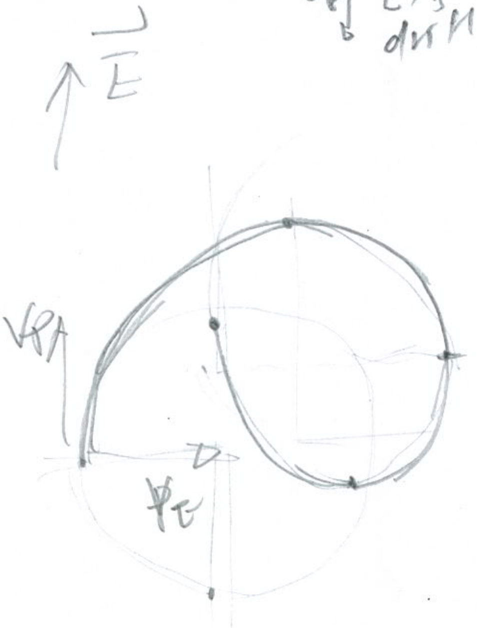
$\omega = \frac{qB}{m} \Rightarrow T = \frac{m}{qB} = \text{const}$

$m\dot{v} = q\vec{E}$

$r_L = \frac{mv}{qB}$

$\left|\dot{r}_L\right| = \frac{m\dot{v}}{qB} = \frac{q\vec{E}(t)}{qB} = \left|\frac{\vec{E}(t)}{B}\right|$

Δr



V_p is a startup drift due to inertia and p 43
occurs only in the first half-cycle of each gyration during which $\vec{E}$ changes

$V_p \rightarrow 0$ with $\frac{\omega}{\omega_c}$

* an oscillating current j_p results from the lag due to the ion inertia.

§ 2.5 Time-Varying B field.

* A magnetic field itself cannot impart energy to a charged particle.

$\because \quad \nabla \times \vec{E} = -\dot{\vec{B}} \quad \Rightarrow$ the $\vec{E}$ associated w/ $\dot{\vec{B}}$ can accelerate particles.

Let $\vec{V}_\perp = \frac{d\vec{l}}{dt} \longrightarrow$ particle trajectory

transverse velocity

$V_\parallel$ is neglected.

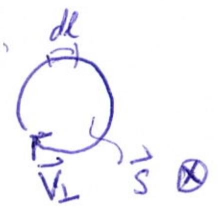
$$m\frac{d\vec{V}}{dt} = q(\vec{E} + \vec{V}\times\vec{B}) \quad \cdot \vec{V}_\perp$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mV_\perp^2\right) = q\vec{E}\cdot\vec{V}_\perp = q\vec{E}\cdot\frac{d\vec{l}}{dt}$$

$$(\vec{V}\times\vec{B})\cdot\vec{V}_\perp = (\underbrace{\vec{V}_\perp\times\vec{B}}_{\perp \vec{V}_\perp})\cdot\vec{V}_\perp \equiv 0$$

Integrate over one period.

$$\delta\left(\frac{1}{2}mV_\perp^2\right) = \int_0^{2\pi/\omega_c} q\vec{E}\cdot\frac{d\vec{l}}{dt}\cdot dt = \oint q\vec{E}\cdot d\vec{l}$$



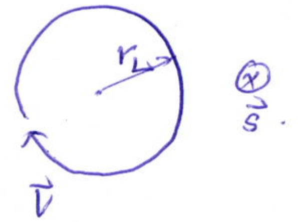
$$\approx q\int_S (\nabla\times\vec{E})\cdot d\vec{S} = -q\int_S \dot{\vec{B}}\, d\vec{S}$$

$\because \omega \ll \omega_c \longrightarrow$

Plasma is diamagnetic.

$\vec{B} \cdot ds < 0$ for ion

$\vec{B} \cdot ds > 0$ for electron



$$\delta\left(\tfrac{1}{2} m v_\perp^2\right) = \pm |q| \dot{B}\, \pi r_L^2 \qquad \int \vec{ds}$$

$$= \pm |q| \dot{B} \pi \left(\frac{v_\perp}{\omega_c}\right)^2$$

$$= \cancel{\pm |q|} \dot{B} \pi \frac{v_\perp^2}{\omega_c} \cdot \frac{m}{\cancel{\pm |q|} B} \cdot \frac{2}{2}$$

$$= \underbrace{\frac{\frac{1}{2} m v_\perp^2}{B}}_{\mu} \cdot \underbrace{\frac{2\pi \dot{B}}{\omega_c}}$$

$$\dot{B} = \frac{dB}{dt} = \frac{\delta B}{\delta t} \approx f_c \delta B$$

$$= \mu \cdot \frac{\dot{B}}{f_c} = \mu \cdot \frac{f_c \delta B}{f_c}$$

$\mu \delta B$

$$= \mu \cdot \delta B$$

$$\because \tfrac{1}{2} m v_\perp^2 = \mu \cdot B \Rightarrow \delta\left(\tfrac{1}{2} m v_\perp^2\right) = \delta(\mu B)$$

$$\therefore \delta(\mu B) = \cancel{\mu \delta B} \Rightarrow \delta\mu \cdot B = 0$$

$$\delta(\mu B) = \delta\mu \cdot B + \cancel{\mu \cdot \delta B}$$

$$\underline{\delta \mu = 0}$$

The magnetic moment is invariant in slowly varying magnetic field.

$$\Rightarrow \frac{\frac{1}{2} m v_\perp^2}{B} \approx \text{const.} \qquad B \uparrow \Rightarrow v_\perp \uparrow$$

B varies $\Rightarrow$ Larmor orbits expand and contract.

$\Rightarrow$ particle loses / gains transverse energy

$S = \pi r_L^2$

$\frac{v_\perp}{\omega_c}$

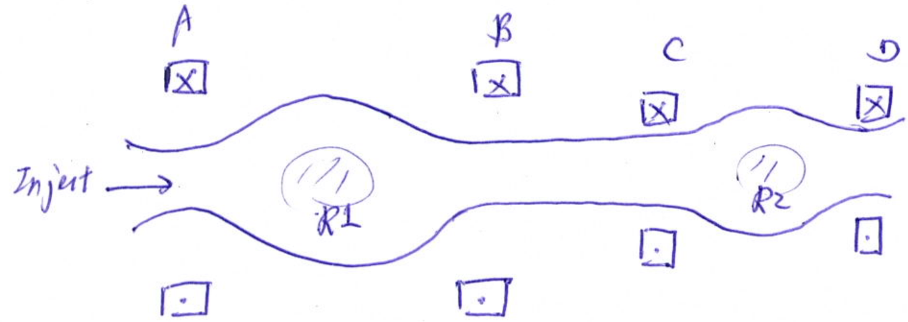
Magnetic Flux:

$$\Phi = B \cdot S = B \pi r_L^2 = B\pi \left(\frac{v_\perp}{\omega_c}\right)^2 = \cancel{B} \pi \frac{v_\perp^2 \cdot m^2}{q^2 B^{\cancel{2}}} \frac{2}{2}$$

$$= \frac{2\pi m}{q^2} \cdot \mu \Rightarrow$$ The magnetic flux through a Larmor orbit is constant.

$B \uparrow \Rightarrow v_\perp \uparrow \Rightarrow E_k(T) \uparrow$

$\Rightarrow$ Adiabatic compression.



① Inject into R1

② $B_A, B_B \uparrow \Rightarrow$ compression $\Rightarrow$ heating

③ $B_A \uparrow \Rightarrow$ push the heated plasma to R_2

④ $B_C, B_D \uparrow \Rightarrow$ further compression $\Rightarrow$ further heating

~~§ 2.6 Summary of guiding center drifts:~~

~~Electric field: V_E~~

§2.6 Summary of guiding center drifts.

Electric field : $\vec{V}_E = \frac{\vec{E}\times\vec{B}}{B^2}$ ($\vec{E}\times\vec{B}$ drift)

General force $\vec{F}$: $\vec{V}_F = \frac{1}{q}\frac{\vec{F}\times\vec{B}}{B^2}$

Gavitational field : $\vec{V}_g = \frac{m}{q}\frac{\vec{g}\times\vec{B}}{B^2}$

Non uniform $\vec{E}$: $\vec{V}_E = \left(1+\frac{1}{4}r_L^2\nabla^2\right)\frac{\vec{E}\times\vec{B}}{B^2}$

Grad-B drift : $\vec{V}_{\nabla B} = \pm\frac{1}{2}V_\perp r_L \frac{\vec{B}\times\nabla B}{B^2}$

Curvature drift : $\vec{V}_R = \frac{m V_\parallel^2}{q}\frac{\vec{R}_c\times\vec{B}}{R_c^2 B^2}$

Curved vacuum field : $\vec{V}_R + \vec{V}_{\nabla B} = \frac{m}{q}\left(V_\parallel^2 + \frac{1}{2}V_\perp^2\right)\frac{\vec{R}_c\times\vec{B}}{R_c^2 B^2}$

Polarization drift : $V_p = \pm\frac{1}{\omega_c B}\frac{d\vec{E}}{dt}$

§ 2.7 Adiabatic invariants.

- Action integral $\oint p\,dq$. p: generalized momentum
  q: generalized coordinate

In classical mechanics, whenever a system has a <u>periodic motion</u>, the action integral $\oint p\,dq$ taken over a period is a constant of the motion.

- Adiabatic invariant: If a slow change is made in the system, so that the motion is not quite periodic, the constant of the motion does not change.
- slow → compared w/ the period of motion.
  → $\oint p\,dq$ is no longer strictly over a closed path.

§ 2.7.1 The first Adiabatic Invariant μ.

$$\mu \equiv \frac{m v_\perp^2}{2B}$$

The periodic motion involved is Larmor gyration.

$\int p\,dq$: p: angular momentum. $m v_\perp r$
q: coordinate θ

$$\oint p\,dq = \int m v_\perp r_L \cdot d\theta = 2\pi r_L m v_\perp = 2\pi \frac{m v_\perp^2}{\omega_c} = 4\pi \frac{m}{|q|}\mu.$$

$\left(\omega_c = \frac{|q|B}{m}\right)$

const.

μ is a const. of the motion as long as $\frac{m}{|q|}$ is not changed.

Previously, we proved the invariant of μ w/ $\omega/\omega_c \ll 1$. ~~to~~ p43 last line

where ω is the frequency of the rate of change of B

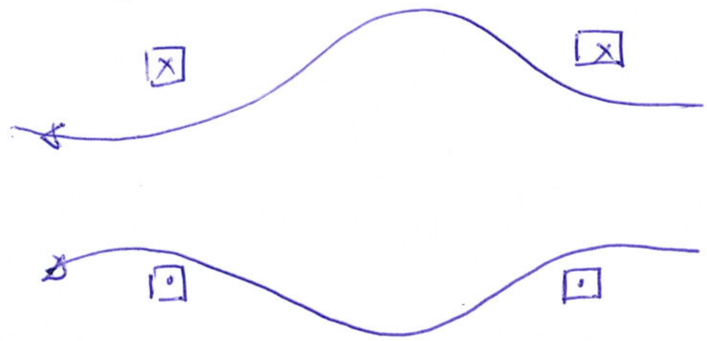
In fact, it valids for $\omega \lesssim \omega_c$.

i.e. μ remains much more nearly constant than B does during one period of gyration.

* Adiabatic invariance of μ is VIOLATED when ω is not small compared w/ ω_c.

Examples:

Magnetic pumping



$$\mu = \frac{mV_\perp^2}{2B}$$

If $B \uparrow$ slowly $\rightarrow V_\perp^2 \uparrow$ increase slowly

$B \sim \rightarrow V_\perp^2 \sim \Rightarrow$ No energy ga

w/ collisions: $V_\perp^2 \longrightarrow V_\parallel^2$ transfer energy from $\perp \rightarrow \parallel$

$\Rightarrow B \uparrow \longrightarrow V_\perp^2 \uparrow \xrightarrow{\text{collisions}} \left\{ \begin{array}{l} V_\parallel^2 \uparrow \\ V_\perp^2 \downarrow \end{array} \right.$ $\mu = \frac{\frac{1}{2}mV_\perp^2}{B}$

$\downarrow$

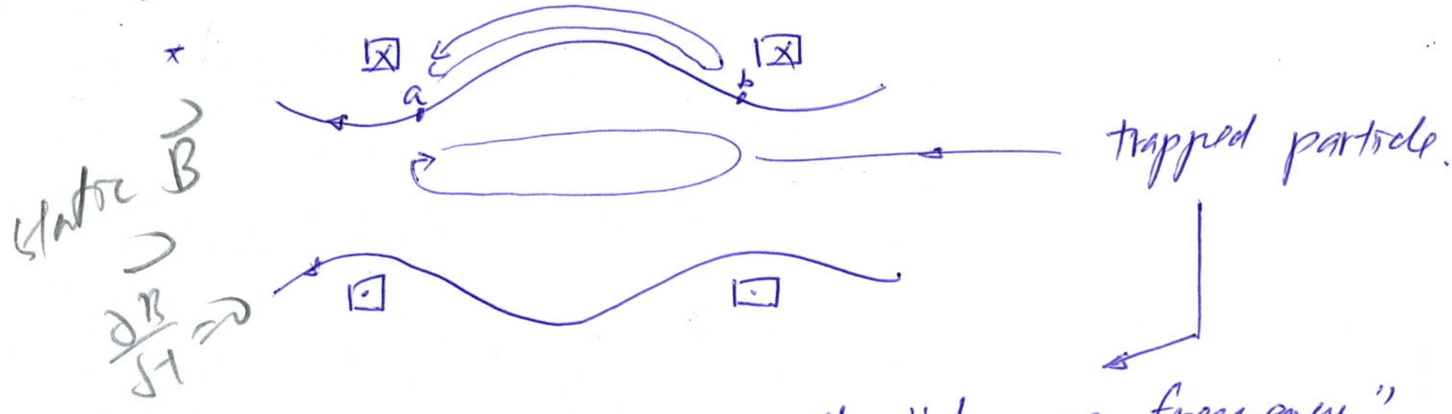
$B \searrow \longrightarrow \left\{ \begin{array}{l} V_\perp^2 \searrow < V_{\perp 0}^2 \\ V_\parallel^2 \text{ doesn't change } (V_\parallel^2 > V_{\parallel 0}^2) \end{array} \right.$

energy is transferre from $B \rightarrow V_\perp \rightarrow V_\parallel$

$\because$ invariant μ is VIOLATED

$\Rightarrow$ plasma can be heated. due to collisions

§ 2.7.2 The second Adiabatic Invariant J. P49



A periodic motion at the "bounce frequency."

$$\Rightarrow \text{Action:} \quad \oint \underbrace{m v_\parallel}_{p} \underbrace{ds}_{dq} \equiv \oint p\,dq.$$

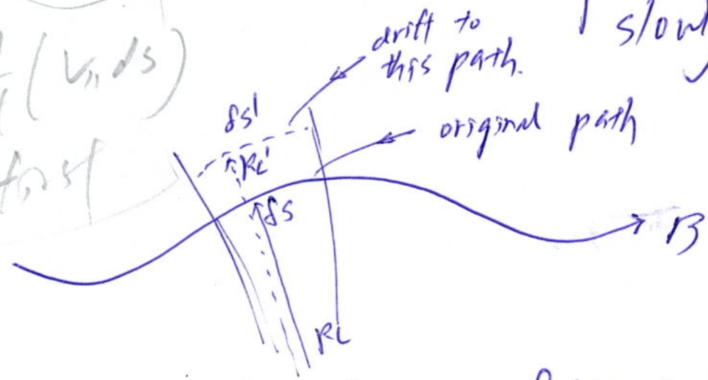
* The guiding center drifts across field lines, the motion is not exactly periodic. The constant of motion becomes an adiabatic invariant:

Longitudinal invariant J.

$$J = \int_a^b v_\parallel \, ds. \quad \leftarrow \text{half cycle between two turning points } a, b.$$

J is invariant in $\begin{cases} \text{a static, nonuniform } B \\ \text{slowly time-varying } B. \end{cases}$

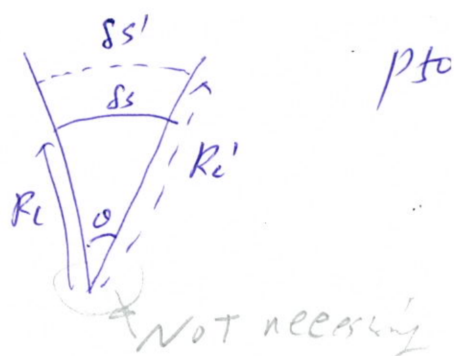
find $\frac{1}{v_\parallel \delta s}\frac{d}{dt}(v_\parallel ds)$ p41 first



Consider $v_\parallel \delta s$ first, δs: a segment of the path along B due to the drift, particle drifts to $\delta s'$ after Δt.

p50

show first. to find $\frac{1}{\delta s}\frac{d\delta s}{dt}$



$$\frac{\delta s}{R_c} = \frac{\delta s'}{R_c'}$$

$$\Rightarrow \frac{\delta s'}{\delta s} = \frac{R_c'}{R_c} \Rightarrow \frac{\delta s'}{\delta s} - 1 = \frac{R_c'}{R_c} - 1$$

$$\Rightarrow \underbrace{\frac{\delta s' - \delta s}{\Delta t\, \delta s}}_{\frac{d\delta s}{dt}} = \frac{R_c' - R_c}{\Delta t R_c}$$

V_{gc} in $\hat{R}$ is $\frac{R_c' - R_c}{\Delta t}$

or $\vec{V_{gc}} \cdot \hat{R_c} = \frac{R_c' - R_c}{\Delta t}$ ✓

The "radial" component of $\vec{V_{gc}}$

$$\vec{V_{gc}} \cdot \frac{\vec{R_c}}{R_c}$$

Note that $\vec{V_{gc}} = \vec{V}_{\nabla B} + \vec{V}_R = \underbrace{\pm \frac{1}{2} V_\perp r_L \frac{\vec{B}\times\nabla B}{B^2}}_{\text{Grad-B drift}} + \underbrace{\frac{mV_\parallel^2}{q}\frac{\vec{R_c}\times\vec{B}}{R_c^2 B^2}}_{\text{Curvature drift.}}$

$\uparrow$ $\perp \vec{R_c}$

$\Rightarrow \cdot \vec{R_c} = 0$

$$\frac{1}{\delta s}\frac{d\delta s}{dt} = \vec{V_{gc}} \cdot \frac{\vec{R_c}}{R_c^2} = \underbrace{\pm\frac{1}{2} V_\perp r_L}_{= \frac{mV_\perp}{qB}} \frac{\vec{B}\times\nabla B}{B^2} \cdot \frac{\vec{R_c}}{R_c^2}$$

$q = \pm|q|$

$$= \frac{1}{2}\frac{mV_\perp^2}{qB^3}(\vec{B}\times\nabla B)\cdot\frac{\vec{R_c}}{R_c^2}$$ → the rate of change of δS as seen by the particle.

to find $\frac{1}{V_\parallel}\frac{dV_\parallel}{dt}$

Total energy: $\mu = \frac{\frac{1}{2}mV_\perp^2}{B}$

⇒ to get $V_\parallel$

$$W = \frac{1}{2}mV_\parallel^2 + \frac{1}{2}mV_\perp^2$$
$$= \frac{1}{2}mV_\parallel^2 + \mu B \equiv W_\parallel + W_\perp$$

$$\Rightarrow V_\parallel = \sqrt{\frac{2}{m}(W - \mu B)}$$, Note that W, μ are invariant, only B changes.

$$\frac{1}{V_\parallel}\frac{dV_\parallel}{dt} = \frac{1}{\sqrt{\frac{2}{m}(W-\mu B)}} \times \frac{1}{2}\sqrt{\frac{2}{m}}\frac{-\mu}{\sqrt{W-\mu B}}\frac{dB}{dt} = -\frac{1}{2}\frac{\mu\dot{B}}{W-\mu B} = -\frac{1}{2}\frac{\mu\dot{B}}{W_\parallel}$$

$$= -\frac{\mu\dot{B}}{mV_\parallel^2}$$ ✗ need to be determined

$$\dot{B} = \frac{dB}{dt} = \cancel{\frac{\partial B}{\partial t}} + \frac{\partial B}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt}$$

$\because$ static B

change of B seen by the particle due to the guiding center motion.

$$= \vec{V}_{gc} \cdot \nabla B = \frac{mV_\parallel^2}{q} \frac{(\vec{R}_c \times \vec{B})}{R_c^2 B^2} \cdot \vec{\nabla} B$$

$$\Rightarrow \frac{\dot{V}_\parallel}{V_\parallel} = -\frac{\mu}{q} \frac{(\vec{R}_c \times \vec{B})}{R_c^2 B^2} \cdot \vec{\nabla} B = -\overbrace{\frac{1}{2} \frac{m}{q} \frac{V_\perp^2}{B}}^{\mu} \frac{(\vec{B} \times \vec{\nabla} B) \cdot \vec{R}_c}{R_c^2 B^2}$$

The fractional change in $V_\parallel \delta s$

$$\frac{1}{V_\parallel \delta s} \cdot \frac{d}{dt}(V_\parallel \delta s) = \frac{1}{\delta s}\frac{d\delta s}{dt} + \frac{1}{V_\parallel}\frac{dV_\parallel}{dt}$$

show first. ☆

$$= \frac{1}{2} \frac{mV_\perp^2}{qB^3} (\vec{B} \times \vec{\nabla} B) \cdot \frac{\vec{R}_c}{R_c^2} - \frac{1}{2} \frac{m}{q} \frac{V_\perp^2}{B} \frac{(\vec{B} \times \vec{\nabla} B) \cdot \vec{R}_c}{R_c^2 B^2}$$

$$= 0$$

$\Rightarrow V_\parallel \delta s = $ const.

the turning points on $\delta s'$ do not coincide w/ intersections of the perpendicular plane

a' a b different ds's, $V_\parallel \approx 0$

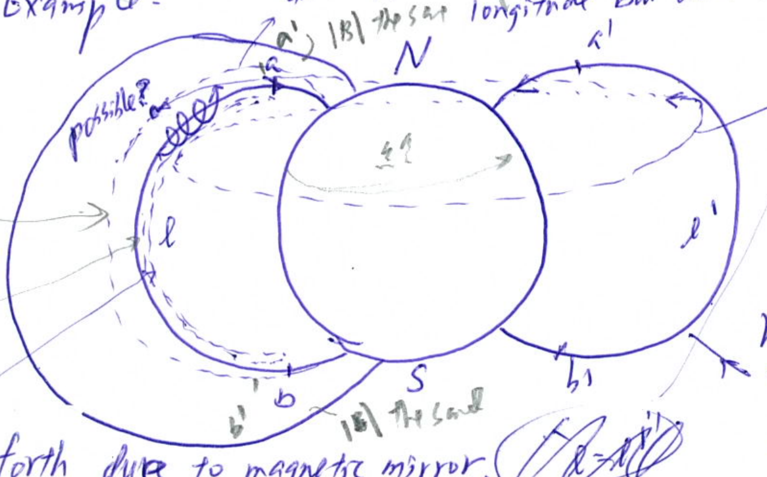
$\because$ At turning point, $V_\parallel \sim 0$.

$\therefore J = \int_a^b V_\parallel \, dS = $ const.

$$J = \int_a^b V_\parallel ds = \int_{a'}^{b'} V_\parallel ds$$

"L-shell", surface w/ the same J.

Example: another fieldline (at the same 經 longitude but different 緯 altitude) ~~altitude~~ $|B|$ the same



slowly drift in longitude around the earth

$\because \mu$ is invariant.

$\therefore |B|$ remains the same at the turning point.

actual field is distorted by such effects as the solar wind.

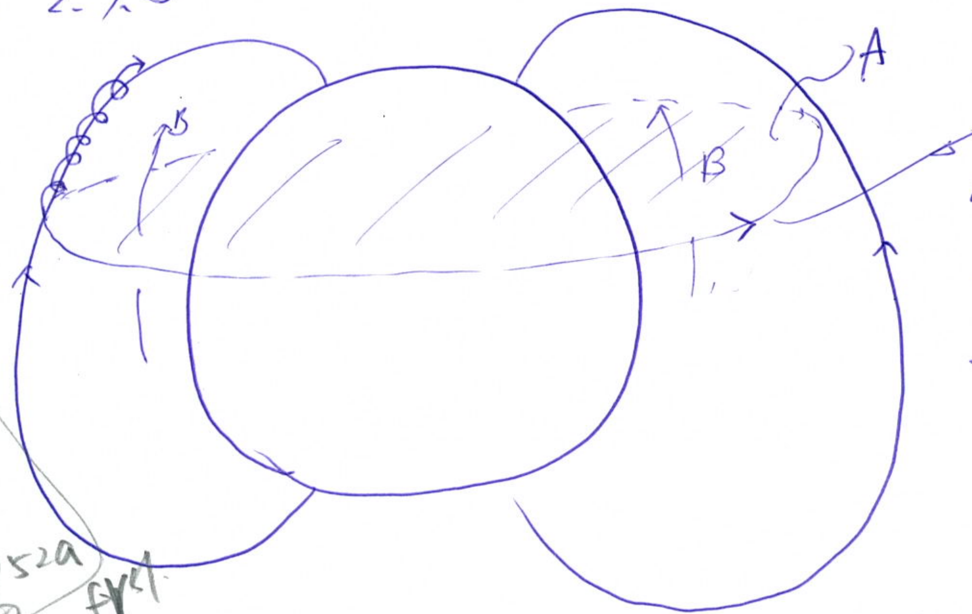
bounce back & forth due to magnetic mirror.

$J = J'$

$J = \int_a^b v_\parallel ds$ invariant : J determines the length of the line of force between turning points, and no two lines have the same length between points with the same $|B|$

$\Rightarrow$ the particle returns to the same line of force even in a slightly asymmetric field.

3. 2.7.3 The third Adiabatic Invariant Φ.



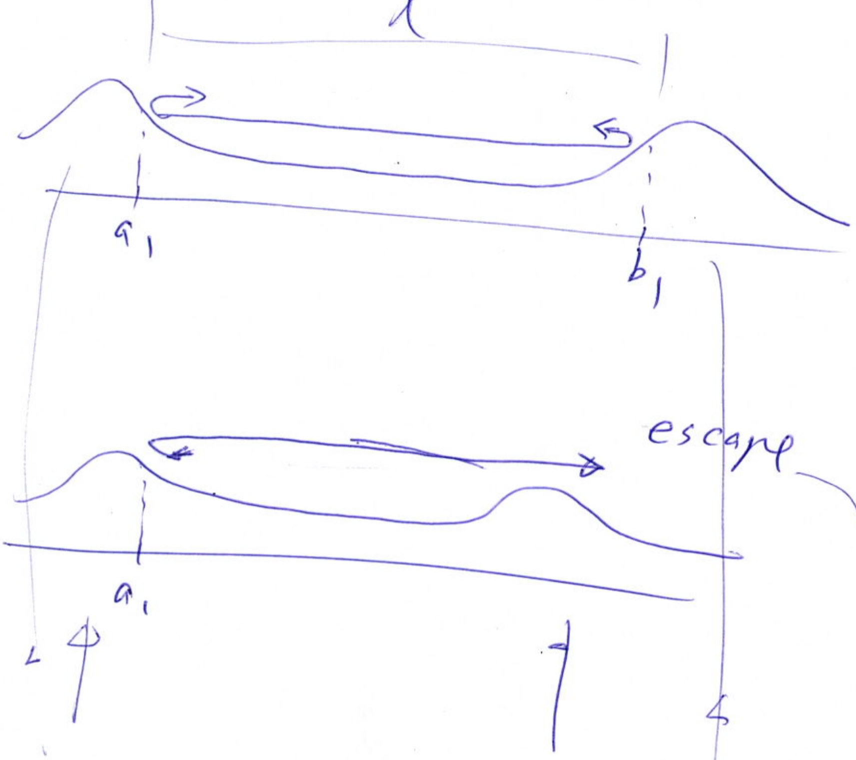
on p52a fig1

$\Phi \equiv \int \vec{B} \cdot d\vec{A}$: total magnetic flux enclose by the drift surface.

$\Rightarrow$ the particle will stay on a surface such that the total number of lines of force enclosed remains const.

$\Rightarrow$ Few applications because most fluctuations of B occur on a time scale short compared w/ the drift period.

Fermi's acceleration mechanism



$$J = \int_a^b v_\parallel \, ds = \text{const.}$$

$l \downarrow \Rightarrow v_\parallel \uparrow$

$\Rightarrow$ acceleration.

$\Rightarrow$ when $v_\parallel$ is too large

Strong magnetic field embedded on two approaching plasma clouds

Fermi acceleration may also be effective in explaining the acceleration of particles by shock waves associated w/ solar flares and supernova explosions.

[pdf Introduction to Plasma Physics

from $\frac{dB}{dt}$

3^{rd} adiabatic invariant Φ

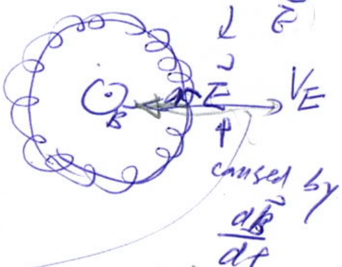
azimuthal $\vec E$ generated from $\dot{\vec B}$

$$\oint_C \vec E \cdot d\vec l = -\int_S \frac{\partial \vec B}{\partial t} \cdot d\vec A$$

$$\int \nabla \times \vec E \cdot d\vec A = -\int \frac{\partial \vec B}{\partial t}\, d\vec A$$

$$\| \quad \oint_C \vec E \cdot d\vec l$$

azimuthal $\vec E$



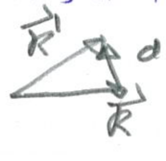
caused by $\frac{d\vec B}{dt}$

$\Rightarrow 2\pi R E = -\frac{dB}{dt} \cdot (\pi R^2)$ drift in $\hat R$

$\Rightarrow E = -\frac{R}{2}\frac{dB}{dt}$

$\Rightarrow \vec E \times \vec B$ drift: $\vec v_E = \frac{E}{B}\hat\phi = -\frac{R}{2B}\frac{dB}{dt}\hat\phi$

$v_E = \frac{dR}{dt} = -\frac{R}{2B}\frac{dB}{dt}$



$\Rightarrow 2\int \frac{dR}{R} = -\int \frac{dB}{B}$

$\Rightarrow 2 \ln R = -\ln B + \text{const}$

$\Rightarrow \ln(BR^2) = \text{const}$

$\Rightarrow \Phi = \pi R^2 B = \text{const.}$

charge particles remain on the same flux tube surface if B change slowly